# ASYNCHRONOUS FAULT DETECTION $H_2$ FILTER FOR MARKOV JUMP LINEAR SYSTEMS

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**Abstract**— This work focuses on the Fault Detection (FD) problem in the Markovian Jump Linear System framework for the discrete-time domain, under the assumption that the Markov chain mode is not directly accessible. This assumption poses new challenges, since the filter responsible for the residue generation no longer depends on the Markov chain mode. For modeling this type of situation, a Hidden Markov chain $(\theta(k), \hat{\theta}(k))$ is considered, with $\theta(k)$ corresponding to the hidden part and $\hat{\theta}(k)$, to the observable part. The main result is the design of an $H_2$ Fault Detection Filter (FDF) that depends only on the estimated mode $\hat{\theta}(k)$, obtained through a formulation based on Linear Matrix Inequalities (LMIs). In order to illustrate the usability of the proposed approach, we consider as an illustrative example a plant with coupled tanks subject to two distinct faults.

**Keywords**— Fault Detection, Markovian Jump Linear System, $H_2$ norm.

## 1 Introduction

Over the last decades, industrial processes have become more complex and the demands for precise, reliable, and secure procedures followed the same path. In order to fulfill those demands, different approaches have been proposed in the literature and, in particular, one of them is the so-called Fault Detection and Isolation (FDI) approach, which primarily detects faults and rearranges the system to minimize the possible losses and/or chances of accidents, see for example (Wang et al., 2019), (Zhou et al., 2017), and (Atitallah et al., 2018). The FDI framework is widely applied to different fields in engineering as presented in the review (Venkatasubramanian et al., 2003).

The FDI scheme is composed of three main structures: the plant itself, a filter and a pre-set threshold. The FDI framework works as follows: the measurements obtained by sensors are transmitted via a network to the filter; the filter generates a residual signal; this residual signal is compared with the pre-set threshold. If the residue surpasses the threshold value, a fault is considered to have occurred, otherwise the system is considered to be operating in the nominal state, see (Patton et al., 2013). From the above, we can draw some hypotheses about the desirable traits the residual filter should have: $i)$ the plant receives three distinct signals: a noise, a known input, and the fault. For increasing the performance of the fault detection approach, the filter must be sensitive to the fault signal and, at the same time, as resilient as possible to the other two signals; $ii)$ the communication between the sensor and the filter must be a full-reliable communication channel since the loss of information has a great impact on the filter performance.

Providing a full-reliable communication channel is possible using different approaches in the communication protocol layers as in (Akyildiz et al., 2002). However, depending on the situation, this task may be inconvenient or, in some cases, even impossible to achieve. Thus, another way to tackle this problem is to consider the Markovian Jump Linear System (MJLS) formulation, since it offers the possibility to associate a specific behavior with a Markov chain mode of operation, making it possible to model the network state.

In the literature there are a number of works on handling both subjects (MJLS and FDI), for example, (Zhong et al., 2005), which designs an $H_\infty$ norm residual filter for discrete-time MJLS and (Wang and Yin, 2017) that considers the synthesis of $H_\infty$ residual filters for continuous-time MJLS. In the former references, the Markov chain modes are assumed to be accessible and, in the latter, the operation modes of the filter are assumed to be unmatched with respect to the system being observed. (Li et al., 2018) presents a Fault Detection Filter (FDF) for a nonlinear MJLS with missing measurements in the discrete-time domain. An FDI for continuous-time MJLS using a geometric approach is presented in (Meskin and Khorasani, 2010). A mode-dependent FDF for discrete-time MJLS for a partially known transition probabilities is provided in (Zhang et al., 2010). Yet, the aforementioned works are based on the traditional premise of complete knowledge of transition probability. Even more importantly, the FDF obtained based on those works also considers that the Markov chain mode is instantly accessible, disregarding the eventual occurrence of mismatches between

the actual Markov chain and the implemented Markov chain, which motivated us for this study.

The hypothesis that the Markov parameter is accessible, in some cases may be seen as an unrealistic assumption. A possible approach for the non-observable case of the Markov parameter would be to design a mode-independent filter. However, depending on the number of Markov chain modes and the system dynamics, the conservadorism added in the optimization problem may lead to unfeasibility.

Another way to solve the FDI problem when the network state is not accessible is to estimate the modes employing a Hidden Markov Model (HMM). In this case, the variable $\theta(k)$ is considered to be the network mode and the variable $\hat{\theta}(k)$, the estimated network mode. The main reason to use the estimated network mode instead of the network mode is based on the difficulty to acquire such information with the necessary speed and precision, as discussed in (Ben-Akiva et al., 2001). Considering that network information is not precise, there is a chance of occurring a mismatch between the actual network mode and the used network mode. Therefore, it is necessary to consider a possible mismatch between the network mode and the estimated one during a FDF design process.

