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$\mathcal{H}_\infty$ STATE-FEEDBACK CONTROL FOR DISCRETE-TIME MARKOV JUMP LUR'E SYSTEMS

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Abstract— We propose LMI conditions for $\mathcal{H}_\infty$ analysis and state-feedback control of discrete-time Markov Jump Lur'e systems. We will also approach two possible situations, either mode-dependent and mode-independent state-feedback control. For the first approach, we consider that the controller has access to the Markov mode $\theta(k)$, on the other hand, for the second approach, we consider that the Markov parameter $\theta(k)$ cannot be read by the controller. A numerical example illustrates the obtained results.

Keywords— Markov jump systems, circle criterion, $\mathcal{H}_\infty$ control, linear matrix inequalities.

1 Introduction

Systems are studied through a mathematical representation of their dynamics. Most representations of physical phenomena need to take nonlinearities into account. Nonlinearities are found in most economical, biological and physical models, such as the survival of different species in a given environment, spread of viruses among a certain population, electronic oscillators and so forth, (Monteiro, 2011).

If such systems are subject to abrupt changes, a single deterministic model may not be enoughly precise. Moreover, such changes may occur in a random fashion and may be described by a stochastic process. Those occurrences are taken into account through their statistics, such as probabilities, expected values, variance among other parameters.

One way to model abrupt changes in dynamic systems is to consider a combination of different subsystems, each representing an operation mode. Each mode is described by a set of linear equations and the randomness is modeled as a jump between the different operation modes. Such modeling is referred to in the literature as *Markov Jump Linear Systems* (MJLS) and has been subject of a large amount of research. We indicate (Costa et al., 2006) and (Costa et al., 2013) and their comprehensive list of references.

The objective of this work is to present *Linear Matrix Inequalities* (LMIs) stability conditions for stability analysis and synthesis of state-feedback control with bounded $\mathcal{H}_\infty$−cost, for discrete-time Markov Jump Lur'e systems. We consider both scenarios for the availability of the Markov mode to the controller: mode-dependent and mode-independent. To achieve that goal, we start representing the Markov System in the Lur'e format, which is a representation of the nonlinear system as the interconnection of two parts: a conventional linear model and a nonlinear function of the output which is negatively fedback into the previous linear representation. Moreover, we complete the system representation by considering a bounded exogenous input, a control input and a performance output. Furthermore, note that the study of Lur'e systems is commonly referred in the literature as the absolute stability problem, (Khalil, 2002).

In this paper, the nonlinearities under consideration are those that belong to a given sector and we consider that, in a first scenario, the Markov mode $\theta(k)$ can be measured by the controller. In a second scenario, the process $\theta(k)$ cannot be measured, thus the design of controller is mode-independent, (Gonçalves et al., 2012). Our aim is to address an extension of the absolute stability problem for $\mathcal{H}_\infty$ state-feedback control using a similar approach as the one presented in (Gonçalves et al., 2012). If the nonlinear feedback of the Lur'e system is substituted by a linear gain, our results recover the ones presented in that paper.

The reference (Gonçalves et al., 2012) also deals with uncertain transition probabilities and cluster observation of the Markov modes. It is possible to extend the LMI conditions presented here to also address both additional constraints.

Convex optimization problems, especially those with LMI constraints, are well established in the control literature (Boyd et al., 1994). There are specialized computer packages that can solve those optimization programs efficiently. Among others we cite: LMI Control Toolbox or SeDuMi with Yalmip parser (Sturm, 1999), (Lofberg, 2005).

