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Observer design and a separation principle for linear discrete-time descriptor sysems ⋆

Lázaro Ismael Hardy Llins * **Daniel Coutinho** **

* *Post-graduation Program in Automation and Systems Engineering (PPGEAS), Universidade Federal de Santa Catarina (UFSC), Florianópolis, SC, 88040-900, (e-mail: hardyllins@gmail.com).*
** *Department of Automation and Systems (DAS), Universidade Federal de Santa Catarina (UFSC), Florianópolis, SC, 88040-900, (e-mail: daniel.coutinho@ufsc.br)*

Abstract: This paper addresses the design of state observers for linear discrete-time descriptor systems. Assuming that the original descriptor system is completely observable, an equivalent (standard) state-space representation of the system is proposed which preserves the system observability. Then, an LMI based approach is proposed for designing a Luenberger-like observer. In addition, a separation principle is demonstrated considering the estimation error dynamics and the closed-loop representation of the original descriptor system. Then, the observer design is extended to cope with model disturbances in an H_∞ sense. The effectiveness of the proposed methodology is illustrated by numerical examples.

Keywords: Descriptor system, State observer, Linear Matrix Inequalities (LMIs), Separation Principle.

1. INTRODUCTION

The introduction of the state space approach in late 50's and early 60's made possible to better interpret the dynamics of a system in order to develop mathematical tools for the analysis and synthesis of control systems. To deal with more complex phenomena, the standard state space representation has been extended to the class of descriptor systems which is also referred in specialized literature as singular systems or the generalized state space representation. Descriptor models are considered in many areas of application such as social-economic and biological systems as well as in many engineering fields (e.g., electrical power systems, aerospace engineering, chemical processes, robotic systems, among others); see, for instance, (Duan, 2010) and references therein.

Certainly, the most powerful tool introduced by the state space approach is the state feedback control law which requires the knowledge of the system state variables. However, in several cases, it is difficult or even impossible to measure all system state variables in order to apply a state feedback controller. To overcome this problem, a widely used strategy is to estimate the system state variables from the knowledge of a few measurements by means of a state observer (Luenberger, 1966). Nowadays, there exists a wide diversity of approaches for designing observers for standard linear and nonlinear systems as, for instance, the seminal work of Kalman and Bucy (1961) and the more recent ones of Bergsten et al. (2002), Choi and Ro (2005) and Khalil and Praly (2014), to cite a few. However, in several applications, the system dynamics is described by descriptor models, which has attracted increasing attention to the design of observer for this class of systems; see, e.g., (Dai, 1988), (Darouach and Boutayeb, 1995) and (Lu and Ho, 2006).

More recently, some observer design results have considered algebraic constraints to derive more tractable design conditions (in terms of linear matrix inequalities – LMIs) for discrete-time descriptor systems having an observable fast dynamics and considering state estimators with a standard state space representation. For instance, Darouach et al. (2010) have proposed H_∞ observer design conditions for a class of discrete-time Lipschitz nonlinear singular systems considering a parametrization of algebraic constraints from the estimation errors, Wang et al. (2012) presented a systemic design approach in terms of LMIs considering a given algebraic constraint derived from the system fast dynamics observability test, and Han et al. (2018) has introduced additional measurements (derived from the system fast dynamics) for designing standard state space estimators in the context of fault detection for linear discrete-time systems.

This work follows the latter observer design approaches but considering an output feedback application. Firstly, an LMI-based result (subject to a linear matrix equality – LME constraint) is proposed for deriving a standard state space observer. Then, a separation principle is proven for the output feedback configuration (i.e., the state feedback of estimated state variables) showing that the observer and the state feedback controller can be independently designed. Then, the observer design is extended to cope with ℓ_2 input disturbances in an H_∞ setting. Numerical

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examples illustrate the application of the proposed approach for output feedback design of discrete-time linear descriptor systems.