Bearing this in mind, we consider here that the Markov chain mode is not accessible. Using the HMM framework, it is possible to design a filter that does not depend on the network state, as in (de Oliveira and Costa, 2017b), but instead, the filter depends on the estimated mode denoted by $\hat{\theta}(k)$, which is provided by a detector.

This work aims to provide conditions for an FDF $H_2$ design that works under the assumption that the network state (Markov chain mode) is not accessible. Motivated by the results presented in (de Oliveira and Costa, 2017b), new Linear Matrix Inequalities (LMI) design conditions are presented. The FDF designed via the LMI formulation provided in our work depends only on detector parameters $\hat{\theta}(k)$, which is the main novelty of this paper.

## 1.1 Organization

This paper is organized as follows: Section 2 presents the necessary theoretical background to understand the problem tackled in this work. Section 3 introduces the Fault Detection Filter problem formulation. Section 4 presents the main theoretical results, Section 5 illustrates the results with an example, and Section 6 concludes the paper with some final comments. In Appendix A, the proof for Theorem 1 is presented.

## 1.2 Notation

The notation used in this manuscript is standard. The transpose of a matrix or vector is denoted by the operator $(\cdot)'$. The operator $Tr(\cdot)$ represents the trace of a square matrix, $A^{-1}$ denotes the inverse of a matrix . A symmetric block in a symmetric matrix is represented by the symbol $(\bullet)$. The symmetric sum is represented by the operator $Her(\cdot)$, as in $Her(A) = A + A'$. The capital letter $I$ denotes the identity matrix. A complete stochastic basis carrying a filtration $\mathcal{F}_k \subset \mathcal{F}$ for $k \in \{0, 1, 2, \dots\}$ is denoted by $(\Omega, \mathcal{F}, \mathcal{F}_k, \mathbb{P})$. The set $\mathbb{K} = \{1, 2, \dots, N\}$ represents the Markov chain states. The mathematical expectation is denoted by the symbol $\mathbb{E}(\cdot)$, and the conditional mathematical expectation is represented by $\mathbb{E}(\cdot|\cdot)$. The convex combination of matrices and weights $\rho_{ij}$ is written as

$$\mathbb{E}_i(X) = \sum_{j=1}^N \rho_{ij} X_j, \text{ for } i \in \mathbb{K},$$

$$\text{and } \sum_{j=1}^N \rho_{ij} = 1, \ \forall \ i.$$

The norm of a stochastic signal $z(k)$ is defined as $\|z\|_2^2 = \sum_{k=0}^\infty \mathbb{E}(z(k)'z(k))$. The set of signals $z(k) \in \mathbb{R}^p$, such that $z(k)$ is $\mathcal{F}_k$ measurable and $\|z\|_2 < \infty$ is indicated by $\mathcal{L}^2$.

## 2 Preliminary

In Section 2 the MJLS, Hidden Markov Modes, Mean Square Stability, and $H_2$ norm are described.

## 2.1 MJLS

A general MJLS formulation is

$$\mathcal{G} : \begin{cases} x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}w(k), \\ z(k) = C_{\theta(k)}x(k) + D_{\theta(k)}w(k), \\ x(0) = x_0, \quad \theta(0) = \theta_0, \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^p$ is the exogenous input vector, and $z(k) \in \mathbb{R}^r$ represents the output vector. We also consider that $w(k) \in \mathcal{L}^2$. We define the transition probability matrix by $\mathbb{P} = [\rho_{ij}]$, where $\rho_{ij} = Pr[\theta(k+1) = j|\theta(k) = i]$ and $\sum_{j=1}^N \rho_{ij} = 1$ for all $i \in \mathbb{K}$. We also define $\mathcal{F}_k$ as the $\sigma$-field generated by $\{x(t), \theta(t); t = 0, \dots, k\}$.

## 2.2 Mean Square Stability

The definition of system (1) being Mean Square Stable (MSS), is given as follows (see (Costa and Marques, 1998)):

*Definition:* System (1) is said to be MSS if, for any initial condition $x(0) = x_0 \in \mathbb{R}^n$, initial distribution $\theta(0) = \theta_0 \in \mathbb{K}$, and $\bar{w}(k) \equiv 0$, we have that $\lim_{k \to \infty} \mathbb{E}\{\bar{x}(k)'\bar{x}(k)\} = 0$.