2 Notation

Throughout the text, $(\cdot)'$ indicates transposed matrices or vectors, $\mathrm{diag}(\cdot)$ indicates a diagonal matrix, $\mathrm{Tr}(\cdot)$ indicates the trace operator, $(\star)$ indicates a block induced by symmetry at symmetric matrices, the operator $\mathbf{E}[\cdot]$ ($\mathbf{E}[\cdot|\cdot]$) is the (conditional) expected value of a random variable, σ is the standard deviation, $\mathbb{R}_+$ $\left(\mathbb{N}_+ \triangleq \{0,1,2,\cdots,\infty\}\right)$ indicates the set of positive real (natural) numbers, and $\mathrm{Her}(A) \triangleq A + A'$. The set of nonlinear functions belonging to given sector is denoted by $\mathbb{F}$. $\|x(k)\|$ is the Euclidean norm of vector $x(k)$. We define $\ell_2^p(\mathfrak{F})$, as the space of all $p-$dimensional square-summable stochastic signals, $x : \mathbb{N}_+ \to \mathbb{R}^p$, such that $\|x\|_2^2 \triangleq \mathbf{E}\left[\sum_{k=0}^{\infty} \|x(k)\|^2\right] < \infty$. Also, we consider the following operators, $\mathcal{E}_i(P) \triangleq \sum_{j\in\mathbb{K}} \pi_{ij} P_j$ and, with a slight abuse of notation, $\mathcal{E}_i(Z) \triangleq \sum_{j\in\mathbb{K}} \pi_{ij} Z_{ij}$ and $\mathcal{L}_i(P) \triangleq A_i'\mathcal{E}_i(P) A_i - P_i$. Finally, we recall the following auxiliary result for symmetric and positive definite matrices $X_i > 0$ and T_i of compatible dimensions, $T_i' X_i^{-1} T_i \geq \mathrm{Her}(T_i) - X_i$, $i \in \mathbb{K}$.

3 Preliminaries

We will assume the probability space $(\Omega, \mathfrak{F}, \mathrm{Prob})$, where Ω is the sample space, $\mathfrak{F}$ is the σ-field into Ω, Prob is the probability measure.

Furthermore, we consider the following Markov jump nonlinear system $\mathcal{G}$:

$$\begin{aligned} x(k+1) &= A_{\theta(k)}x(k) + E_{\theta(k)}p(k) + J_{\theta(k)}w(k),\\ q(k) &= G_{\theta(k)}x(k),\\ z(k) &= F_{\theta(k)}x(k) + H_{\theta(k)}w(k),\\ p(k) &= -\phi\left(q(k)\right), \end{aligned} \tag{1}$$

where $x : \mathbb{N}_+ \to \mathbb{R}^n$ is the state, $z : \mathbb{N}_+ \to \mathbb{R}^s$ is the performance output, $w : \mathbb{N}_+ \to \mathbb{R}^r$ exogenous input, $p : \mathbb{N}_+ \to \mathbb{R}^m$ is an input to the linear part of the system, $q : \mathbb{N}_+ \to \mathbb{R}^m$ is the output signal, and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinearity between $q(k)$ and $p(k)$. It is important to notice that such input and output signals $p(k)$ and $q(k)$ may not be signals with physical meaning, but rather they may be mathematically defined so that the nonlinear system can be cast in the Lur'e framework. The nonlinear function ϕ is such that, $\phi(0) = 0$, which ensures the origin $x = 0$ is an equilibrium point. As it is common in the treatment of passive and/or positive real systems, we consider that the dimension $m \in \mathbb{N}_+$ of inputs and outputs is the same (Khalil, 2002).

Additionally, we say that $\phi(\cdot)$ belongs to the sector $[0, \mathcal{S}]$, with $\mathcal{S} \triangleq \mathrm{diag}(\kappa_1, \cdots, \kappa_m)$, $\kappa_j > 0$ for $j \in \{1, 2, \cdots, m\}$, if

$$\left(\phi\left(q(k)\right) - \mathcal{S}q(k)\right)' \phi\left(q(k)\right) \leq 0,\ \forall k \in \mathbb{N}_+. \tag{2}$$

The random variable $\theta(k) \in \mathbb{K} \triangleq \{1,\ 2,\ \cdots,\ N\}$ is governed by a Discrete-time Markov chain (DTMC) (Costa et al., 2006), therefore, the stochastic process $\{\theta(k),\ k \in \mathbb{N}_+\}$ is such that

$$\mathrm{Prob}(\theta(k+1) = j|\theta(k) = i) = \pi_{ij}, \tag{3}$$

with transition probability matrix given by $\Pi = [\pi_{ij}]$ where

$$\pi_{ij} \geq 0,\ \sum_{j\in\mathbb{K}} \pi_{ij} = 1,\ \forall i \in \mathbb{K}.$$

In the following subsection, we define stochastic stability and $\mathcal{H}_\infty$ bound for our system.