Notation: $\mathbb{C}$ is the set of complex numbers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real matrices, $\|\cdot\|$ is the Euclidean vector norm, I_n is the $n \times n$ identity matrix, 0_n and $0_{m\times n}$ are the $n \times n$ and $m \times n$ matrices of zeros, respectively, and diag$\{\cdots\}$ denotes a block-diagonal matrix. For a real matrix S, S^T denotes its transpose, He(S) stands for $S + S^T$, rank(S) is the rank of S, and $S > 0$ (≥ 0) means that S is symmetric and positive-definite (positive semi-definite). For a symmetric block matrix, $\star$ stands for the transpose of the blocks outside the main diagonal block. For a nonnegative integer number k and a vector sequence $f(k)$, its ℓ_2 norm is defined as $\|f(k)\|_{\ell_2} = \sqrt{\sum_{k=0}^{\infty} f(k)^T f(k)}$.

2. PROBLEM STATEMENT

Consider the following discrete-time linear time-invariant (LTI) descriptor system

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k),\ x(0) = x_0, \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input vector, $y(k) \in \mathbb{R}^p$ is the output vector, and A, B, C and E are known real matrices with appropriate dimensions with E allowed to be singular and satisfying rank$(E) = r \leq n$.

The nonsingularity of E induces some complexity in the behavior of system (1). Thus, in the following, some definitions regarding the solution of LTI descriptor systems are introduced.

Definition 2.1. Zhang et al. (2008) Consider system (1) and let z be a complex scalar and $E_0 \in \mathbb{R}^{n_r \times n}$ be a matrix such that $E_0 E = 0_{n_r\times n}$ and rank$(E_0) = n_r$, with $n_r = n - r$.

(1) An initial condition $x_0 \in \mathbb{R}^n$ is said to be consistent if $E_0 A x_0 + E_0 B u(0) = 0$ holds.
(2) The pair (E, A) is said to be regular if det$(zE - A)$ is not identically zero.
(3) The pair (E, A) is said to be causal if the degree of det$(zE - A)$ is equal to rank$(E) = r$.
(4) The system is said to be asymptotically stable if $\rho(E, A) < 1$, where $\rho(E, A)$ stands for the generalized spectral radius.
(5) The system is said to be admissible if it is regular, causal and asymptotically stable.

With respect to system (1), we assume the following:

Assumption 2.1. There exists a real matrix $K \in \mathbb{R}^{m\times n}$ such that system (1), with the following control law

$$u(k) = -Kx(k), \tag{2}$$

is admissible.

Assumption 2.2. System (1) is completely observable, that is, the following two conditions hold (Feng and Yagoubi, 2017):

(i) $\text{rank}\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n,$

(ii) $\text{rank}\left(\begin{bmatrix} (zE - A) \\ C \end{bmatrix}\right) = n,\ \forall\ z \in \mathbb{C} : 1 \leq |z| < \infty.$

In this paper, we are interested in designing a state observer to provide an estimate $\hat{x}(k)$ of $x(k)$ such that

- the estimation error
$$e(k) = x(k) - \hat{x}(k) \tag{3}$$
converges to zero as $k \to \infty$, and
- the closed-loop system of (1) with
$$u(k) = -K\hat{x}(k), \tag{4}$$
is admissible for a given K satisfying assumption 2.1.

To accomplish the above, in the next sections, we will firstly develop an LMI based approach to design an observer having a standard state-space representation and then establish a separation principle for the proposed observer, the descriptor system of (1) and the state-feedback control law of (4). In addition, we extend the proposed observer to cope with input disturbances in an H_∞ sense.

3. OBSERVER DESIGN

Notice from assumption 2.2-(i) that there exist real matrices $T \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{n\times p}$ such that the following holds:

$$TE + RC = I_n. \tag{5}$$

Furthermore, according to (Wang et al., 2012), there always exists a pair T, R satisfying (5) with rank$(T) = n$.

Then, we introduce an algebraic model transformation which yields a standard state-space representation of system (1) based on (5). To this end, post-multiplying (5) by $x(k + 1)$ yields:

$$\begin{aligned} x(k+1) &= TEx(k+1) + RCx(k+1) \\ &= TEx(k+1) + Ry(k+1) \\ &= TAx(k) + TBu(k) + Ry(k+1) \end{aligned}$$

leading to the following standard state-space representation

$$\begin{aligned} x(k+1) &= \tilde{A}x(k) + \tilde{B}u(k) + Ry(k+1) \\ y(k) &= Cx(k),\ x(0) = x_0, \end{aligned} \tag{6}$$

where $\tilde{A} = TA$ and $\tilde{B} = TB$.