### 2.3 $H_2$ Norm

Assuming that (1) is MSS, the $H_2$ norm is calculated via

$$\|\mathcal{G}\|_2^2 = \sum_{s=1}^p \sum_{i=1}^N \mu_i \|z^{s,i}\|_2^2, \qquad (2)$$

where $\mu_i$ is the initial Markov chain state and $z^{s,i}$ represents the output $z(0)$, $z(1)$, ... obtained when $i$) $x(0) = 0$ and the input is given by $w(k) = e_s \delta(k)$, where $e_s \in \mathcal{R}^p$ is the s-th column of the identity matrix $p \times p$ and $\delta$ is the unitary impulse, (Costa et al., 1997); $ii$) $\theta_0 = i \in \mathbb{K}$ with probability $\mu_i = P(\theta_0 = i \in \mathbb{K})$. In (Costa et al., 2006), it was shown that if the Markov Chain is stationary, meaning that $\mu_i = \rho_i$, where $\rho_i$ is the stationary distribution of the Markov chain, the norm defined in (2) can also be defined as $\|\mathcal{G}\|_2^2 = \lim_{k \to \infty} \mathbb{E}(z'(k)z(k))$, where $z(k)$ is the system output and $w(k)$ represents a white noise in the broad sense, and also is independent from $\theta(k)$ and $x(0) = x_0$.

## 3  Problem Formulation

In this section, the problem formulation and the description for each component in the FDI scheme used in the paper is presented. The MJLS formulation for the FDI problem is

$$G : \begin{cases} x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k) \\ \qquad\qquad + B_{d\theta(k)}d(k) + B_{f\theta(k)}f(k), \\ y(k) = C_{\theta(k)}x(k) + D_{d\theta(k)}d(k) \\ \qquad\qquad + D_{f\theta(k)}f(k), \\ x(0) = x_0, \quad \theta(0) = \theta_0, \end{cases} \quad (3)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^q$ is the measured output vector, $u(k) \in \mathbb{R}^m$ is the known input vector, $d(k) \in \mathbb{R}^p$ is the exogenous input vector and $f(k) \in \mathbb{R}^t$ is the fault vector, which is considered as an unknown function of time. We also consider that $f(k)$, and $d(k) \in \mathcal{L}^2$. Observe that system (3) depends on the index $\theta(k)$, which is the Markov chain mode.

The goal in the present paper is to design an FDF, which is responsible for generating the residue signal $r(k)$. The FDF is defined as

$$F : \begin{cases} \eta(k+1) = A_{\eta\hat{\theta}(k)}\eta(k) + M_{\eta\hat{\theta}(k)}u(k) \\ \qquad\qquad + B_{\eta\hat{\theta}(k)}y(k), \\ r(k) = C_{\eta\hat{\theta}(k)}\eta(k) + D_{\eta\hat{\theta}(k)}y(k), \\ \eta(0) = \eta_0, \end{cases} \quad (4)$$

whereby $\eta(k) \in \mathbb{R}^\eta$ represents the filter states, and $r(k) \in \mathbb{R}^t$ is the filter residual vector. We point out that this filter structure depends exclusively on the detector mode $\hat{\theta}(k)$.

**Assumption 1.** We consider that the index $\hat{\theta}$ is the switching variable, which is obtained using a Hidden Markov Chain, $\hat{\theta} \in \mathbb{M}$, as in (Ross, 2014). Consider that $\hat{\mathcal{F}}_k$ is the $\sigma$-field generated by $\{\bar{x}(0), \theta(0), \hat{\theta}(0), \ldots, \bar{x}(k), \theta(k)\}$. We have that $\hat{\theta}(k) \in \{1, \ldots, M\}$ is linked to $\theta(k)$ in the following way: $P(\hat{\theta}(k) = l|\hat{\mathcal{F}}_k) = P(\hat{\theta}(k) = l|\theta(k)) = \alpha_{\theta(k)l}, l \in \mathbb{M}$, whereby $\sum_{l=1}^M \alpha_{il} = 1$ for each $i \in \mathbb{N}$. Therefore, $\alpha_{il}$ represents the probability of the detector to emit signal $l \in \mathbb{M}_i$ given that $\theta(k) = i$ where the sets $\mathbb{M}_i$ can be characterized as $\mathbb{M}_i = \{l \in \mathbb{M}; \alpha_{il} > 0\}, \bigcup_{i=1}^N \mathbb{M}_i = \mathbb{M}$.

A possible way to improve the FDF performance is to consider a weight matrix during the design process, as used in (Chen and Patton, 2000; Zhong et al., 2005; Zhong et al., 2003). As described in (Chen and Patton, 2000), the weight matrix improves the FDF performance for a specific frequency range. Herein, the weight matrix $\mathcal{W}$ is denoted by

$$\mathcal{W} : \begin{cases} x_f(k+1) = A_w x_f(k) + B_w f(k), \\ \hat{f}(k) = C_w x_f(k) + D_w f(k), \\ x_f(0) = 0, \end{cases} \quad (5)$$

where $x_f(k) \in \mathbb{R}^t$ is the weight matrix state, $f(k)$ is the same signal as in (3), and $\hat{f}(k) \in \mathbb{R}^t$ is the weighted fault signal.