3.1 Stochastic stability and $\mathcal{H}_\infty-$cost

The following definitions are important for obtaining analysis and synthesis conditions for the absolute stability problem. The reader can find more information on (Costa et al., 2006) and its references.

We consider the Markov jump Lur'e system (1) with initial conditions $x(0) = x_0$, $\theta(0) = \theta_0$, and matrices $A_{\theta(k)}$, $E_{\theta(k)}$, $G_{\theta(k)}$ $F_{\theta(k)}$, $J_{\theta(k)}$ and $H_{\theta(k)}$ with appropriate dimensions. In order to simplify the notation, we will write $X_{\theta(k)=i} = X_i,\ \forall\ i \in \mathbb{K}$.

Definition 1 (Stochastic Stability). *System $\mathcal{G}$, with null inputs, initial conditions $x(0) = x_0$ and $\theta(0) \in \mathbb{K}$, is said to be stochastically stable when*

$$\mathbf{E}\left[\sum_{k=0}^{\infty} \|x(k)\|^2\right] < \infty. \tag{4}$$

On the other hand, we assume that system (1) is subject to a stochastic disturbance $w = \{w(k),\ k \geq 0\ :\ w \in \ell_2^r(\mathfrak{F})\}$.

Definition 2 ($\mathcal{H}_\infty-$norm). *The $\mathcal{H}_\infty$ norm of stochastically stable system $\mathcal{G}$ is defined as the least $\gamma > 0$ such that,*

$$\|z\|_2 < \gamma \|w\|_2. \tag{5}$$

4 Main Results

The purpose of this section is to present two important results: analysis and state feedback control design with guaranteed $\mathcal{H}_\infty-$cost, for discrete-time Markov jump Lur'e system.

4.1 Stochastic stability of Lur'e systems

Consider the dynamical system (1).

Theorem 1 ($\mathcal{H}_\infty$−cost of Lur'e system). *The system* (1)*, with* $\phi \in [0, \mathcal{S}]$ *and* $x(0) = 0$ *is stochastically stable with guaranteed* $\mathcal{H}_\infty$ *cost* $\gamma \in \mathbb{R}_+$*, if there exists feasible solutions* $P_i > 0$, $i \in \mathbb{K}$*, for the LMIs*

$$\boldsymbol{A}_i' \begin{bmatrix} \mathcal{E}_i(P) & 0 \\ 0 & I \end{bmatrix} \boldsymbol{A}_i - \begin{bmatrix} P_i & \star & \star \\ -\mathcal{S}G_i & 2I & \star \\ 0 & 0 & \gamma^2 I \end{bmatrix} < 0, \tag{6}$$

where

$$\boldsymbol{A}_i = \begin{bmatrix} A_i & -E_i & J_i \\ F_i & 0 & H_i \end{bmatrix}, \forall i \in \mathbb{K}.$$

In the affirmative case, that guarantees

$$\max_{\mathbb{F}} \|z\|_2 < \gamma \|w\|_2, \ \mathbb{F} = \{\phi(q(k)) \in [0, \mathcal{S}]\}. \tag{7}$$

Proof: We assume that inequality (6) holds, we also consider the following stochastic Lyapunov function $v_{x(k),\theta(k)=i} = x(k)'P_i x(k)$. After multiplying inequality (6) to the right by $\begin{bmatrix} x' & \phi' & 0' \end{bmatrix}$ and to the left by its transpose, we obtain

$$\begin{aligned} & x' \left(A_i'\mathcal{E}_i(P)A_i - P_i\right) x + \phi' \left(-2I + E_i'\mathcal{E}_i(P)E_i\right) \phi + \\ & + \mathrm{Her}\left(\phi' \left(-E_i'\mathcal{E}_i(P)A_i + \mathcal{S}G_i\right) x\right) < 0. \end{aligned} \tag{8}$$

Grouping the terms of inequality (8)

$$\underbrace{\mathbf{E}\left[x(k+1)'P_{\theta(k+1)}x(k+1)|x,\theta(k)\right] - x'P_{\theta(k)}x}_{\Delta\left(v_{x(k),\theta(k)}\right)} < \underbrace{\mathrm{Her}\left(\left(\phi - \mathcal{S}q\right)'\phi\right)}_{\leq 0}, \tag{9}$$

