INVARIANCE AND CONTRACTIVITY OF POLYHEDRA FOR LINEAR CONTINUOUS-TIME SYSTEMS WITH SATURATING CONTROLS

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Abstract: The study of the positive invariance and contractivity properties of polyhedral sets with respect to (w.r.t) linear systems subject to control saturation is addressed. The analysis of the nonlinear behavior of the closed-loop saturated system is made by dividing the state space in regions of saturation. In each one of these regions, the system evolution can be represented by the that of a linear system with an additive constant disturbance. From this representation, a sufficient algebraic condition relative to the positive invariance of a polyhedral set is given. In a second stage, using the same system representation, a necessary and sufficient condition to the contractivity of a compact polyhedral set with respect to the trajectories of the system stated. In this case, it is shown that a Lyapunov function can be associated with the polyhedral set and the local asymptotic stability of the saturated closed-loop system inside the set is guaranteed. From these results, an algorithm based on linear programming is proposed to determine such positively invariant and contractive polyhedral sets.
Keywords : control saturation, positive invariance, contractivity, local stability, polyhedral Lyapunov function.

Invariância e Contratividade de Regiões Poliedrais para Sistemas Lineares Contínuos no Tempo sujeitos à Saturação de Controle

Resumo: Este trabalho aborda as propriedades de invariância positiva e contratividade de conjuntos poliedrais com relação à sistemas lineares sujeitos à saturação de controle. A análise do comportamento não linear do sistema em malha fechada é feita a partir de uma divisão do espaço de estados em regiões de saturação. Em cada uma destas regiões, o comportamento do sistema pode ser representado por aquele de um sistema linear sujeito à ação de uma perturbação aditiva constante. A partir desta representação, é estabelecida uma condição algébrica suficiente para a garantia de invariância positiva de um domínio poliedral. Em um segundo momento, usando a mesma representação, estabelece-se uma uma condição necessária e suficiente para a contratividade de uma região poliedral compacta com relação às trajetórias do sistema. Neste caso, é mostrado que uma função de Lyapunov pode ser associada ao sistema em malha fechada garantindo a estabilidade assintótica dentro da região poliedral. A partir destes resultados, um algoritmo baseado em programação linear é proposto para a determinação de tais regiões poliedrais positivamente invariantes e contrativas.
Palavras chaves: saturação de controle, invariância positiva, contratividade, estabilidade local, função de Lyapunov poliedral.

1 INTRODUCTION

The problem of control constraints appears in most of industrial control systems. Due to physical or security reasons, the actuators cannot drive unlimited energy to the controlled plants. This fact can be modeled as a saturation block in the closed-loop scheme of the system.

Control systems are often linearly designed. The plant is represented by a local linear model and the control law is in general a linear state or output feedback. The modern theory of linear control furnishes efficient techniques and methodologies to compute such control laws that guarantee both the stability and some performance and robustness requirements with respect to (w.r.t) the linear closed-loop model of the process. In general, this kind of design does not directly consider the bounds on the control inputs. In this case, the control saturation can be source of parasitic equilibrium points and limit cycles, or even, can lead the closed-loop system to an unstable behavior. This fact has motivated, in the last years, the development of analysis and design techniques considering the control bounds and saturation occurrence (see for example (Bernstein and Michel, 1995)).

An interesting approach proposed in the literature consists in considering a set D_0 of admissible initial states and in determining a feedback control law that guarantees the positive invariance of a set $P \supset D_0$ such that P is contained in the region of linear behavior of the closed-loop system. The set P is considered as a *linear local region of stability*. This problem is also known in the literature as *saturation avoidance problem* or *linear constrained regulator problem* (see for example (Benzaouia and Hmamed, 1993), (Bitsoris, 1991), (Milani, 1994), (Vassilaki and Bitsoris, 1989),(Tarbouriech and Burgat,1994), for continuous-time systems and (Castelan *et al.*, 1996) with references therein for the discrete-time case). However, by this method if the set D_0 is relatively large, the calculated control law can degrade the performances of the closed-loop non-saturated system or the solution may not exist. Then, an alternative approach consists in computing a control law satisfying certain performance and ro-

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bustness requirements and analyzing, a posteriori, the stability of the closed-loop system when submitted to control saturation. In this case, given the control law, the designer is interested in determining regions of initial states that can be driven asymptotically to the origin by considering the possibility of control saturation. These regions can be viewed as *zones of safe operation* of the closed-loop system. From this analysis the designer can therefore evaluate if the determined zone contains the set D_0, i.e., if the computed control law guarantees that all the trajectories emanating from D_0 converge asymptotically to the origin. In this context, the determination of ellipsoidal type regions was addressed in (Burgat and Tarbouriech, 1996),(Kim and Bien, 1994),(Gutman and Hagander, 1985).

The aim of this paper is to propose a method of analysis that guarantees the local asymptotic stability, in polyhedral sets, of linear multivariable continuous-time systems subject to control saturation. First the positive invariance of polyhedral sets w.r.t saturated systems is addressed. A sufficient condition for the positive invariance of a given polyhedral set w.r.t this kind of systems is stated. This condition can be viewed as a generalization of the classical relations of positive invariance for linear systems without control saturation (Bitsoris, 1991),(Castelan and Hennet, 1992). Since the positive invariance is not sufficient to guarantee the local asymptotic stability , in a second stage, a necessary and sufficient condition to guarantee the contractivity of a polyhedral set is stated. Consequently, a polyhedral Lyapunov function (see (Blanchini, 1995) and (Kiendl *et al.*,1992)) can be associated with the contractive set and the local asymptotic stability of the saturated closed-loop system is ensured. Finally, from these conditions an algorithm based on linear programming is proposed to compute homothetical expansions of contractive polyhedra over the zone of nonlinear behavior of the closed-loop system. This method, that can be used to determine polyhedral regions of stability, is illustrated by an example.