In addition, the observability of (1) implies the observability of (6) for a nonsingular matrix T. To demonstrate this point, recall that the following holds:

$$\text{rank}\left(\begin{bmatrix} zE - A \\ C \end{bmatrix}\right) = n,\ \forall\ z \in \mathbb{C} : 1 \leq |z| < \infty.$$

Hence, the observability of (6) follows from the fact that (Guo et al., 2019)

$$
\begin{aligned}
n &= \operatorname{rank}\left(\begin{bmatrix} zE - A \\ C \end{bmatrix}\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} T & zR \\ 0 & I \end{bmatrix}\begin{bmatrix} zE - A \\ C \end{bmatrix}\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} zTE + zRC - \tilde{A} \\ C \end{bmatrix}\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} z(TE + RC) - \tilde{A} \\ C \end{bmatrix}\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} zI - \tilde{A} \\ C \end{bmatrix}\right), \ \forall\, z \in \mathbb{C} : 1 \le |z| < \infty.
\end{aligned}
$$

In view of the above developments, the following observer is proposed:

$$
\begin{aligned}
\hat{x}(k+1) &= \tilde{A}\hat{x}(k) + \tilde{B}u(k) + Ry(k+1) \\
&\qquad + L(y(k) - \hat{y}(k)) \qquad (7) \\
\hat{y}(k) &= C\hat{x}(k),\ \hat{x}(0) = \hat{x}_0,
\end{aligned}
$$

where $\hat{x} \in \mathbb{R}^n$ is the observer state, $\hat{y}$ is an estimate of $y(k)$ and $L \in \mathbb{R}^{n\times p}$ is to be designed such that the estimation error $e(k)$ as defined in (3) converges to zero as $k \to \infty$.

Hence, before introducing the main result of this section which establishes an LMI condition for designing the observer gain L as well as the matrices T and R, consider the following error dynamics which can be easily derived from (3), (6) and (7):

$$
e(k+1) = (TA - LC)e(k),\ e(0) = e_0 = x_0 - \hat{x}_0. \qquad (8)
$$

Theorem 1. Consider the error system in (8) under assumption 2.2. Let $E_0 \in \mathbb{R}^{n_r\times n}$ be a matrix such that $E_0E = 0_{n_r\times n}$, with $\operatorname{rank}(E_0) = n_r$, and $n_r = n - r$. Suppose there exist matrices $P = P^T \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{n\times n}$, $L_z \in \mathbb{R}^{n\times p}$, $T_a \in \mathbb{R}^{r\times n}$, $T_b \in \mathbb{R}^{n_r\times n_r}$ and $R_b \in \mathbb{R}^{n_r\times p}$ such that the following constraints are satisfied.

$$
T_zE + R_zC - Z = 0 \qquad (9)
$$

$$
T_b + T_b^T > 0 \qquad (10)
$$

$$
\begin{bmatrix} P - Z - Z^T & T_zA - L_zC \\ \star & -P \end{bmatrix} < 0 \qquad (11)
$$

where

$$
T_z = \begin{bmatrix} T_a \\ T_bE_0 \end{bmatrix}, \quad R_z = \begin{bmatrix} 0_{r\times p} \\ R_b \end{bmatrix}. \qquad (12)
$$

Then, the matrices Z and T_z are nonsingular; the error system in (8) is asymptotically stable, with

$$
L = Z^{-1}L_z, \quad T = Z^{-1}T_z\ ; \qquad (13)
$$

and the equality constraint in (5) holds with T as above and $R = Z^{-1}R_z$.

Proof 1. Suppose that (9), (10) and (11) hold for some P, Z, L_z, T_a, T_b and R_b. Then, notice from (11) that $P > 0$ and $Z + Z^T - P > 0$. Then, Z is nonsingular and let $V(k) = e(k)^TPe(k)$ be a Lypunov function candidate.