Therefore, the inequality (9) guarantees the Lur'e system stochastic stability. After applying the Schur complement to inequality (6), we multiply it to the left by $\begin{bmatrix} x' & \phi' & w' \end{bmatrix}$ and to the right by its transpose to get,

$$\begin{aligned} & x' \left(\mathcal{L}_i(P) + F_i'F_i\right) x + \\ & + \phi' \left(-2I + E_i'\mathcal{E}_i(P)E_i\right) \phi + \\ & + \mathrm{Her}\left(\phi' \left(-E_i'\mathcal{E}_i(P)A_i + \mathcal{S}G_i\right) x\right) + \\ & + \mathrm{Her}\left(w' \left(-J_i'\mathcal{E}_i(P)E_i\right) \phi\right) + \\ & + \mathrm{Her}\left(w' \left(J_i'\mathcal{E}_i(P)A_i + H_i'F_i\right) x\right) + \\ & + w' \left(-\gamma^2 I + J_i'\mathcal{E}_i(P)J_i + H_i'H_i\right) w < 0, \end{aligned} \tag{10}$$

Grouping the terms of inequality (10), we have,

$$\Delta\left(v_{x(k),\theta(k)}\right) + \|z(k)\|^2 - \gamma^2\|w(k)\|^2 < \underbrace{\mathrm{Her}\left(\left(\phi - \mathcal{S}q\right)'\phi\right)}_{\leq 0}, \tag{11}$$

where

$$\Delta\left(v_{x(k),\theta(k)}\right) = \mathbf{E}\left[x(k+1)'P_{\theta(k+1)}x(k+1)|x,\theta(k)\right] - v_{x(k),\theta(k)}.$$

Summing up (11) for $k = 0$ to ∞ and applying the expectation operator, remembering that the system is stochastically stable, we get

$$\underbrace{\mathbf{E}\left[\sum_{k=0}^{\infty} \Delta\left(v_{x(k),\theta(k)}\right)\right]}_{=0} + \|z\|_2^2 - \gamma^2\|w\|_2^2 < 0, \tag{12}$$

guaranteeing, in the worst case

$$\max_{\mathbb{F}} \|z\|_2 < \gamma \|w\|_2, \tag{13}$$

concluding the proof. □

4.2 State-feedback control of Discrete-time Markov jump Lur'e system

Consider the following closed-loop Lur'e system, with the following control signal $u : \mathbb{N}_+ \to \mathbb{R}^t$, on the probability space $(\Omega, \mathfrak{F}, \mathrm{Prob})$,

$$\begin{aligned} x(k+1) &= \tilde{A}_{\theta(k)}x(k) + E_{\theta(k)}p(k) + J_{\theta(k)}w(k), \\ q(k) &= G_{\theta(k)}x(k), \\ p(k) &= -\phi\left(q(k)\right), \\ z(k) &= \tilde{F}_{\theta(k)}x(k) + H_{\theta(k)}w(k), \end{aligned} \tag{14}$$

where,

$$\tilde{A}_i \triangleq A_i + B_iK_i, \ \tilde{F}_i \triangleq F_i + L_iK_i, \ \forall i \in \mathbb{K}, \tag{15}$$

where, B_i, L_i, H_i, with appropriate dimensions. We also consider $\tilde{A}_i$ and $\tilde{F}_i$ are closed-loop matrices, that are obtained by adding a control input $u(k) = K_{\theta(k)}x(k)$ to the system (1). The next result addresses the $\mathcal{H}_\infty$ mode-dependent controller.

Theorem 2 ($\mathcal{H}_\infty$ State-feedback Control). *The closed-loop system* (14) *with* $\phi(q) \in [0, \mathcal{S}]$ *is stochastically stable with guaranteed* $\mathcal{H}_\infty$ *cost* $\gamma \in \mathbb{R}_+$*, if there exists a feasible solution in the set* $\mathbb{S} \triangleq \{X_i > 0, Y_i, T_i, Z_{ij}, W_i, \forall\, i, j \in \mathbb{K}\}$ *for the LMIs*

$$\begin{bmatrix} Z_{ij} & \star \\ W_i & X_j \end{bmatrix} > 0, \tag{16}$$

$$\begin{bmatrix} \mathrm{Her}\left(T_i\right) - X_i & \star & \star & \star & \star \\ -\mathcal{S}G_iT_i & 2I & \star & \star & \star \\ 0 & 0 & \gamma^2 I & \star & \star \\ A_iT_i + B_iY_i & -E_i & J_i & \mathrm{Her}\left(W_i\right) - \mathcal{E}_i\left(Z\right) & \star \\ F_iT_i + L_iY_i & 0 & H_i & 0 & I \end{bmatrix} > 0. \tag{17}$$