Notations. For any vector $x \in \Re^n$, $x \geq 0$ means that all the components of x, denoted $x_{(i)}$, are nonnegative. For two vectors x, y of $\Re^n$, the notation $x \geq y$ means that $x_{(i)} - y_{(i)} \geq 0$, $\forall i = 1,...,n$, and $\ll x, y \gg = \sum_{i=1}^{n} x_{(i)} y_{(i)}$. $|x|$ is the vector composed by the absolute values of the components of x. For any real matrix M, square or not, $M_{(i)}$ denotes its ith line and $M_{(i,l)}$ the entry m_{il}. M^T and $\mathcal{K}er M$ denote respectively the transpose and the null space of M. For a polyhedral set $\mathcal{S}$, $int(\mathcal{S})$, $\partial\mathcal{S}$ and $\partial_i\mathcal{S}$ denote respectively the interior, the boundary and the ith facet of $\mathcal{S}$. Given a vector $x \in \Re^n$, $diag(x)$ denotes an n-order diagonal matrix generated from x. $1_m \triangleq [1\ 1 \ldots 1]^T \in \Re^m$.

2 POLYHEDRAL SETS

In this section we state some definitions about polyhedral sets that will be used in the sequel.

A *polyhedral set* in the state space is a finite intersection of half-spaces and can be defined as follows:

$$S(G, w) = \{x \in \Re^n ; Gx \leq w\}\ ,\ G \in \Re^{g*n}\ ,\ w \in \Re^g \qquad (1)$$

In particular, if $w_{(i)} > 0$, $\forall i = 1, \ldots, g$, $x = 0$ belongs to the interior of the polyhedral set, i.e., $0 \in int(S(G, w))$.

Notice that a polyhedral set is always a closed and convex set of $\Re^n$. A compact polyhedral set is also called a *polytope* (Bronsted, 1983).

The *i-facet* (Bronsted, 1983) (or the *i-side*) of polyhedral set $S(G, w)$ is defined as:

$$\partial_i S(G, w) \triangleq \{x \in \Re^n\ ;\ G_{(i)} x = w_{(i)},\ G_{(k)} x \leq w_{(k)} \text{ for } k \neq i\}$$

The *polyhedral cone generated by the i-facet of* $S(G, w)$ is the set defined as:

$$\mathcal{K}_i \triangleq \{x \in \Re^n\ ;\ \frac{G_{(k)} x}{w_{(k)}} \leq \frac{G_{(i)} x}{w_{(i)}}\ ,\ \ \forall k = 1, \ldots, g,\ \ k \neq i\} \qquad (2)$$

The cone $\mathcal{K}_i$ can also be defined from the vertices of $S(G, w)$ that belongs to facet $\partial_i S(G, w)$. The vector $G_{(i)}^T \in \Re^n$ is the *normal vector* to the of facet $\partial_i S(G, w)$, that is,

$$\ll G_{(i)}^T, x \gg - w_{(i)} = 0 \ \ \forall x \in \partial_i S(G, w)$$

From a given polyhedral set it is possible to construct a family of polyhedra by applying an homothety to the primal polyhedral set:

$$S(G, w\alpha) = \{x \in \Re^n ; Gx \leq w\alpha\},\ \ \alpha \in \Re,\ \ \alpha > 0$$

The positive scalar α is called *coefficient of homothety*. The sets $S(G, w\alpha)$, with $\alpha \in (0, 1]$, are called *internal homothetic* sets of $S(G, w)$.

Associated to a polyhedral set $S(G, w)$ we have the Minkowski Functional (Blanchini, 1995),(Kiendl *et al.*,1992),(Sznaier, 1993):

$$\mathcal{V}(x) = \max_i\ \{G_{(i)} x / w_{(i)}\} \qquad (3)$$

It follows that if $x \in \partial S(G, \alpha w)$, $\mathcal{V}(x) = \alpha$. Besides, $\forall x \in \mathcal{K}er G$ it follows that $\mathcal{V}(x) = 0$. Note that in the case where $S(G, w)$ is a polytope, $\mathcal{V}(x)$ is a norm. Otherwise, $\mathcal{V}(x)$ is a semi-norm.

3 PROBLEM STATEMENT

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (4)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $A \in \Re^{n*n}$ and $B \in \Re^{n*m}$.

The control vector is subject to linear constraints that define a compact polyhedral region Ω in the control space:

$$\Omega \triangleq \{u \in \Re^m\ ;\ -u_{min} \leq u \leq u_{max}\} \qquad (5)$$

with $u_{min(i)}, u_{max(i)} \geq 0$, for $i = 1, \ldots, m$.

Consider now a classical saturating state feedback control law:

$$u(t) = sat(Fx(t))\ \ ,\ \ F \in \Re^{m*n}$$

where the ith component of the control vector, $i = 1, \ldots, m$, is defined as follows:

$$u_{(i)}(t) = \begin{cases} -u_{min(i)} & \text{if } F_{(i)} x(t) < -u_{min(i)} \\ F_{(i)} x(t) & \text{if } -u_{min(i)} \leq F_{(i)} x(t) \leq u_{max(i)} \\ u_{max(i)} & \text{if } F_{(i)} x(t) > u_{max(i)} \end{cases} \qquad (6)$$

The closed-loop system is given by

$$\dot{x}(t) = Ax(t) + Bsat(Fx(t)) \qquad (7)$$

It is worth to notice that the polyhedral region $S(F, u_{min}, u_{max})$ defined in the state space as

$$S(F, u_{min}, u_{max}) \stackrel{\triangle}{=} \{x \in \Re^n \ ; \ -u_{min} \leq Fx \leq u_{max}\} \quad (8)$$

is the *region of linearity* of the closed-loop system. In $S(F, u_{min}, u_{max})$, saturation does not occur and the evolution of the closed-loop system is given by

$$\dot{x}(t) = (A + BF)x(t) \quad (9)$$

We define now the positive invariance and contractivity properties of a set $S(G, w)$ with respect to system (7).