*Remark:* It is important to point out that $\mathcal{W}$ works as a tuning tool, consequently $\mathcal{W}$ is not implemented, nor the signal $f(k)$ is accessible. It is possible to ignore the presence of $\mathcal{W}$ by setting the matrices that compose $\mathcal{W}$ as $A_w = I$, $B_w = 0$, $C_w = 0$, and $D_w = I$.

Considering the description of all component in the FDI approach, and $r_e(k) = r(k) - \hat{f}(k)$, the equivalent system can be written in the augmented form as

$$\mathcal{G}_{aug} : \begin{cases} \bar{x}(k+1) = \tilde{A}_{\theta(k),\hat{\theta}(k)}\bar{x}(k) + \tilde{B}_{\theta(k),\hat{\theta}(k)}\bar{w}(k), \\ r_e(k) = \tilde{C}_{\theta(k),\hat{\theta}(k)}\bar{x}(k) + \tilde{D}_{\theta(k),\hat{\theta}(k)}\bar{w}(k), \end{cases} \quad (6)$$

where the augmented state is $\bar{x}(k) = [x(k)' \; \eta(k)' \; x_f(k)^T]'$, and $\bar{w}(k) = [u(k)' \; d(k)' \; f(k)']'$. To simplify the notation hereafter $\theta(k) = i$, $\hat{\theta}(k) = l$.

$$\tilde{A}_{i,l} = \begin{bmatrix} A_i & 0 & 0 \\ B_{\eta l}C_i & A_{\eta l} & 0 \\ 0 & 0 & A_{wf} \end{bmatrix},$$

$$\tilde{B}_{i,l} = \begin{bmatrix} B_i & B_{di} & B_{fi} \\ M_{\eta l} & B_{\eta l}D_{di} & B_{\eta l}D_{fi} \\ 0 & 0 & B_{wf} \end{bmatrix},$$

$$\tilde{C}_{i,l} = \begin{bmatrix} D_{\eta l}C_i & C_{\eta l} & -C_{wf} \end{bmatrix},$$

$$\tilde{D}_{i,l} = \begin{bmatrix} 0 & D_{\eta l}D_{di} & D_{\eta l}D_{fi} - D_{wf} \end{bmatrix}.$$

The main goal here is to obtain matrices $A_{\eta l}$, $B_{\eta l}$, $C_{\eta l}$, $D_{\eta l}$, $M_{\eta l}$ such that the residual generator (4) is MSS when $u(0) = 0$, $d(0) = 0$ and $f(0) = 0$

and minimizes the $H_2$ upper bound to be defined next.

The goal to consider the $H_2$ norm is to design a FDF that generates a residual signal which is sensible to fault signals with impulsive behavior. This assumption that the $H_2$ norm increases the FDF sensibility against the fault signal is motivated by the physical interpretation of the $H_2$ norm, which is the sum of the energy dissipated by the impulsive exogenous input.

Assuming that (6) is MSS, the $H_2$ norm is calculated via

$$\|\mathcal{G}_{aug}\|_2^2 = \sum_{s=1}^{p} \sum_{i=1}^{N} \mu_i \|r_e^{s,i}\|_2^2 < \gamma \quad (7)$$

Minimizing the energy dissipated in $r_e(k) = r(k) - \hat{f}(k)$ means that the signal $r$ must have a similar energy dissipation behavior when compared to the weighted fault signal $\hat{f}(k)$. To calculate the upper bound $\gamma$ for the norm $H_2$ for partially known mode can be carried out by the LMI constraints presented in (Costa et al., 2015).

As mentioned previously the FDI scheme is divided into two stages, the first one is the residual generation, and the latter is the residue signal evaluation. In order to perform the evaluation of the residue signal, an evaluation function that depends on the residue signal is defined as $J(k)$, and the threshold is obtained by using the evaluation function. Both were used in (Zhong et al., 2005). We consider $L$ the evaluation time step. The evaluation functions are set as

$$J(k) \triangleq \sqrt{\sum_{k=k_0}^{k_0+L} r^T(k)r(k)}, \quad (8)$$

$$J_{th} \triangleq \sup_{0 \neq w(k) \in \mathcal{L}^2,\, 0 \neq u(k) \in \mathcal{L}^2,\, f=0} J(k), \quad (9)$$

whereby $k_0$ denotes the initial evaluation instant. Considering both equations, the fault occurrence may be detected as follows $J(k) < J_{th}$ represents the nominal condition, and $J(k) \geq J_{th}$ represents the fault occurrence.