In the affirmative case,

$$K_i = Y_iT_i^{-1}, \tag{18}$$

guarantees that,

$$\max_{\mathbb{F}} \ \|z\|_2 < \gamma \|w\|_2, \ \mathbb{F} \triangleq \{\phi(q(k)) \in [0, \mathcal{S}]\}. \tag{19}$$

Proof: We assume that LMIs (16) and (17) hold. From LMI (16), we have, $Z_{ij} > W_i' (X_j)^{-1} W_i$. Multiplying these inequalities by π_{ij} and summing up for all $j \in \mathbb{K}$ we obtain

$$\begin{aligned}\sum_{j\in\mathbb{K}} \pi_{ij} Z_{ij} > W_i' \sum_{j\in\mathbb{K}} \pi_{ij} (X_j)^{-1} W_i \\ \geq \mathrm{Her}(W_i) - \left(\mathcal{E}_i\left(X^{-1}\right)\right)^{-1},\end{aligned} \tag{20}$$

where $\left(\mathcal{E}_i\left(X^{-1}\right)\right)^{-1} = \left(\sum_{j\in\mathbb{K}} \pi_{ij} (X_j)^{-1}\right)^{-1}$, and rewriting the inequality (20),

$$\left(\mathcal{E}_i\left(X^{-1}\right)\right)^{-1} \geq \mathrm{Her}(W_i) - \mathcal{E}_i(Z). \tag{21}$$

On the other hand, notice that $\mathrm{Her}(T_i) - X_i \leq T_i' X_i^{-1} T_i, \forall i \in \mathbb{K}$, also consider, (21) and (15),

$$\begin{bmatrix} T_i' X_i^{-1} T_i & \star & \star & \star & \star \\ -\mathcal{S}G_i T_i & 2I & \star & \star & \star \\ 0 & 0 & \gamma^2 I & \star & \star \\ \tilde{A}_i T_i & -E_i & J_i & \left(\mathcal{E}_i\left(X^{-1}\right)\right)^{-1} & \star \\ \tilde{F}_i T_i & 0 & H_i & 0 & I \end{bmatrix} > 0, \tag{22}$$

note that, $X_i = P_i^{-1}, \forall i \in \mathbb{K}$, so applying the congruence transformation $\mathrm{diag}\left(T_i^{-1}, I, I, I, I\right)$,

$$\begin{bmatrix} P_i & \star & \star & \star & \star \\ -\mathcal{S}G_i & 2I & \star & \star & \star \\ 0 & 0 & \gamma^2 I & \star & \star \\ \tilde{A}_i & -E_i & J_i & \mathcal{E}_i(P)^{-1} & \star \\ \tilde{F}_i & 0 & H_i & 0 & I \end{bmatrix} > 0, \tag{23}$$

after applying Schur complement to inequality (23), we have,

$$\tilde{\boldsymbol{A}}_i' \begin{bmatrix} \mathcal{E}_i(P) & 0 \\ 0 & I \end{bmatrix} \tilde{\boldsymbol{A}}_i - \begin{bmatrix} P_i & \star & \star \\ -\mathcal{S}G_i & 2I & \star \\ 0 & 0 & \gamma^2 I \end{bmatrix} < 0, \tag{24}$$

where,

$$\tilde{\boldsymbol{A}}_i = \begin{bmatrix} \tilde{A}_i & -E_i & J_i \\ \tilde{F}_i & 0 & H_i \end{bmatrix}.$$