Definition 1 *: The polyhedral set $S(G, w)$ is positively invariant w.r.t system (7) if and only if $\forall x(0) \in S(G, w)$ the corresponding state trajectory of system (7) is confined in $S(G, w)$. In other words,*

$$\forall x(0) \in S(G, w) \Rightarrow x(t) \in S(G, w), \forall t > 0$$

In particular, since $x(t)$ is a continuous function, the positive invariance of $S(G, w)$ implies that every trajectory that originates in $S(G, w)$ does not escape from $S(G, w)$.

Definition 2 *: The polyhedron $S(G, w)$ defined by (1) is said to be contractive w.r.t the trajectories of system (7) if, $\forall x(t) \in S(G, w)$, $\forall t \geq 0$, $\forall \tau > 0$, the implication*

$$x(t) \in \partial S(G, w\alpha) \Rightarrow x(t + \tau) \in int(S(G, w\alpha))$$

holds $\forall \alpha \in (0, 1]$.

The interpretation of this definition is the following. If, at instant t, $x(t)$ belongs to the boundary of an interior homothetic of $S(G, w)$ ($S(G, w\alpha)$ with $0 < \alpha \leq 1$) it follows that, at the instant $(t+\tau)$, with τ infinitesimal, $x(t+\tau)$ belongs to the interior of this homothetic. In this case, we can say that $x(\infty)$ belongs to $\lim\limits_{\alpha \to 0} S(G, w\alpha)$.

When $S(G, w) \subset S(F, u_{min}, u_{max})$ the analysis of the positive invariance and the asymptotic stability in $S(G, w)$ w.r.t system (7) is equivalent to the analysis of these properties w.r.t. system (9). The conditions that guarantee the positive invariance and the asymptotic stability in this case were widely studied in the literature (see for example, (Benzaouia and Hmamed, 1993),(Bitsoris, 1991),(Castelan and Hennet, 1992),(Tarbouriech and Burgat, 1994),(Vassilaki and Bitsoris, 1989)).

When $S(G, w) \not\subset S(F, u_{min}, u_{max})$, these properties have to be studied by considering the nonlinear behavior of the closed-loop system (7). In this work we are interested in this case. Hence, the following problems will be treated in the sequel :

- **Problem 1 :** Determine conditions to guarantee the positive invariance of $S(G, w)$ w.r.t the saturated system (7).
- **Problem 2 :** Determine conditions to guarantee the contractivity and the asymptotic stability of the system (7) in $S(G, w)$.

4 SATURATED SYSTEM REPRESENTATION

Before studying the positive invariance, contractivity, and stability properties of polyhedral sets related to linear systems with saturating controls, we define a mathematical representation for this kind of systems.

The representation chosen consists in dividing the state space in regions called *regions of saturation*. A region of saturation is defined by the intersection of half-spaces of type $F_{(i)}x \leq d_{(i)}$ or $-F_{(i)}x \leq d_{(i)}$, where $d_{(i)}$ can be $u_{min(i)}$,$-u_{min(i)}$, $u_{max(i)}$ or $-u_{max(i)}$. For a system with m inputs, there exists 3^m regions of saturation. Considering $j = 1, \ldots, 3^m$, the jth region of saturation is a polyhedral set denoted generically as

$$S(R_j, d_j) = \{x \in \Re^n \ ; \ R_j x \leq d_j\} \quad (10)$$

where $d_j \in \Re^{l_j}$ is defined from the entries of u_{max}, $-u_{max}$, u_{min} et $-u_{min}$, and $R_j \in \Re^{l_j * n}$ is defined from the rows of F and $-F$ (see figure 1 and the numerical example in section 8).

We show now that inside each region of saturation, system (7) can be modeled as a linear system with an additive constant disturbance.

Consider a vector $\xi \in \Re^m$ such that each entry $\xi_{(i)}$, $i = 1, \ldots, m$, takes the values 1, 0 or -1 as follows :

- If $u_{(i)}(t) = u_{max(i)}$ then $\xi_{(i)} = 1$, that is, $x(t)$ is such that $F_{(i)}x(t) > u_{max(i)}$.
- If $u_{(i)}(t) = F_{(i)}x(t)$ then $\xi_{(i)} = 0$, that is, $x(t)$ is such that $-u_{min(i)} \leq F_{(i)}x(t) \leq u_{max(i)}$.
- If $u_{(i)}(t) = -u_{min(i)}$ then $\xi_{(i)} = -1$, that is, $x(t)$ is such that $F_{(i)}x(t) < -u_{min(i)}$.

Hence, each vector ξ represents a possible combination between saturated and non-saturated control entries. Note that there are 3^m different vectors ξ: $\xi_j \in \Re^m$ for $j = 1, \ldots, 3^m$ and it is possible to associate each vector ξ_j to a specific region of saturation $S(R_j, d_j)$. Notice also that the region corresponding to $\xi_j = 0$ is the polyhedron $S(F, u_{min}, u_{max})$. In the other regions there is at least one control entry that is saturated.