## 4 Main Results

In this section the main results are presented, which are the LMI constraints for the $H_2$ norm to design an FDF that considers the Markov chain mode $\theta(k)$ not accessible and depends only on the estimated state $\hat{\theta}(k)$, as presented in (4).

**Theorem 1** *There exists a filter in the form of (4) such that $\|G_{aug}\|_2 < \gamma$ if there exist symmetric matrices $Z_i$, $X_i$, $M_{il}$, $W_{il}$, $E_i$, and matrices $\Delta_l$, $O_l$, $F_l$, $G_l$, $R_l$ with compatible dimensions that satisfy the LMI constraints (10)-(13).*

$$\sum_{i=1}^{N} \sum_{l \in \mathbb{M}_i} \mu_i \alpha_{il} Tr(W_{il}) < \gamma,$$

$$(10)$$

$$\begin{bmatrix} Z_i & \bullet & \bullet \\ Z_i & X_i & \bullet \\ 0 & 0 & E_i \end{bmatrix} > \sum_{l \in \mathbb{M}_i} \alpha_{il} \begin{bmatrix} M_{il} \end{bmatrix},$$

$$(11)$$

$$\begin{bmatrix} & \begin{bmatrix} W_{il} \end{bmatrix} & \\ \mathbb{E}_i(Z)B_i & \mathbb{E}_i(Z)B_{di} & \mathbb{E}_i(Z)B_{fi} \\ R_lB_i + H_l & R_lB_{di} + \Delta_lD_{di} & R_lB_{fi} + \Delta_lD_{fi} \\ 0 & 0 & \mathbb{E}_i(E)B_w \\ 0 & G_lD_{di} & G_lD_{fi} - D_w \end{bmatrix}$$

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \mathbb{E}_i(Z) & \bullet & \bullet & \bullet \\ 0 & \nu_{il} & \bullet & \bullet \\ 0 & 0 & \mathbb{E}_i(E) & \bullet \\ 0 & 0 & 0 & I \end{bmatrix} > 0,$$

$$(12)$$

$$\begin{bmatrix} & \begin{bmatrix} M_{il} \end{bmatrix} & \\ \mathbb{E}_i(Z)A_i & \mathbb{E}_i(Z)A_i & 0 \\ R_lA_i + \Delta_lC_i + O_l & R_lA_i + \Delta_lC_i & 0 \\ 0 & 0 & \mathbb{E}_i(E)A_w \\ G_lC_i + F_l & G_lC_i & -C_w \end{bmatrix}$$

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \mathbb{E}_i(Z) & \bullet & \bullet & \bullet \\ 0 & \nu_{il} & \bullet & \bullet \\ 0 & 0 & \mathbb{E}_i(E) & \bullet \\ 0 & 0 & 0 & I \end{bmatrix} > 0,$$

$$(13)$$

*where $\nu_{il} = Her(R_l) + \mathbb{E}_i(Z) - \mathbb{E}_i(X)$. If a feasible solution is obtained, the matrices that compose the filter are $A_{\eta l} = -R_l^{-1}O_l$, $B_{\eta l} = -R_l^{-1}\Delta_l$, $M_{\eta l} = -R_l^{-1}H_l$, $C_{\eta l} = F_l$, $D_{\eta l} = G_l$.*

**Proof:** The proof is presented in the Appendix A.

*Remark*: The situation where the detector matrix has equal rows $\alpha_{il} = \alpha_l \ \forall \ i \ \in \ \mathbb{N}$, represents the worst case scenario, since the detector cannot properly distinguish the modes of operation, see, for instance, the works (de Oliveira and Costa, 2017b) and (de Oliveira and Costa, 2018). In this situation the filter, as similarly presented in (de Oliveira and Costa, 2017b), is mode-independent.

## 5 Numerical Example

In this section we present a numerical example based on a system of coupled tanks taken from (Feedback Instruments Ltd., 2013), in which we want to estimate the flow. The coupled tanks system, (Patton et al., 2013), is a multiple variable plant, used to illustrate numerical fault detection problems. The parameters for the simulated coupled tanks system were obtained from the Coupled Multi-Tanks System model number 33-041, (Feedback Instruments Ltd., 2013). The state-space matrices (3) in the discrete-time domain, obtained by using a Zero-Order-Hold discretization process with sample time of $0.5s$, are given
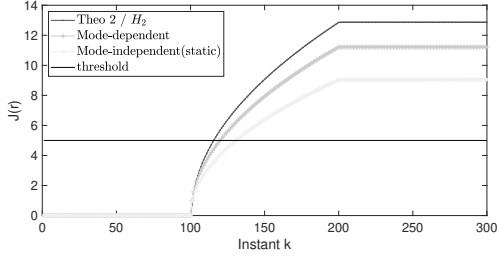
Figure 1: Evaluation function $J(k)$.

