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Novel Gaussian State Estimator based on H_2 Norm and Steady-State Variance ⋆

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Abstract: This paper proposes a novel state estimator for discrete-time linear systems with Gaussian noise. The proposed algorithm is a fixed-gain filter, whose observer structure is more general than Kalman one for linear time-invariant systems. Therefore, the steady-state variance of the estimation error is minimized. For white noise stochastic processes, this performance criterion is reduced to the square H_2 norm of a given linear time-invariant system. Then, the proposed algorithm is called *observer* H_2 *filter* (OH_2F). This is the standard Wiener-Hopf or Kalman-Bucy filtering problem. As the Kalman predictor and Kalman filter are well-known solutions for such a problem, they are revisited.

Keywords: Linear time-invariant systems, Gaussian random variables, H_2 norm, steady-state variance, Kalman filter.

1. INTRODUCTION

Under Gaussian and linear assumptions, the Kalman filter (KF) (Kalman, 1960) is the minimum-variance state estimator. Its observer structure corrects the state forecasts according to the measurement error, called *innovation*, which is weighted by the Kalman gain. The KF assumes that all uncertainties are characterized by Gaussian random variables (GRVs), whose probability density functions (PDFs) are completely defined by the two first moments of a random variable. In turn, the Kalman predictor (KP) is an algorithm derived from a modification on the KF structure. The difference between them is the time instant in which the measurement information is acquired and assimilated in the estimator; the time instant affects both measurement sequence and measurement model. The KF uses the past and current measurements, while the KP uses only the past measurements (Teixeira, 2008; de Paula et al., 2019).

Time-varying gain algorithms, as the KF, are suitable to follow the linear time-varying (LTV) system evolution, since its dynamics change over time. When a system is linear time invariant (LTI), its dynamics are composed by two parcels, namely transient and steady state. Initially, a varying gain algorithm should be used due to the transient. After the transient, the varying gain converges to the constant gain. In this case, a fixed gain could be used instead. Moreover, if the transient is not significant for a given application, as the examples reviewed in Tang et al. (2019a,b), then a fixed-gain algorithm can be employed during all time. The main advantage of fixed-gain algorithms is to reduce the computational burden of varying gain algorithms, since the observer gain is obtained only once.

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In this paper, we propose a fixed-gain filter for discrete-time LTI state-space uncertain systems, called OH_2F. This algorithm is inspired in Tang et al. (2019b), whose observer structure is more general than the KP one. The filter proposed in Tang et al. (2019b) assumes that the uncertainties are characterized by *zonotopes*, which are centrally symmetric polytopes defined by *center* and *generator matrix*. In such work, the observer is built introducing two unknown matrices in the KP observer structure. Moreover, the constant design matrices, including the observer gain, are computed via *bounded real lemma* (Boyd et al., 1994; Xu and Lam, 2006), that is, computing the H_∞ norm of the error system based on linear matrix inequalities (LMIs).

Unlike Tang et al. (2019b), the OH_2F assumes that the uncertainties are characterized by GRVs. In this case, we are interested in representing uncertainties by means of probability regions. In addition, the constant design matrices are computed by solving the H_2 norm of the error system based on LMIs. To achieve that, we relate GRVs to white noise, which is a special case of wide-sense stationary (WSS) stochastic processes (Kay, 1993). For white noise inputs, minimizing the H_2 norm corresponds to minimizing the steady-state variance of the estimation error (Khargonekar et al., 1996). Thereby, the state variance is assumed bounded instead of the noise realizations; this enables the usage of GRVs. Due to significant similarities, we also discuss the KP, KF, and their stationary version, here called H_2P and H_2F, respectively. We show that the H_2P is a special case of the OH_2F. At the end, we use a numerical example to compare their performance.

2. PRELIMINARIES

2.1 Stochastic Processes

A random variable is a mapping from the sample space Ω into the measurable set $\mathbb{S}$. If $\mathbb{S} \subset \mathbb{N}$, then the random variable is said to be discrete. If $\mathbb{S} \subset \mathbb{R}$, then the random variable is said to be continuous. Such classifications do not change if any random variables are related to discrete-time or continuous-time systems.

Let X be an n-dimensional random variable with realizations $x \in \mathbb{R}^n$. The mean and covariance matrix of X are given by $\hat{x} = \mathrm{E}[X]$ and $P^{\mathrm{xx}} = \mathrm{cov}(X, X) \triangleq \mathrm{E}[(X - \hat{x})(X - \hat{x})^{\mathrm{T}}]$, respectively, where $\mathrm{E}[\bullet]$ is the expected value operator. A GRV X is defined by its Gaussian PDF, $p(x)$, and can be completely characterized by its mean and covariance, $X \sim \mathcal{N}(\hat{x}, P^{\mathrm{xx}})$.