Now, adopting a strategy similar to that of last proof, we multiply the inequality (24) on the left by $\begin{bmatrix} x' & \phi' & 0' \end{bmatrix}$ and on the right by its transpose,

$$\begin{aligned} & x'\left(\tilde{A}_i' \mathcal{E}_i(P) \tilde{A}_i - P_i\right) x + \phi' \left(-2I + E_i' \mathcal{E}_i(P) E_i\right) \phi + \\ & + \mathrm{Her}\left(\phi'\left(-E_i' \mathcal{E}_i(P) \tilde{A}_i + \mathcal{S}G_i\right) x\right) < 0, \end{aligned} \tag{25}$$

grouping the terms of inequality (25), we have,

$$\underbrace{\mathbf{E}\left[x(k+1)' P_{\theta(k+1)} x(k+1) | x, \theta(k)\right] - x' P_{\theta(k)} x}_{\Delta\left(v_{x(k),\theta(k)}\right)} < \underbrace{\mathrm{Her}\left((\phi - \mathcal{S}q)' \phi\right)}_{\leq 0}, \tag{26}$$

note that, $v_{x(k),\theta(k)} = x(k)' P_{\theta(k)} x(k)$ is the stochastic Lyapunov function. Therefore, inequality (26) guarantees the stochastic stability of closed-loop Lur'e system. After, pre and post multiplying the inequality (24) by $\begin{bmatrix} x' & \phi' & w' \end{bmatrix}$,

$$\begin{aligned} & x'\left(\mathcal{L}_i(P) + \tilde{F}_i' \tilde{F}_i\right) x + \\ & + \phi' \left(-2I + E_i' \mathcal{E}_i(P) E_i\right) \phi + \\ & + \mathrm{Her}\left(\phi'\left(-E_i' \mathcal{E}_i(P) \tilde{A}_i + \mathcal{S}G_i\right) x\right) + \\ & + \mathrm{Her}\left(w'\left(-J_i' \mathcal{E}_i(P) E_i\right) \phi\right) + \\ & + \mathrm{Her}\left(w'\left(J_i' \mathcal{E}_i(P) \tilde{A}_i + H_i' F_i\right) x\right) + \\ & + w'\left(-\gamma^2 I + J_i' \mathcal{E}_i(P) J_i + H_i' H_i\right) w < 0, \end{aligned} \tag{27}$$

where $\mathcal{L}_i(P) = \tilde{A}_i' \mathcal{E}_i(P) \tilde{A}_i - P_i$. Grouping the terms of inequality (27), we have

$$\begin{aligned} & \underbrace{\mathbf{E}\left[v_{x(k+1),\theta(k+1)} \middle| x(k), \theta(k)\right] - v_{x(k),\theta(k)}}_{\Delta\left(v_{x(k),\theta(k)}\right)} + \\ & + \|z(k)\|^2 - \gamma^2 \|w(k)\|^2 < \underbrace{\mathrm{Her}\left((\phi - \mathcal{S}q)' \phi\right)}_{\leq 0}, \end{aligned} \tag{28}$$

summing in k and applying the expectation operator, remembering that the closed-loop system is stochastically stable,

$$\underbrace{\mathbf{E}\left[\sum_{k=0}^{\infty} \Delta\left(v_{x(k),\theta(k)}\right)\right]}_{=0} + \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0, \tag{29}$$

guaranteeing, in the worst case

$$\max_{\mathbb{F}} \|z\|_2 < \gamma \|w\|_2, \tag{30}$$

concluding the proof. □

In order to find the lowest guaranteed cost γ, it is possible to solve the following convex optimization problem

$$\inf_{\mathbb{S}} \{\gamma : \text{ subject to } (16) - (17)\},$$

and the controller can be obtained from (18) guaranteeing (19).

For the mode-independent control problem, we consider the following closed-loop matrices,

$$\tilde{A}_i \triangleq A_i + B_i K, \; \tilde{F}_i \triangleq F_i + L_i K. \tag{31}$$

The following corollary provides conditions to determine mode-independent feedback control gains.