Firstly, it will be shown that the matrix T is nonsingular by showing that T_z defined in (13) is nonsingular. To this end, taking the definitions of R_z and T_z in (12) into account, it follows from (9) that

$$
\begin{bmatrix} T_a & 0 \\ 0 & R_b \end{bmatrix}\begin{bmatrix} E \\ C \end{bmatrix} = Z.
$$

Since Z is nonsingular, one can infer that $\operatorname{rank}(T_a) = \operatorname{rank}(T_aE) = r$ and $\operatorname{rank}(R_b) = n_r$. In addition, it follows from (10) that T_b is nonsingular, which implies that $\operatorname{rank}(T_bE_0) = n_r$. Thus, the two row-partitions T_a and T_bE_0 of T_z in (12) are full row-rank matrices. As $T_z \in \mathbb{R}^{n\times n}$, to complete the proof of the nonsingularity of T_z it will be shown that the r rows of T_a are linearly independent of the n_r rows of T_bE_0. Hence, by contradiction, suppose that the rank of the composite matrix T_z is smaller than n. This implies there exist vectors $\zeta = [\zeta_1\ \cdots\ \zeta_r]^T$ and $\theta = [\theta_1\ \cdots\ \theta_{n_r}]^T$, with at least one of the ζ_i's and one of the θ_i's being nonzero, such that $\zeta^TT_a + \theta^TT_bE_0 = 0$. Considering that $E_0E = 0$, post-multiplying the latter equation by E leads to $\zeta^TT_aE = 0$, which is a contradiction because $\zeta \neq 0$ and T_aE is a full row-rank matrix.

Then, consider the Linear Matrix Equality (LME) in (9) and let $T = Z^{-1}T_z$ and $R = Z^{-1}R_z$. Pre-multiplying (9) by Z^{-1} leads to $TE + RC = I_n$.

Next, notice that (11) can be cast as follows:

$$
\operatorname{diag}\{P, -P\} + Z_aA_a + A_a^TZ_a^T < 0, \qquad (14)
$$

where $Z_a = \begin{bmatrix} Z^T & 0 \end{bmatrix}^T$ and $A_a = [-I_n\ \ (TA - LC)]$. Now, let $\xi = \begin{bmatrix} e(k+1)^T & e(k)^T \end{bmatrix}^T$. Then, pre- and post-multiplying (14) by ξ^T and ξ, respectively, yields

$$
V(k+1) - V(k) = e(k+1)^TPe(k+1) - e(k)^TPe(k) < 0,
$$

since $A_a\xi = 0$ from (8). The rest of this proof follows directly from the Lyapunov theory; see, for instance, Koenig et al. (2008), Buzurovic et al. (2019).

Remark 1. Notice that Theorem 1 is in terms of linear matrix inequalities (LMIs) subject to a linear matrix equality (LME) constraint, which can be solved using standard software packages. In particular, the LMI parser YALMIP (Lofberg, 2004) allows for translating LME constraints into LMIs, which can be efficiently solved using standard LMI solvers such as SeDuMi (Sturm, 1999) and SDPT3 (Toh et al., 1999).

In practice, the observer defined in (7) can be implement considering its one step delayed version, that is:

$$
\begin{aligned}
\hat{x}(k) &= (TA - LC)\hat{x}(k-1) + TBu(k-1) + Ry(k) \\
&\qquad + Ly(k-1) \qquad (15)
\end{aligned}
$$

Notice that the current estimate $\hat{x}(k)$ is obtained from current and past measurements (i.e., $u(k-1)$, $y(k)$ and $y(k-1)$). However, for feedback purposes, it may exist an algebraic loop when the algebraic states are a function of the control input since in this case $y(k)$ will be indirectly determined by the current control $u(k) = K\hat{x}(k)$ by noting that

$$
E_0Ax(k) + E_0Bu(k) = 0
$$

holds for all $k \ge 0$, where E_0^T is a basis for the null space of E^T. In order to avoid the solution of the algebraic loop for each k, we also assume the following with respect to system (1):

Assumption 3.1. The control input matrix B is such that $E_0B = 0$.

Remark 2. It can be shown using the singular value representation of (1) and the system reduced order model (see, e.g., Duan (2010)) that the above assumption implies that:

(i) the output vector signal $y(k)$ is only a function of the dynamic states of (1), which are defined by $Ex(k)$; and

(ii) the algebraic state variables, defined by $E_0x(k)$, do not depend on the control signal.

4. SEPARATION PRINCIPLE

We show in the following that the proposed observer preserves the separation principle, that is, the matrices K and L of the following closed-loop system

$$\begin{cases} Ex(k{+}1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k), \\ \hat{x}(k{+}1) &= (TA{-}LC)\hat{x}(k) + TBu(k) \\ & \quad +Ry(k+1) + Ly(k) \\ u(k) &= -K\hat{x}(k) \end{cases} \tag{16}$$

can be designed independently provided that (5) holds.