From the definition of ξ_j, the motion of the system (7) inside the region $S(R_j, d_j)$ can be described by the following linear dynamical equation:

$$\dot{x}(t) = (A + Bdiag(1_m - |\xi_j|)F)x(t) + Bu(\xi_j) \quad (11)$$

where

$$u_{(i)}(\xi_j) \stackrel{\triangle}{=} \begin{cases} -u_{min(i)} & \text{if} \quad \xi_{j(i)} = -1 \\ 0 & \text{if} \quad \xi_{j(i)} = 0 \\ u_{max(i)} & \text{if} \quad \xi_{j(i)} = 1 \end{cases} \quad (12)$$

Generically, if $x(t) \in S(R_j, d_j)$ it follows that (Rocha, 1994):

$$\dot{x}(t) = \bar{A}_j x(t) + p_j \quad (13)$$

with $\bar{A}_j = A + Bdiag(1_m - |\xi_j|)F$ and $p_j = Bu(\xi_j)$.

5 POSITIVE INVARIANCE

In this section conditions to ensure the positive invariance of a polyhedron $S(G, w)$ w.r.t the saturated system (7) are studied. These conditions are established from the representation of the saturated system described in the above section.

Definition 3 *: Let the set $\mathcal{J}$ be the set of indices j such that $(S(R_j, d_j) \cap \partial S(G, w))) \neq \emptyset$. The regions $S(R_j, d_j)$ such that $j \in \mathcal{J}$ are called* target regions

For each target region define a polyhedral set $S(D_j, s_j) \triangleq S(R_j, d_j) \cap \partial S(G, w)$, $D_j \triangleq \begin{bmatrix} \bar{G}_j \\ R_j \end{bmatrix}$, $s_j \triangleq \begin{bmatrix} \bar{w}_j \\ d_j \end{bmatrix}$, where $\bar{G}_j \in \Re^{g_j * n}$ and $\bar{w}_j \in \Re^{g_j}$ correspond to the facets of $S(G, w)$ that have a nonempty intersection with $S(R_j, s_j)$. Figure 1 depicts the regions $S(R_j, d_j)$ (notation R_j) and $S(D_j, s_j)$ (notation D_j) for a second order system with two control entries.

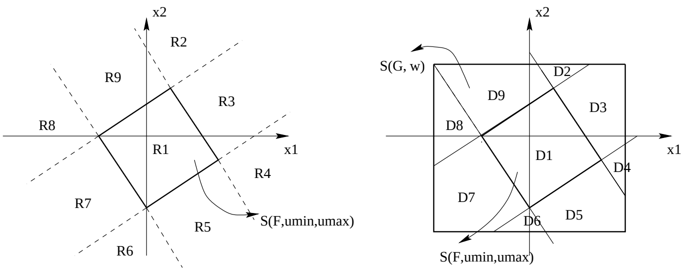

Figure 1: Representation of $S(R_j, d_j)$ and $S(D_j, s_j)$ for $n = m = 2$

Lemma 1 *: The set $S(G, w)$ is positively invariant w.r.t system (7) if*

$$G_{(i)}\dot{x}(t) < 0 \quad \forall x(t) \in \partial_i S(G, w), \;\; \forall i = 1, \ldots, g \tag{14}$$

Proof: Since the state trajectory is continuous, for the positive invariance of $S(G, w)$ it is sufficient to guarantee the admissibility of any infinitesimal motion starting from any point on the boundary of this domain (Castelan and Hennet, 1992). Hence, if at instant $t = t_0$ $x(t_0)$ belongs to the facet $\partial_i S(G, w)$ it suffices that the time derivative vector $\dot{x}(t_0)$ points towards the interior of $S(G, w)$ (see (Berman *et al.*, 1989) pp.65-66). Note that $G_{(i)}^T$ points always towards the exterior of the polyhedral set. Applying this reasoning to all points of $\partial S(G, w)$ the condition for the positive invariance is equivalent to:

$$\ll G_{(i)}^T, \dot{x}(t) \gg < 0 \quad \forall x(t) \in \partial_i S(G, w), \;\; \forall i = 1, \ldots, g$$

which is equivalent to condition (14). □

Theorem 1 *: The set $S(G, w)$ is positively invariant w.r.t system (7) if, for each target region j, there exists a matrix $H_j \in \Re^{g_j * (g_j + l_j)}$, with $H_{j(i,l)} \geq 0$ if $i \neq l$, such that :*

$$\begin{aligned} H_j D_j &= \bar{G}_j \bar{A}_j & (15) \\ H_j s_j &< -\bar{G}_j p_j & (16) \end{aligned}$$

Proof: For each target region of saturation j, consider the following set of linear programs:

$$LP_{j,i} : \begin{cases} y_{j(i)} = \min\limits_{H_{j(i)}} H_{j(i)} s_j + \bar{G}_{j(i)} p_j \\ \qquad \text{subject to} \\ H_{j(i)} D_j = (\bar{G}_j \bar{A}_j)_{(i)} \\ H_{j(i,l)} \geq 0 \quad \text{if } l \neq i. \end{cases} \tag{17}$$

where $H_{j(i)}^T \in \Re^{g_j + l_j}$. Let $H_{j(i)}^*$ be the optimal solution of $LP_{j,i}$. The satisfaction of relations (15) and (16) implies that:

$$H_{j(i)}^* s_j + \bar{G}_{j(i)} p_j < 0 \tag{18}$$

By duality (see for example (Luenberger, 1984), each of programs $LP_{j,i}$ is equivalent to the followig one :

$$DLP_{j,i} : \begin{cases} y_{j(i)} = \max\limits_{x} \bar{G}_{j(i)} \bar{A}_j x + \bar{G}_{j(i)} p_j \\ \qquad \text{subject to} \\ \bar{G}_{j(i)} x = \bar{w}_{j(i)} \\ \bar{G}_{j(l)} x \leq \bar{w}_{j(l)} \quad l = 1, \ldots, g_j \;;\; l \neq i \\ R_j x \leq d_j \end{cases} \tag{19}$$