by

$$A = \begin{bmatrix} -0.0239 & -0.0127 \\ 0.0127 & -0.0285 \end{bmatrix}, \quad B = \begin{bmatrix} 0.71 \\ 0 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0.001 \\ 0.001 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$D_d = \begin{bmatrix} 0.05 \\ 0.05 \end{bmatrix}, B_f = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},$$

$$D_f = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad \begin{matrix} A_w = 0.5, & B_w = 0.25, \\ C_w = 0.6, & D_w = 0.75. \end{matrix}$$

*The Communication Failure model:* To model the communication failure, the strategy Zero-Input, (Schenato, 2009), is implemented. The matrix that denotes the communication failure is $C_i$, whereby $C_1$ represents the nominal network communication and $C_2$ denotes the communication loss state.

*Fault Signal model:* The fault signal is a step signal with magnitude equal to 1, and starting at $k = 100$.

Here we present the results obtained in a Monte Carlo simulation using FDF obtained by Theorem 1. Additionally, a comparison between the FDF proposed and the mode-dependent FDF from (Zhong et al., 2005) and a static FDF is made. For the next test, the transition matrices and detector matrices are given by $\mathbb{P} = [0.8\ 0.2;\ 0.4\ 0.6]$, and $\Psi = [0.6\ 0.4;\ 0.75\ 0.25]$. For Theorem 1 the matrices obtained are

$$A_{\eta 1} = \begin{bmatrix} -0.00 & 0.01 \\ -0.00 & 0.00 \end{bmatrix}, A_{\eta 2} = \begin{bmatrix} 0.08 & -0.18 \\ 0.03 & -0.08 \end{bmatrix},$$

$$B_{\eta 1} = \begin{bmatrix} 0.003 & -0.008 \\ 0.001 & -0.003 \end{bmatrix}, B_{\eta 2} = \begin{bmatrix} -0.44 & 1.00 \\ -0.20 & 0.44 \end{bmatrix},$$

$$M_{\eta 1} = \begin{bmatrix} 0.00 \\ 0.05 \end{bmatrix}, M_{\eta 2} = \begin{bmatrix} -0.01 \\ 0.03 \end{bmatrix},$$

$$C_{\eta 1} = \begin{bmatrix} -0.15 & 0.35 \end{bmatrix}, C_{\eta 2} = \begin{bmatrix} -0.47 & 1.07 \end{bmatrix},$$

$$D_{\eta 1} = \begin{bmatrix} 1.13 & -2.53 \end{bmatrix}, D_{\eta 2} = \begin{bmatrix} 1.83 & -4.11 \end{bmatrix}.$$

Observing the curves related to the mode-dependent FDF and the static FDF in Fig.1, allow us to state that neglecting the network behavior during the design process indeed impacts in the FDF performance. But, even when the network behavior is accounted (mode-dependent), the possible mismatch also has an impact on the performance. Recall that the main premise of this paper is the consideration of the partial observation of the Markov chain to model the network mode mismatch. Hence, this example illustrates that under this premise, the proposed design technique for the FDF provides a good alternative for the fault detection problem.

## 6 Conclusion

We studied the fault detection problem associated with the Markovian jump linear system in the discrete-time domain considering the partial access of the Markov chain mode. The main results are the design of $H_2$ MJLS filters. Notice that this filter is responsible for generating the residual signal in the FDI problem. As illustrated in the numerical results, all the approaches provided are viable solutions to the FDF problem. The next step along this line of research is to change the problem formulation, and use the concept of $H_-$ index to maybe improve the filter performance.

## 7 Acknowledgment

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## A    Appendix

Proof Theorem 1: Fixing the following structure for the matrices

$$\tilde{P}_i = \begin{bmatrix} X_i & \bullet & \bullet \\ U_i' & \hat{X}_i & \bullet \\ 0 & 0 & P_i^{33} \end{bmatrix}, \ \tilde{P}_i^{-1} = \begin{bmatrix} Y_i & \bullet & \bullet \\ V_i' & \hat{Y}_i & \bullet \\ 0 & 0 & P_i^{33\ -1} \end{bmatrix},$$

(14)

$$\mathbb{E}_i(\tilde{P})^{-1} = \begin{bmatrix} \hat{T}_{1i} & \bullet & \bullet \\ \hat{T}_{2i}^T & \hat{T}_{3i} & \bullet \\ 0 & 0 & \hat{T}_{4i} \end{bmatrix},$$