Corollary 3 ($\mathcal{H}_\infty$ State-feedback Control). *The closed-loop system* (14) *with* $\phi(q) \in [0, \mathcal{S}]$ *is stochastically stable with guaranteed* $\mathcal{H}_\infty$ *cost* $\gamma \in \mathbb{R}_+$ *if there exists a feasible solution in the set*

$$\mathbb{S} \triangleq \{X_i > 0,\, Y,\, T,\, Z_{ij},\, W_i,\, \forall\, i,\, j \in \mathbb{K}\},$$

$$\begin{bmatrix} Z_{ij} & \star \\ W_i & X_j \end{bmatrix} > 0, \tag{32}$$

$$\begin{bmatrix} \mathrm{Her}\,(T) - X_i & \star & \star & \star & \star \\ -\mathcal{S}G_iT & 2I & \star & \star & \star \\ 0 & 0 & \gamma^2 I & \star & \star \\ A_iT + B_iY & -E_i & J_i & \mathrm{Her}\,(W_i) - \mathcal{E}_i\,(Z) & \star \\ F_iT + L_iY & 0 & H_i & 0 & I \end{bmatrix} > 0. \tag{33}$$

In the affirmative case,

$$K = YT^{-1}, \tag{34}$$

guarantees that, in closed-loop,

$$\max_{\mathbb{F}} \; \|z\|_2 < \gamma \, \|w\|_2 \,,\; \mathbb{F} \triangleq \{\phi(q) \in [0,\, \mathcal{S}]\}\,. \tag{35}$$

Proof: We assume that LMIs (32) and (33) hold. Inequalities (32) are equivalent to (16). On the other hand, notice that $\mathrm{Her}(T) - X_i \leq T'X_i^{-1}T$, $\forall i \in \mathbb{K}$, also consider (31),

$$\begin{bmatrix} T'X_i^{-1}T & \star & \star & \star & \star \\ -\mathcal{S}G_iT & 2I & \star & \star & \star \\ 0 & 0 & \gamma^2 I & \star & \star \\ \tilde{A}_iT & -E_i & J_i & \left(\mathcal{E}_i\left(X^{-1}\right)\right)^{-1} & \star \\ \tilde{F}_iT & 0 & H_i & 0 & I \end{bmatrix} > 0, \tag{36}$$

note that, $X_i = P_i^{-1}$,$\forall i \in \mathbb{K}$, so applying the congruence transformation $\mathrm{diag}\left(T^{-1},\, I,\, I,\, I,\, I\right)$,

$$\begin{bmatrix} P_i & \star & \star & \star & \star \\ -\mathcal{S}G_i & 2I & \star & \star & \star \\ 0 & 0 & \gamma^2 I & \star & \star \\ \tilde{A}_i & -E_i & J_i & \mathcal{E}_i\,(P)^{-1} & \star \\ \tilde{F}_i & 0 & H_i & 0 & I \end{bmatrix} > 0, \tag{37}$$

so notice that inequality (37) is equivalent to (23), therefore, if inequalities (32) and (33) had feasible solution, then (16) and (17) will also have, concluding the proof. □

Again, to find the lowest guaranteed cost γ, it is possible to solve the following convex optimization problem

$$\inf_{\mathbb{S}} \{\gamma : \text{ subject to } (32) - (33)\},$$

and the controller can be obtained from (34) guaranteeing (35).

5 Numerical Example

We study a Lur'e system taken from (Gonzaga and Costa, 2014) in our numerical example. We consider a following adaptation: nonlinear function $\phi(q(k))$ and sector κ are mode-independent. The transition probability matrix Π and initial probability distribution which is used in simulation, $\mu = \mathrm{Prob}\,(\theta_0 = i)$, are given by,

$$\left[\; \Pi \mid \mu \;\right] = \left[\begin{array}{cc|c} 0.6000 & 0.4000 & 0.3333 \\ 0.2000 & 0.8000 & 0.6667 \end{array}\right].$$

With the system matrices,

$$\left[\begin{array}{c|c|c} A_1 & E_1 & B_1 \\ \hline A_2 & E_2 & B_2 \\ \hline G_1 & 0 & 0 \\ \hline G_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c|c} 0.4 & 0.4 & 1.0 & 0.5 \\ 0.2 & 1.0 & 1.2 & 0.5 \\ \hline 1.1 & 0.6 & 1.2 & 0.7 \\ 0.3 & 0.4 & 1.0 & 0.5 \\ \hline 0.9 & 0.5 & 0.0 & 0.0 \\ \hline -1.0 & 0.7 & 0.0 & 0.0 \end{array}\right],$$

$$\left[\begin{array}{c|c} J_1' & H_1 \\ \hline J_2' & H_2 \end{array}\right] = \left[\begin{array}{cc|c} 0.10 & 0.12 & 0.5 \\ \hline 0.10 & 0.11 & 0.1 \end{array}\right],$$

$$\left[\; F_i \mid L_i \;\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \end{array}\right], i \in \mathbb{K}.$$