To this end, consider the system representation in (6). Then, the closed-loop system in (16) can be cast as follows:

$$x(k+1) = \tilde{A}x(k) - \tilde{B}K\hat{x}(k) + Ry(k+1), \tag{17}$$

$$\hat{x}(k+1) = (\tilde{A}{-}LC{-}\tilde{B}K)\hat{x}(k) + LCx(k) + Ry(k+1), \tag{18}$$

or in the following compact representation:

$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A} & -\tilde{B}K \\ LC & (\tilde{A}{-}LC{-}\tilde{B}K) \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} + \begin{bmatrix} R \\ R \end{bmatrix} y(k+1). \tag{19}$$

Hence, pre-multiplying the above by $\begin{bmatrix} I_n & -I_n \\ I_n & 0_n \end{bmatrix}$ leads to

$$\begin{bmatrix} e(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} (\tilde{A}{-}LC) & -(\tilde{A}{-}LC) \\ \tilde{A} & -\tilde{B}K \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ R \end{bmatrix} y(k+1), \tag{20}$$

since

$$\begin{bmatrix} I_n & -I_n \\ I_n & 0_n \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} = \begin{bmatrix} e(k) \\ x(k) \end{bmatrix}. \tag{21}$$

Next, by noting that

$$\begin{bmatrix} 0_n & I_n \\ -I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ I_n & 0_n \end{bmatrix} = I_{2n},$$

we obtain from (21) the following:

$$\begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ -I_n & I_n \end{bmatrix} \begin{bmatrix} e(k) \\ x(k) \end{bmatrix}.$$

Hence, we can rewritten (20) in the following state-space form:

$$\begin{bmatrix} e(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} (\tilde{A} - LC) & 0 \\ \tilde{B}K & (\tilde{A} - \tilde{B}K) \end{bmatrix} \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} + \begin{bmatrix} 0 \\ R \end{bmatrix} y(k+1). \tag{22}$$

Then, taking into account that $y(k+1) = Cx(k+1)$, the representation in (22) can be cast in the following descriptor form:

$$\begin{bmatrix} I_n & 0_n \\ 0_n & (I{-}RC) \end{bmatrix} \begin{bmatrix} e(k{+}1) \\ x(k{+}1) \end{bmatrix} = \begin{bmatrix} (\tilde{A}{-}LC) & 0 \\ \tilde{B}K & (\tilde{A}{-}\tilde{B}K) \end{bmatrix} \begin{bmatrix} e(k) \\ x(k) \end{bmatrix}.$$

Next, pre-multiplying the above by diag$\{I_n, T^{-1}\}$ yields

$$\begin{bmatrix} I_n & 0_n \\ 0_n & E \end{bmatrix} \begin{bmatrix} e(k{+}1) \\ x(k{+}1) \end{bmatrix} = \begin{bmatrix} \tilde{A} - LC & 0 \\ BK & (A{-}BK) \end{bmatrix} \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \tag{23}$$

Notice that the dynamic matrix of the augmented system (23) is lower block triangular. Hence, the eigenvalues of estimation error sub-system can be freely assigned. Assuming that the matrix $(\tilde{A} - LC)$ is Schur stable, the closed-loop dynamics will be as follows

$$Ex(k+1) = (A - BK)x(k) + BKe(k). \tag{24}$$

Since $e(k)$ vanishes to zero as $k \to \infty$, the admissibility of (24) is ensured by Assumption 2.1.

5. H_∞ OBSERVER DESIGN

Suppose that the dynamics of system (1) is subject to an ℓ_2 exogenous disturbance vector $w \in \mathbb{R}^q$, that is:

$$Ex(k+1) = Ax(k) + Bu(k) + B_w w(k), \; Ex(0) = 0, \tag{25}$$

where $B_w \in \mathbb{R}^{n \times q}$.

In this section, we present a methodology for designing the observer in (7) such that the effects of $w(k)$ in the error dynamics are minimized in an H_∞ sense. To this end, considerer the following system:

$$\mathcal{G} : \begin{cases} e(k{+}1) &= (TA{-}LC)e(k){+}B_w w(k), \\ s(k) &= C_s e(k) + D_s w(k), \; e(0) = 0, \end{cases} \tag{26}$$

where $s(k) \in \mathbb{R}^h$ is the performance output, and C_s and D_s are given real matrices with appropriate dimensions.