(15)

and the linearization matrices

$$\tau_i = \begin{bmatrix} I & I & 0 \\ V_i'Y_i^{-1} & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad (16)$$

$$\iota_i = \begin{bmatrix} \hat{T}_{1i}^{-1} & \mathbb{E}_i(X) & 0 \\ 0 & \mathbb{E}_i(U)' & 0 \\ 0 & 0 & \mathbb{E}_i(P^{33}) \end{bmatrix}, \qquad (17)$$

we get that

$$\tau_i'\tilde{P}_i\tau_i = \begin{bmatrix} Y_i^{-1} & Y_i^{-1} & 0 \\ Y_i^{-1} & X_i & 0 \\ 0 & 0 & P_i^{33} \end{bmatrix},$$

$$\iota_i'\mathbb{E}_i(\tilde{P})^{-1}\iota_i = \begin{bmatrix} \hat{T}_{1i}^{-1} & \bullet & \bullet \\ \hat{T}_{1i}^{-1} & \mathbb{E}_i(X) & \bullet \\ 0 & 0 & \mathbb{E}_i(P^{33}) \end{bmatrix}. \quad (18)$$

The matrix $\mathbb{E}_i(\tilde{P})^{-1}$, as explained in (Gonçalves et al., 2010), depends nonlinearly on $\mathbb{E}_i(\tilde{P})$. The matrix $\iota_i'\mathbb{E}_i(\tilde{P})^{-1}\iota_i$ is linearized by considering $U_i = -\hat{X}_i$ and observing that (14) provides $U_i = -\hat{X}_i = Y_i^{-1} - X_i = Z_i - X_i$, which enables us to rewrite $\iota_i^T\mathbb{E}_i(\tilde{P})^{-1}\iota_i$ as

$$\iota_i'\mathbb{E}_i(\tilde{P})^{-1}\iota_i = \begin{bmatrix} \mathbb{E}_i(Z) & \bullet & \bullet \\ \mathbb{E}_i(Z) & \mathbb{E}_i(X) & \bullet \\ 0 & 0 & \mathbb{E}_i(P^{33}) \end{bmatrix}. \quad (19)$$

Considering the constraintw (12), (11) and (13), and $U_i = Z_i - X_i$, $\hat{X}_i = -U_i$, $V_i'Y_i^{-1} = I$ and from (11) we are able to say that $\mathbb{E}_i(X) - \mathbb{E}_i(Z)$ is invertible due the $X_i > Z_i$. This observation also allows us to write $R_l(\mathbb{E}_i(X) - \mathbb{E}_i(Z))^{-1}R_l' \geq Her(R_l) + \mathbb{E}_i(Z) - \mathbb{E}_i(X)$, (see (de Oliveira et al., 1999)), such that

$$\begin{bmatrix} & & [W_{il}] & \\ \mathbb{E}_i(Z)B_i & \mathbb{E}_i(Z)B_{di} & \mathbb{E}_i(Z)B_{fi} \\ R_lB_i + H_l & R_lB_{di} + \Delta_lD_{di} & R_lB_{fi} + \Delta_lD_{fi} \\ 0 & 0 & \mathbb{E}_i(E)B_{wf} \\ 0 & G_lD_{di} & G_lD_{fi} - D_{wf} \end{bmatrix}$$
$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \mathbb{E}_i(Z) & \bullet & \bullet & \bullet \\ 0 & \Pi_{il} & \bullet & \bullet \\ 0 & 0 & \mathbb{E}_i(E) & \bullet \\ 0 & 0 & 0 & I \end{matrix} \Bigg] > 0$$
$$(20)$$

$$\begin{bmatrix} & & [M_{il}] & \\ \mathbb{E}_i(Z)A_i & \mathbb{E}_i(Z)A_i & 0 \\ R_lA_i + \Delta_lC_i + O_l & R_lA_i + \Delta_lC_i & 0 \\ 0 & 0 & \mathbb{E}_i(E)A_{wf} \\ G_lC_i + F_l & G_lC_i & -C_{wf} \end{bmatrix}$$
$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \mathbb{E}_i(Z) & \bullet & \bullet & \bullet \\ 0 & \Pi_{il} & \bullet & \bullet \\ 0 & 0 & \mathbb{E}_i(E) & \bullet \\ 0 & 0 & 0 & I \end{matrix} \Bigg] > 0,$$
$$(21)$$