Let $x_{j(i)}^*$ be the optimal solution of $DLP_{j,i}$. From duality, it follows also that

$$\bar{G}_{j(i)} \bar{A}_j x_{j(i)}^* + \bar{G}_{j(i)} p_j = H_{j(i)}^* s_j + \bar{G}_{j(i)} p_j < 0 \tag{20}$$

Hence, from (19), $\forall x(t) \in (\partial_i S(G, w) \cap S(D_j, s_j)) \triangleq \{x \in S(D_j, s_j)\,;\, R_j x \leq d_j\,,\; \bar{G}_{j(i)} x = \bar{w}_{j(i)}\,,\; \bar{G}_{j(l)} x \leq \bar{w}_{j(l)}, \forall l \neq i\}$, $\forall i = 1, \ldots, g_j$, $\forall j \in \mathcal{J}$, it follows

$$\bar{G}_{j(i)}(\bar{A}_j x(t) + p_j) < 0 \tag{21}$$

Therefore, the satisfaction of (15) and (16), for all target region j, with $H_{j(i,l)} \geq 0$ if $i \neq l$, implies that

$$G_{(i)}\dot{x}(t) < 0 \quad \forall x(t) \in \partial_i S(G, w), \;\; \forall i = 1, \ldots, g$$

which, from Lemma 1, ensures the positive invariance of $S(G, w)$ w.r.t. the system (7). □

Remark 1 *: Note that the classical strict positive invariance relations for the linear closed-loop system (9) (i.e. when $S(G, w) \subseteq S(F, u_{min}, u_{max})$) (Bitsoris, 1991),(Castelan and Hennet, 1992):*

$$\begin{aligned} HG &= G(A + BF) & (22) \\ Hw &< 0 & (23) \end{aligned}$$

with $H_{(i,l)} \geq 0$ if $i \neq l$, are a particular case of relations (15) and (16). In fact, if $S(G, w) \subseteq S(F, u_{min}, u_{max})$ the only target region is $S(G, w)$. In this case $\bar{G} = G$, $\bar{A}_j = (A + BF)$, $p_j = 0$, $D_j = G$, $s_j = w$.

6 CONTRACTIVITY AND LOCAL ASYMPTOTIC STABILITY

The conditions stated in Theorem 1 guarantee that $\forall x(0) \in S(G, w)$, the corresponding trajectories of the saturated system (7) are confined in $S(G, w)$, $\forall t \geq 0$. However, this property is not sufficient to ensure the convergence of the trajectories to the origin. Of course, when $S(G, w)$ is contained in $S(F, u_{min}, u_{max})$ and is positively invariant, the asymptotic stability in $S(G, w)$ is guaranteed if all the eigenvalues of $(A + BF)$ are in the open left half-plane (Vassilaki and Bitsoris, 1989). However, this is only a necessary condition if $S(G, w) \not\subseteq S(F, u_{min}, u_{max})$. In this case, since the behavior of the system is nonlinear, the possible existence of limit cycles and/or parasitic equilibrium points inside $S(G, w)$ have to be considered. Hence, before concluding that the polyhedron $S(G, w)$ is also a region of asymptotic stability for system (7) it is necessary to eliminate these possibilities. The verification of the existence of equilibrium points, different from the origin, inside $S(G,w)$ is trivial. Nevertheless, in general, it is not easy to verify if there exist limit cycles inside $S(G,w)$.

In this case, a conservative, but relatively easy, way of ensuring the asymptotic stability of the system inside a positively invariant set consists in guaranteeing the contractivity of this set.

Consider the Minkowski functional defined in (3) along the trajectories of system (7) (Blanchini, 1995),(Kiendl *et al.*,1992),(Sznaier, 1993).

Define the upper right Dini derivative of $\mathcal{V}(x(t))$ as follows (Rouche, 1977):

$$D^+\mathcal{V}(x(t)) = \lim_{\tau\to 0^+}\sup \frac{\mathcal{V}(x(t+\tau)) - \mathcal{V}(x(t))}{\tau}$$

Since $\mathcal{V}(x(t))$ is a continuous function, $D^+\mathcal{V}(x(t))$ is well-defined and its existence is guaranteed (Rouche, 1977)

If $S(G,w)$ is compact the following Lemma can be stated.

Lemma 2 *: A compact polyhedron $S(G,w)$ is contractive w.r.t. system (7) if and only if*

$$D^+\mathcal{V}(x(t)) < 0 \tag{24}$$

$\forall x(t) \in S(G,w)\ ,\ \ x(t) \neq 0\ ,\ \ \forall t \geq 0.$

Proof: *Sufficiency:* Suppose that $x(t) \in \partial S(G,\alpha w)$, $\alpha \in (0,1]$. In this case, $\mathcal{V}(x(t)) = \alpha$. Hence, if (24) holds, it follows that $\mathcal{V}(x(t+\tau)) < \alpha$, that is, $x(t+\tau) \in intS(G,\alpha w)$ where τ is a positive infinitesimal. Since this is valid for all $x(t) \in S(G,w)$ contractivity is proven according to Definition 2.
Necessity: Suppose that $S(G,w)$ is contractive and $D^+\mathcal{V}(x(t)) \geq 0$ for some $x(t) \in \partial S(G,\alpha w)$, $\alpha \in (0,1]$. In this case, $\mathcal{V}(x(t+\tau)) \geq \alpha$ and it follows that $x(t+\tau) \notin intS(G,w)$ which contradicts the assumption that $S(G,w)$ is contractive. □

The results in the sequel are stated by supposing that $S(G,w)$ is compact.