Then, we are interested in this section in designing the matrices L, T and R such that an upper-bound γ on the ℓ_2-gain, denoted as $\|\mathcal{G}\|_\infty$, of system (26) is minimized, where

$$\|\mathcal{G}\|_\infty = \sup_{0 \not\equiv w(k) \in \ell_2} \frac{\|s(k)\|_{\ell_2}}{\|w(k)\|_{\ell_2}} \tag{27}$$

The following result provides a bound γ on $\|\mathcal{G}\|_\infty$ while guaranteeing that the error dynamics is asymptotically stable for $w(k) \equiv 0$ and a nonzero initial condition.

Theorem 2. Consider the error system in (26) under assumption 2.2. Let $E_0 \in \mathbb{R}^{n_r \times n}$ be a matrix such that $E_0 E = 0_{n_r \times n}$, with rank$(E_0) = n_r$. Suppose there exist matrices $P = P^T \in \mathbb{R}^{n \times n}$, $Z \in \mathbb{R}^{n \times n}$, $L_z \in \mathbb{R}^{n \times p}$, $T_a \in \mathbb{R}^{r \times n}$, $T_b \in \mathbb{R}^{n_r \times n_r}$ and $R_b \in \mathbb{R}^{n_r \times p}$, and a positive scalar γ such that the following constraints are satisfied.

$$T_z E + R_z C - Z = 0 \tag{28}$$

$$T_b + T_b^T > 0 \tag{29}$$

$$\begin{bmatrix} P{-}Z{-}Z^T & T_z A{-}L_z C & Z B_w & 0 \\ \star & -P & 0 & C_s^T \\ \star & \star & -\gamma I_q & D_s^T \\ 0 & C_s & D_s & -\gamma I_h \end{bmatrix} < 0 \tag{30}$$

where

$$T_z = \begin{bmatrix} T_a \\ T_b E_0 \end{bmatrix}, \quad R_z = \begin{bmatrix} 0_{r \times p} \\ R_b \end{bmatrix}. \tag{31}$$

Then, the following holds:

(i) The matrices Z and T_z are nonsingular;

(ii) The equality constraint in (5) holds with T as in (32) and $R = Z^{-1} R_z$;

(*iii*) The unforced system in (26) (i.e., $w(k) \equiv 0$) with

$$L = Z^{-1}L_z, \quad T = Z^{-1}T_z\ , \tag{32}$$

is asymptotically stable; and

(*iv*) $\|\mathcal{G}\|_\infty \le \gamma$.

Proof 2. Assume there exist matrices P, Z, T_a, T_b and R_b and a scalar γ satisfying (28)-(30). The proof of items (*i*) and (*ii*) follows from the proof of Theorem 1.

Next, notice that (11) can be cast as follows from the Schur's complement:

$$\text{diag}\{P, -P, -\gamma I_q\} + \text{He}(Z_b A_b) + \gamma^{-1} C_b^T C_b < 0, \tag{33}$$

where $Z_b = \begin{bmatrix} Z^T & 0 & 0\end{bmatrix}^T$, $A_b = [-I_n \quad (TA-LC) \quad B_w]$ and $C_b = [0 \quad C_s \quad D_s]$. Now, let

$$\eta = \begin{bmatrix} e(k+1)^T & e(k)^T & w(k)^T \end{bmatrix}^T.$$

Then, pre- and post-multiplying (33) by η^T and η, respectively, yields

$$V(k+1) - V(k) + \frac{1}{\gamma} s(k)^T s(k) - \gamma w(k)^T w(k) < 0,$$

since $A_b \eta = 0$ and $C_b \eta = s(k)$ from (26). The rest of this proof follows the bounded real lemma for discrete-time systems (de Souza and Xie, 1992).

6. NUMERICAL EXAMPLES

In this section, we shall illustrate the results through two numerical examples.

6.1 State observer design and separation principle

Consider the discrete-time LTI descriptor system defined in (1) with the following matrices:

$$E = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.153 & 0.045 & 0.069 \\ 0.156 & 0.252 & 0.156 \\ 0.153 & -0.171 & -0.636 \end{bmatrix},$$

$$B = [1\ 1\ 0]^T, \quad C = [0\ 0\ 1]\,. \tag{34}$$

Notice that the above matrices imply that system (1) is regular and causal and satisfies assumptions 2.1, 2.2 and 3.1.