where $\Pi_{il} = R_l(\mathbb{E}_i(Z) - \mathbb{E}_i(X))^{-1}R_l^T$. Recalling that $O_l = -R_lA_{\eta l}$, $\Delta_l = -R_lB_{\eta l}$, $H_l = -R_lM_{\eta l}$, $F_l = C_{\eta l}$, $G_l = D_{\eta l}$. As in (de Oliveira and Costa, 2017b), $\hat{T}_{1i}^{-1} = \mathbb{E}_i(X) - \mathbb{E}_i(U)\mathbb{E}_i(\hat{X})^{-1}\mathbb{E}_i(U)^T$, and since $\mathbb{E}_i(U) = -\mathbb{E}_i(\hat{X})$ we get that $\hat{T}_{1i}^{-1} = \mathbb{E}_i(Z) = \mathbb{E}_i(X) + \mathbb{E}_i(U)$. Define the matrix $Q_{il}$ as,

$$Q_{il} = \begin{bmatrix} I_n & I_n & 0 \\ 0 & (R_l^{-1})'(\mathbb{E}_i(X) - \mathbb{E}_i(Z)) & 0 \\ 0 & 0 & I \end{bmatrix}. \quad (22)$$

Applying congruence transformations $diag(I, Q_{il}, I)$ and $diag(I, I, I, Q_{il}, I)$, respectively, in (20) and (21) we obtain the constraints below (similarly as presented in (Gonçalves et al., 2010))

$$\begin{bmatrix} & [W_{il}] & \\ \mathbb{E}_i(Z)B_i & \mathbb{E}_i(Z)B_{di} \\ \mathbb{E}_i(U)B_i + \mathbb{E}_i(U)M_{\eta l} & \mathbb{E}_i(U)B_{di} + \mathbb{E}_i(U)B_{\eta l}D_{di} \\ 0 & 0 \\ 0 & G_lD_{di} \end{bmatrix}$$
$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \mathbb{E}_i(Z)B_{fi} & \mathbb{E}_i(Z) & \bullet & \bullet & \bullet \\ \mathcal{S}_{il} & \mathbb{E}_i(Z) & \mathbb{E}_i(X) & \bullet & \bullet \\ \mathbb{E}_i(E)B_{wf} & 0 & 0 & \mathbb{E}_i(E) & \bullet \\ G_lD_{fi} - D_{wf} & 0 & 0 & 0 & I \end{matrix} \Bigg] > 0$$
$$(23)$$

$$\begin{bmatrix} & [M_{il}] & \\ \mathbb{E}_i(Z)A_i & \mathbb{E}_i(Z)A_i \\ \mathcal{Q}_{il} & \mathbb{E}_i(U)A_i + \mathbb{E}_i(U)B_{\eta l}C_i \\ 0 & 0 \\ G_lC_i + F_l & G_lC_i \end{bmatrix}$$
$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \mathbb{E}_i(Z) & \bullet & \bullet & \bullet \\ 0 & \mathbb{E}_i(Z) & \mathbb{E}_i(X) & \bullet & \bullet \\ \mathbb{E}_i(E)A_{wf} & 0 & 0 & \mathbb{E}_i(E) & \bullet \\ -C_{wf} & 0 & 0 & 0 & I \end{matrix} \Bigg] > 0.$$
$$(24)$$

where $\mathcal{S}_{il} = \mathbb{E}_i(U)B_{fi} + \mathbb{E}_i(U)B_{\eta l}D_{fi}$ and $\mathcal{Q}_{il} = \mathbb{E}_i(U)A_i + \mathbb{E}_i(U)B_{\eta l}C_i + \mathbb{E}_i(U)A_{\eta l}$. The constraint (11), (23) and (24) can also be described as

$$\tau_i^T\tilde{P}_i\tau_i > \sum_{l \in \mathbb{M}_i} \alpha_{il}\tau_i^T\tilde{R}_{il}\tau_i, \qquad (25)$$

$$\begin{bmatrix} W_{il} & \bullet & \bullet \\ \iota_i^T\tilde{B}_{il}\tau_i & \iota_i^T\mathbb{E}_i(\tilde{P})^{-1}\iota_i & \bullet \\ \tilde{D}_{il} & 0 & I \end{bmatrix} > 0, \qquad (26)$$

$$\begin{bmatrix} \tau_i^T\tilde{R}_{il}\tau_i & \bullet & \bullet \\ \iota_i^T\tilde{A}_{il}\tau_i & \iota_i^T\mathbb{E}_i(\tilde{P})^{-1}\iota_i & \bullet \\ \tilde{C}_{il} & 0 & I \end{bmatrix} > 0. \qquad (27)$$

Applying the congruence transformations $\tau_i^{-1}$, $diag(I, \iota_i^{-1})$ and $diag(\tau_i^{-1}, \iota_i^{-1}, I)$ in (25) (26), and (27), respectively, we end up with the similar LMIs constraints as in (de Oliveira and Costa, 2017a), concluding the proof.