Lemma 3 *: A compact polyhedron $S(G,w)$ is contractive w.r.t. system (7) if and only if*

$$G_{(i)}\dot{x}(t) < 0 \tag{25}$$

$\forall x(t) \in \partial_i S(G,w\alpha),\ \forall \alpha \in (0,1],\ \forall i = 1,\ldots,g\ ,\forall t \geq 0.$

Proof: From Lemma 2 the contractivity of $S(G,w)$ is equivalent to satisfy:

$$D^+\mathcal{V}(x(t)) = \lim_{\tau\to 0^+} \frac{\max_i\left\{\frac{G_{(i)}x(t+\tau)}{w_{(i)}}\right\} - \max_i\left\{\frac{G_{(i)}x(t)}{w_{(i)}}\right\}}{\tau} < 0 \tag{26}$$

Since $x(t)$ is a continuous function and by supposing that at instant t, $x(t) \in \partial_l S(G,w)$ it follows that (26) is equivalent to:

$$D^+\mathcal{V}(x(t)) = \lim_{\tau\to 0^+} \frac{G_{(l)}x(t+\tau) - G_{(l)}x(t)}{\tau} < 0 \tag{27}$$

Expanding $x(t+\tau)$ in Taylor's serie we have:

$$x(t+\tau) = x(t) + \tau\frac{\dot{x}(t)}{1!} + \tau^2\frac{x^{(2)}(t)}{2!} + \ldots + \frac{x^{(n)}(t)}{n!} + \mathcal{O}_{n+1} \tag{28}$$

From (27), (28) can be re-written as follows:

$$D^+\mathcal{V}(x(t)) = \lim_{\tau\to 0^+}\frac{\tau G_{(l)}\dot{x}(t)}{\tau w_{(l)}} + \lim_{\tau\to 0^+}\left(\frac{\tau^2 G_{(l)}x^{(2)}(t)}{\tau w_{(l)}2!} + \ldots + \frac{\tau^n G_{(l)}x^{(n)}(t)}{\tau w_{(l)}n!} + \ldots\right) = \frac{G_{(l)}\dot{x}(t)}{w_{(l)}} \tag{29}$$

whence $D^+\mathcal{V}(x(t)) < 0$ if and only if $G_{(l)}\dot{x}(t) < 0$.

Since this reasoning can be applied $\forall x(t) \in \partial_i S(G,w\alpha),\ \forall \alpha \in (0,1],\ \forall i = 1,\ldots,g\ ,\forall t \geq 0$, it follows that condition (25) is equivalent to have $D^+\mathcal{V}(x(t)) < 0$, $\forall x(t) \in S(G,w)$, $\forall t > 0$ which proves the lemma. □

Note that this result is similar to the one of Lemma 1. The difference here is that the *strict* inequality

$$\ll G_{(i)}^T, \dot{x}(t) \gg < 0$$

must be ensured for all $x(t)$ belonging to the polyhedral cone $\mathcal{K}_i$ defined by the facet $\partial_i S(G,w)$.

We state now the main result of this section.

Let $\bar{\mathcal{J}}$ be the set of indices j such that $S(G,w) \cap S(R_j,d_j) \neq \emptyset$. Define $\mathcal{I}_j$ as the set of indices i such that the cone $\mathcal{K}_i$ has a nonempty intersection with the region $S(R_j,d_j)$, i.e., $\mathcal{I}_j \triangleq \{i\ ;\ \mathcal{K}_i \cap S(R_j,d_j) \neq \emptyset\}$.

Theorem 2 *: Consider the following linear programs*

$$LP_{j,i} = \left\{\begin{array}{c} y_{j(i)} = \max_x G_{(i)}\bar{A}_j x + G_{(i)}p_j \\ \textit{subject to} \\ x \in (\mathcal{K}_i \cap S(R_j,d_j) \cap S(G,w)) \end{array}\right. \tag{30}$$

Define $y_j \triangleq \max\{y_{j(i)}\ ;\ i \in \mathcal{I}_j\}$. A compact polyhedron $S(G,w)$ is contractive w.r.t system (7) if and only if the following conditions hold:

(i) $y_{j(i)} < 0$ for each $j \in \bar{\mathcal{J}}$ such that $S(R_j,d_j) \neq S(F,u_{min},u_{max})$

(ii) $y_{j(i)} = 0$ for j such that $S(R_j,D_j) = S(F,u_{min},u_{max})$ and, in this case, the optimal solution of each linear program $LP_{j,i}$ is unique and obtained for $x = 0$.

Proof:
Sufficiency: For all $x(t) \in S(G,w)$, $x(t) \neq 0$, it follows that $x(t)$ belongs at least to one facet of $S(G,w\alpha)$, $0 < \alpha \leq 1$. In other words, $x(t) \in \partial_i S(G,w\alpha)$ and it follows that $x(t) \in \mathcal{K}_i$. Moreover $x(t)$ belongs to some region of saturation $S(R_j,d_j)$, $j \in \bar{\mathcal{J}}$. Thus, $i \in \mathcal{I}_j$ and $x(t) \in (\mathcal{K}_i \cap S(R_j,d_j) \cap S(G,w))$. Hence, since $x(t)$ is supposed to be different from zero, if conditions (i) and (ii) hold, it follows that

$$G_{(i)}\bar{A}_j x(t) + G_{(i)}p_j < 0$$

i.e., $G_{(i)}\dot{x}(t) < 0$. Since this reasoning can be applied $\forall x(t) \in S(G,w)$, from Lemma 3 the contractivity of $S(G,w)$ is guaranteed if conditions (i) and (ii) are verified.
Necessity: Suppose that $S(G,w)$ is contractive and condition (i) or (ii) is not verified. Then, for some $i \in \mathcal{I}_j, j \in \bar{\mathcal{J}}$, there may exist $x(t) \in (\mathcal{K}_i \cap S(R_j,d_j) \cap S(G,w))$, $x(t) \neq 0$, such

that $G_{(i)}(\bar{A}_j x(t) + p_j) \geq 0$. In this case $x(t) \in \partial_i S(G, w\alpha)$, for some $0 < \alpha \leq 1$, and it follows that $G_{(i)}\dot{x}(t) \geq 0$. Since $x(t) \in S(G, w)$, this contradicts the assumption that $S(G, w)$ is contractive and thus the necessity of the condition is proven. □