Then, we firstly design a state-feedback controller considering the system reduced-order model. To this end, we applied the command *place* of *Matlab* for the reduced order model given by

$$\xi(k+1) = \begin{bmatrix} 0.8543 & -0.5590 \\ 0.3556 & 0.0677 \end{bmatrix} \xi(k) + \begin{bmatrix} -4.5798 \\ -0.1598 \end{bmatrix} u(k) \tag{35}$$

such that the closed-loop poles are located in 0.4 and 0.6 leading to

$$K = [0.0109\ -0.0178\ -0.0089]\,. \tag{36}$$

Next, Theorem 1 is applied to determine the state observer (i.e., the matrices L, T and R) using the parser *YALMIP* and solver *SeDuMi* which yields

$$L = \begin{bmatrix} -0.1977 \\ 0.3934 \\ -0.0188 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0000 \\ -0.5000 \\ 1.0000 \end{bmatrix}, \tag{37}$$

$$T = \begin{bmatrix} 0.7265 & -0.7265 & 0.2735 \\ 0.6263 & -0.1263 & -0.6263 \\ -0.0123 & 0.0123 & 0.0123 \end{bmatrix}.$$

Figures 1, 2 and 3 show respectively the states of the closed-loop system with $u(k) = Kx(k)$ (i.e., state-feedback) and $u(k) = K\hat{x}(k)$ (i.e., output feedback) as well as the estimation error trajectory $e(k) = x(k) - \hat{x}(k)$ considering an admissible initial condition given by $x(0) = [0.7156\ -1.4189\ 0.1103]^T$ with $\hat{x}(0) = 0$. Notice the good performance achieved by the output feedback control law $u(k) = K\hat{x}(k)$ when compared to the state feedback controller $u(k) = Kx(k)$ demonstrating the efficiency of the proposed methodology.

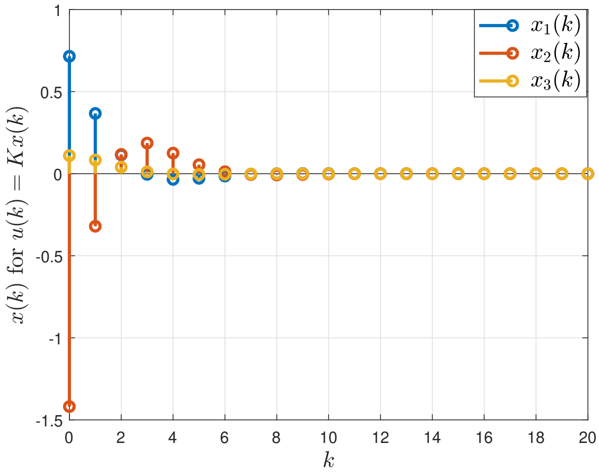

Figure 1. State trajectories of the closed-loop system with $u(k) = Kx(k)$ (state feedback controller).

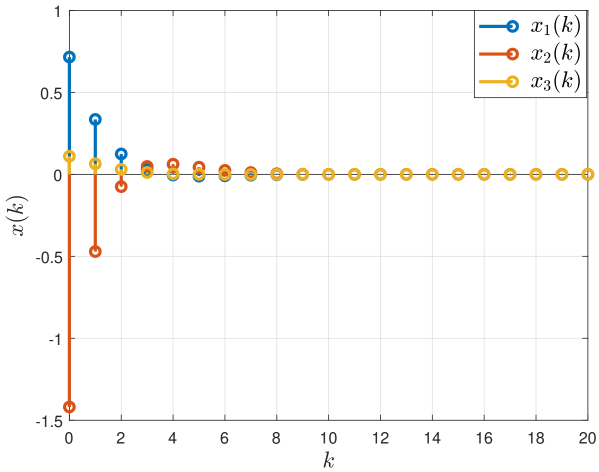

Figure 2. State trajectories of the closed-loop system with $u(k) = K\hat{x}(k)$ (output feedback controller).