Solving the optimizations problems proposed in **Theorem 2** and **Corollary 3**, considering the sector $[0,\, 0.7]$, we have the controllers,

$$\left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right] = \left[\begin{array}{cc} -0.9352 & -0.9818 \\ \hline -1.2190 & -0.9116 \end{array}\right],$$

which guarantee the upper bound for $\mathcal{H}_\infty-$cost $\gamma < 0.5075$. In order to illustrate that fact, we perform a Monte-Carlo realization of 2,500 samples, each lasting 400 discrete time units with nonlinear function $\phi(q(k)) = 0.35q(k)(1 + \cos(25q(k)))$, and disturbance input,

$$w(k) = \begin{cases} \sin\left(\dfrac{2\pi}{200}k\right), & 0 \leq k \leq 200, \\ 0, & 200 < k \leq 400. \end{cases}$$

The chosen signal is bounded and deterministic to simplify the simulation, though it could be random as indicated in Definition 2.

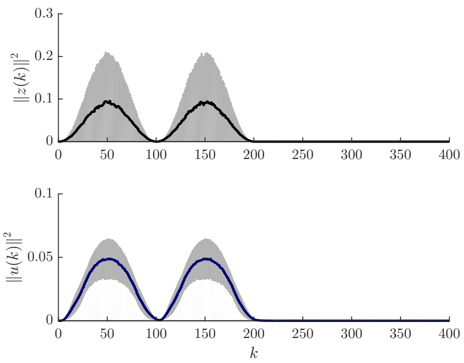

Figure 1: Mean of $\|z(k)\|^2$ (black) and $\|u(k)\|^2$ (blue) evaluated at all times k, for mode-dependent controller, the gray region is $\mathbf{E}(\|z(k)\|^2) \pm \sigma$ and $\mathbf{E}(\|u(k)\|^2) \pm \sigma$.

We obtained the $\mathcal{H}_\infty$ norm $\gamma_{\text{simulated}} = 0.3013$. On the other hand, for the mode-independent controller

$$K = \begin{bmatrix} -1.3198 & -0.4633 \end{bmatrix},$$

we can guarantee the upper bound for $\mathcal{H}_\infty-$cost $\gamma < 1.5075$.

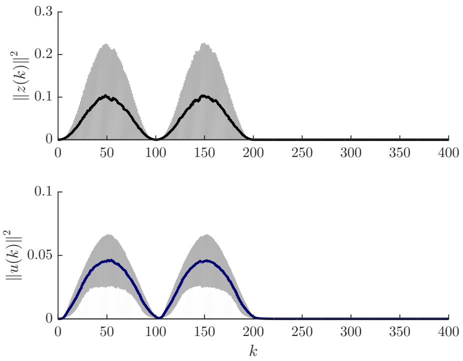

Figure 2: Mean of $\|z(k)\|^2$ (black) and $\|u(k)\|^2$ (blue) evaluated at all times k, for mode-independent controller, the gray region is $\mathbf{E}(\|z(k)\|^2) \pm \sigma$ and $\mathbf{E}(\|u(k)\|^2) \pm \sigma$.

Figure 1 shows the mean-square behavior of both performance output and the control input for mode-dependent controller, on the other hand, in Figure 2 shows the mean-square behavior of both performance output and the control input for mode-independent controller.

Running 2,500 Monte-Carlo realizations, with the same nonlinear function, we calculated the value $\gamma_{\text{simulated}} = 0.3166$. Note that the norms obtained through the simulations, for both mode-dependent and mode-independent controllers, are less than the guaranteed costs, as expected.

6 Conclusion

In this paper, we present conditions for the stochastic stability test, as well as for the design of controllers, both with upper bounds for $\mathcal{H}_\infty-$cost. Such controllers may or may not measure Markov parameter, $\theta(k)$, i.e, we design mode-dependent and mode-independent controllers. The paper ends with a numerical example that illustrates the obtained results.

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