The basic idea of Theorem 2 consists in analyzing the closed-loop system trajectories in some sub-domains of $S(G, w)$. These sub-domains are delimited by a polyhedral cone $\mathcal{K}_i$ and a region of saturation $S(R_j, d_j)$. In each one of these sub-domains we verify if for every $x(t)$ ($x(t) \neq 0$) belonging to the domain one obtains $D^+\mathcal{V}(x(t)) < 0$. This test is accomplished by solving linear programs like (30). Hence, since $S(G, w)$ is supposed to be compact, if conditions (i) and (ii) are verified, we can conclude that $D^+\mathcal{V}(x(t)) < 0, \forall x(t) \in S(G, w), x(t) \neq 0$.

Note that if $S(G, w)$ is compact $\lim\limits_{\alpha \to 0} S(G, w\alpha) = \{0\}$. Thus, the contractivity of $S(G, w)$ implies the asymptotic convergence to the origin of all the trajectories emanating from $S(G, w)$. In fact, since $S(G, w)$ is supposed to be compact and contains the origin, the function $D^+\mathcal{V}(x(t))$ defined by (3) is a strictly decreasing Lyapunov function for system (7) in $S(G, w)$. The following corollary can be stated.

Corollary 1 *: If condition of Theorem 2 holds, then*

(i) System (7) is locally asymptotically stable in $S(G, w)$.

(ii) The polyhedral function $\mathcal{V}(x(t)) = \max\limits_i \{\frac{G_{(i)}x(t)}{w_{(i)}}\}$ is a strictly decreasing Lyapunov function for system (7) in $S(G, w)$.

Remark 2 *: If $S(G, w)$ is unbounded, Theorem 2 cannot be directly applied. Suppose, however, in this case that there exists $\alpha \leq 1$ such that $S(G, w\alpha) \subseteq S(F, u_{min}, u_{max})$, this implies that $\mathcal{K}erG \subseteq \mathcal{K}erF$. Thus, a necessary condition to the contractivity of $S(G, w)$ is that $\mathcal{K}erG$ is an $A-invariant$ subspace (Castelan and Hennet, 1992) and it follows that the pair (G, A) is nonobservable. Define $\mathcal{N}$ the unobservable subspace of (G, A). The projection of $S(G, w)$ in the subspace $\mathcal{N}^\perp$ (annihilator of $\mathcal{N}$) along $\mathcal{N}$, denoted by $S(G_o, w_o)$, is a compact polyhedron. Then it can be proven, since $\mathcal{K}erG \subseteq \mathcal{K}erF$, that the contractivity of $S(G, w)$ w.r.t system (7) is equivalent to the contractivity of $S(G_o, w_o)$ w.r.t a reduced-order saturated system obtained from the observable part of pair (G, A) and then Theorem 2 can be applied (this is an extension of the result presented in (Castelan* et al.*, 1996). Note, however, that in this case the contractivity does not imply the local asymptotic stability. For this, it is also necessary that pair (G, A) is detectable.*

7 DETERMINATION OF LOCAL ASYMPTOTIC STABILITY REGIONS

In this section, we consider that a contractive compact polyhedron $S(G, w) \subseteq S(F, u_{min}, u_{max})$ w.r.t system (9) was computed by one of the methods proposed in the literature (see for instance (Blanchini and Miani, 1996),(Castelan and Hennet, 1992),(Vassilaki and Bitsoris, 1989)). We propose now an algorithm to calculate the maximum coefficient of homothesis, δ_{max}, for which $S(G, \delta_{max} w)$ preserves the property of contractivity w.r.t the saturated system (7). In this case, note that, by hypothesis, the condition (ii) of Theorem 2 is automatically verified.

Algorithm

- **Step 0 -** Initialize : $\delta = \delta_0$. Choose a computational accuracy: $precision$.
- **Step 1 -** Determine $\bar{\mathcal{J}}$ w.r.t $S(G, \delta w)$. For each $j \in \bar{\mathcal{J}}$ solve the following linear programs $\forall i \in \mathcal{I}_j$:

$$y_{j(i)} = \max_x G_{(i)}\bar{A}_j x + G_{(i)}p_j \quad \text{subject to} \quad \begin{cases} R_j x \leq d_j \\ Gx \leq \delta w \\ (\frac{G_{(k)}}{w_{(k)}} - \frac{G_{(i)}}{w_{(i)}})x \leq 0 \ \ \forall k \neq i \end{cases} \tag{31}$$

- **Step 2 :** If conditions (i) and (ii) of Theorem 2 hold, goto step 4. Otherwise goto step 3.
- **Step 3 :** Decrease δ and return to step 2.
- **Step 4 :** If the difference between the δ of this iteration and the above iterations is greater than the chosen accuracy, increase δ. Otherwise stop: $\delta_{max} = \delta$.

From Corollary 1 the obtained set $S(G, \delta_{max} w)$ is a region of asymptotic stability. The proposed algorithm can be viewed as a tool to generate polyhedral regions of local stability and thus to approximate the region of attraction of the origin (Suarez et al., 1991) for system (7). This approximation can be improved by considering, for example, the union of different polyhedra obtained by the application of the proposed algorithm. In this case, the final domain may be non-convex.