6.2 H_∞ observer design

This example illustrates the application of Theorem 2 for designing a state observer. To this end, consider the error system dynamics as defined in (26) with (34) and the following matrices:

$$B_w = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad C_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_s = \begin{bmatrix} 0 \\ 0.5 \\ 0.25 \end{bmatrix}.$$

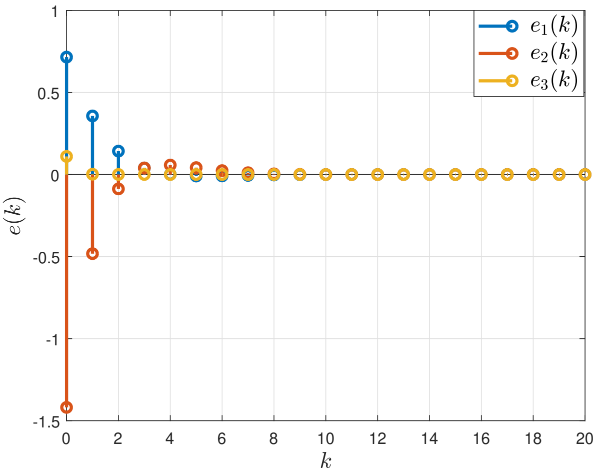

Figure 3. Estimation error trajectory for $e(0) = x(0)$.

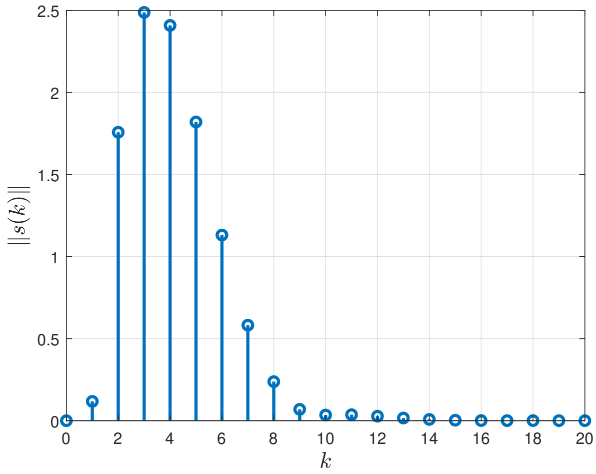

Figure 4. Norm of the output performance variable $\|s(k)\|$.

Firstly, we design an H_∞ state-feedback controller considering the system reduced-order model as defined in (35) with an additional disturbance input given by $[-7.2479\ 1.2118]^T\, w(k)$ and the following performance output

$$s_\xi(k) = \begin{bmatrix} -0.2521 & 0.9677 \\ -1.0129 & -0.4061 \\ -0.1381 & 0.2484 \end{bmatrix} \xi(k) + \begin{bmatrix} 0.00 \\ 0.50 \\ 2.50 \end{bmatrix} w(k)$$

Then, we apply a standard LMI based H_∞ control design technique leading to the following control gain:

$$K = [-0.0212\ \ -0.0315\ \ -0.0157]\,. \tag{38}$$

Next, the following optimization problem (i.e., Theorem 2)

$$\min_{P,Z,L_z,T_z,R_z,\gamma} \gamma :\ \text{(28)-(30)}$$

is applied to design the observer which yields:

$$L = \begin{bmatrix} -0.1171 \\ 0.7681 \\ 0.1981 \end{bmatrix},\quad R = \begin{bmatrix} 0.0000 \\ -0.5000 \\ 1.0000 \end{bmatrix}, \tag{39}$$

$$T = \begin{bmatrix} 0.7804 & -0.7804 & 0.2195 \\ 1.2670 & -0.7670 & -1.2670 \\ 0.3235 & -0.3235 & -0.3235 \end{bmatrix},\ \gamma = 2.5808.$$

Figure 4 shows the norm of the performance output vector considering the closed-loop system with $u(k) = K\hat{x}(k)$, a disturbance input $w(k) = 0.5^k \sin(0.5k)$ and an initial condition $x(0) = 0$, which demonstrates the effectiveness of the proposed approach.

7. CONCLUDING REMARKS

This paper has presented an LMI based technique for the observer design of LTI discrete-time descriptor systems. In addition, it has been shown a separation principle aiming an output feedback implementation. The observer design technique is extended to cope with additive ℓ_2 disturbances. The simulations have demonstrated the potentials of the proposed approach as a tool to design output feedback controllers for discrete-time descriptor systems.

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