8 NUMERICAL EXAMPLE

Consider system (4)-(5) described by the following matrices (Kim and Bien, 1994):

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix} \ ; \ B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$u_{min} = u_{max} = \begin{bmatrix} 5 & 2 \end{bmatrix}^T$$

A stabilizing state feedback matrix F and a positively invariant set $S(G, w) \subset S(F, u_{min}, u_{max})$ are given by :

$$F = \begin{bmatrix} -0.7283 & -0.0338 \\ -0.0135 & -1.3583 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \ ; \ w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Matrix F and control constraints define nine regions of saturation. Since the polyhedra $S(G, w)$ and $S(F, u_{min}, u_{max})$ are symmetric, we can analyze only five of these regions:

Region 1 ($\xi_1 = [0 \ \ 0]^T \quad \Leftrightarrow$ Reg. linearity):

$$\bar{A}_1 = A + BF = \begin{bmatrix} -3.5415 & -0.2690 \\ 0.0865 & -4.3583 \end{bmatrix} \ ; \ p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} F \\ -F \end{bmatrix} \ ; \ d_1 = \begin{bmatrix} 5 \\ 2 \\ 5 \\ 2 \end{bmatrix}$$

Region 2 ($\xi_2 = [0 \quad -1]^T$):

$$\bar{A}_2 = \begin{bmatrix} -3.5415 & -0.2690 \\ 0.1 & -3 \end{bmatrix} \;;\; p_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} -0.7283 & -0.0338 \\ 0.7283 & 0.0338 \\ -0.0135 & -1.3583 \end{bmatrix} \;;\; d_2 = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix}$$

Region 3 ($\xi_3 = [-1 \quad -1]^T$):

$$\bar{A}_3 = A \;;\; p_3 = \begin{bmatrix} -25 \\ -2 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} -0.7283 & -0.0338 \\ -0.0135 & -1.3583 \end{bmatrix} \;;\; d_3 = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

Region 4 ($\xi_4 = [-1 \quad 0]^T$):

$$\bar{A}_4 = \begin{bmatrix} 0.1 & -0.1 \\ 0.0865 & -4.3583 \end{bmatrix} \;;\; p_4 = \begin{bmatrix} -25 \\ 0 \end{bmatrix}$$

$$R_4 = \begin{bmatrix} -0.7283 & -0.0338 \\ -0.0135 & -1.3583 \\ 0.0135 & 1.3583 \end{bmatrix} \;;\; d_4 = \begin{bmatrix} -5 \\ 2 \\ 2 \end{bmatrix}$$

Region 5 ($\xi_5 = [1 \quad -1]^T$):

$$\bar{A}_5 = A \;;\; p_5 = \begin{bmatrix} 25 \\ -2 \end{bmatrix}$$

$$R_5 = \begin{bmatrix} 0.7283 & 0.0338 \\ -0.0135 & -1.3583 \end{bmatrix} \;;\; d_5 = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

By applying the algorithm described in section 7 one obtains $\delta_{max} = 124.99$ ($precision = 0.01$). This value gives the maximal homothetical set of $S(G, w)$ which is positively invariant and contractive w.r.t the closed-loop saturated system. Since $S(G, \delta_{max} w)$ is bounded, it is a domain of asymptotic stability and *safe operation*, for system (7). It is worth to notice that the maximal homothetical set of $S(G, w)$ contained in the region of linearity $S(F, u_{min}, u_{max})$ is obtained for $\delta = 1.45$. Figure 2 depicts $S(G, \delta_{max} w)$ and the regions of saturation.

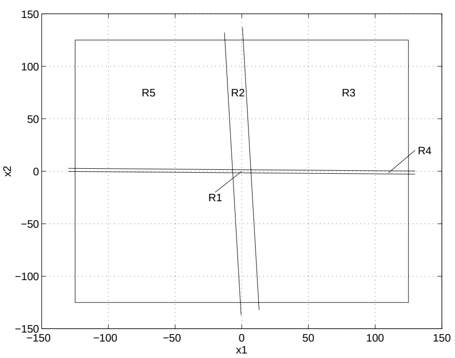

Figure 2: $S(G, \delta_{max} w)$ and the regions of saturation

9 CONCLUSION

In this paper, the properties of positive invariance and contractivity of polyhedral sets with respect to continuous-time linear systems with saturating controls were studied. First, it was given a sufficient algebraic condition for the positive invariance of polyhedral sets having nonempty intersection with the nonlinear behavior region of the saturated system. In a second moment, a necessary and sufficient condition was stated in order to guarantee also the contractivity of a compact polyhedral set. In this case, it was shown that there exists a Lyapunov polyhedral function, strictly decreasing, for all the states belonging to the considered polyhedral set. Consequently, the local asymptotic stability of the saturated system is ensured.

An algorithm based on linear programming was proposed to generate homothetical expansions of a positively invariant and contractive set w.r.t. the non-saturated system over the region of nonlinear behavior. The obtained set is a positively invariant and contractive set for the saturated system and therefore a set of nonlinear behavior. Since the exact determination of the region of attraction of the origin is, in general, not possible for saturated systems, the use of the proposed algorithm can be seen as an interesting way to compute approximations of this region.

The results presented in this paper considered the case of state feedback. Nevertheless, the application of these results to the case of output feedback (static or dynamic) is straightforward. In this case, we have to redefine the region of linearity and the regions of saturation in function of the matrices that define the considered feedback. The proposed approach should also allow to treat the problem of saturated systems with both additive and input disturbances. This will be addressed by the authors in a forthcoming publication.

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