The affine transformation and summation of uncorrelated GRVs are computed as

$$LX + m \sim \mathcal{N}\left(L\hat{x} + m, LP^{\mathrm{xx}}L^{\mathrm{T}}\right), \tag{1}$$
$$X + W \sim \mathcal{N}\left(\hat{x} + \hat{w}, P^{\mathrm{xx}} + P^{\mathrm{ww}}\right), \tag{2}$$

where $L \in \mathbb{R}^{b\times n}$, $m \in \mathbb{R}^b$, and $W \sim \mathcal{N}(\hat{w}, P^{\mathrm{ww}})$.

Without loss of generality, a *stochastic process* (SP) is defined to be an infinite sequence of random variables $\{X\} = \{X_0, X_1, \ldots, X_k, \ldots\}$, with one random variable X_k for each time instant $k \in \mathbb{N}$, and each realization of the SP takes on a value that is represented as an infinite sequence of numbers, that is, $\{x\} = \{x_0, x_1, \ldots, x_k, \ldots\}$ (Kay, 1993). An SP is called *stationary* if its probabilistic description does not change with respect to the time origin. An SP is called WSS if: (i) its mean $\mathrm{E}[X_k]$ is constant over time k, and (ii) its covariance $\mathrm{cov}\left(X_{k_1}, X_{k_2}\right)$ depends only upon the time difference, that is, $k_2 - k_1$. Then, a WSS SP is *ergodic* in the mean.

Here, a *white noise* is defined as a WSS SP with: (i) zero mean $\mathrm{E}[X_k] = 0$, (ii) constant covariance $\mathrm{cov}\left(X_k, X_{k+\tau}\right) = Q$ for $\tau = 0$, and (iii) uncorrelated samples $\mathrm{cov}\left(X_k, X_{k+\tau}\right) = 0_{n\times n}$ for $\tau \neq 0$, where $0_{n\times n}$ represents the null matrix.

The power spectral density (PSD) quantifies the distribution of power of a given SP over the frequency. Here, we are interested in discrete-time signals. The PSD matrix of a *zero mean* WSS SP $\{W\}$ is given by

$$S^{\mathrm{w}}(\omega) = \sum_{\tau=-\infty}^{+\infty} Q_\tau \mathrm{e}^{-j\omega\tau}, \tag{3}$$

where $Q_\tau = \mathrm{cov}(W_\tau, W_0)$ is the autocorrelation matrix, $\omega \in [-\pi, \pi]$ is the angular frequency, and $k = 0$ is the time origin of the SP.

If the SP $\{W\}$ is a white noise with $Q = \mathrm{I}_{b\times b}$, where $\mathrm{I}_{b\times b}$ is the identity matrix, then, its PSD is $S^{\mathrm{w}}(\omega) = \mathrm{I}_{b\times b}$. That is, the PSD of the white noise is constant over its frequency spectrum.

2.2 Linear Matrix Inequalities

In the following lemma, the procedure to compute the H_2 norm of a given LTI system via LMIs is revisited.

Lemma 1. (Boyd et al., 1994) Consider the discrete-time state-space system described by

$$x_k = \mathcal{A}x_{k-1} + \mathcal{B}w_{k-1}, \tag{4}$$
$$z_k = \mathcal{C}x_k, \tag{5}$$

where $w_{k-1} \in \mathbb{R}^b$ is the disturbance input, $\mathcal{A} \in \mathbb{R}^{a\times a}$, $\mathcal{B} \in \mathbb{R}^{a\times b}$, and $\mathcal{C} \in \mathbb{R}^{c\times a}$. If there exists the symmetric and positive-definite matrix $P = P^{\mathrm{T}} \succ 0_{a\times a}$, then the square H_2 norm of (4)-(5), $\psi > 0$, satisfies

$$\begin{aligned} \min \quad & \psi \\ \text{s.t.} \quad & \mathcal{A}^{\mathrm{T}}P\mathcal{A} - P + \mathcal{C}^{\mathrm{T}}\mathcal{C} \prec 0_{a\times a},\ \mathrm{tr}\left(\mathcal{B}^{\mathrm{T}}P\mathcal{B}\right) < \psi, \end{aligned} \tag{6}$$

where $\mathrm{tr}(\bullet)$ is the trace of a square matrix, and "$\prec$" means a negative-definite matrix. □

The following lemma uses the PSD of $\{W\}$ to minimize the steady-state variance of the estimation error. In Khargonekar et al. (1996), a similar result is formulated for continuous-time SPs, but here it is formulated for discrete-time SPs.

Lemma 2. Consider the discrete-time error system described as (4)-(5), where $w_{k-1} \in \mathbb{R}^b$ is a realization of the white noise $\{W\}$ at time $k - 1$.

Assume that the PSD of $\{W\}$ is $S^{\mathrm{w}} = \mathrm{I}_{b\times b}$, and denote $G \in \mathbb{R}^{c\times b}$ the transfer matrix from the noise input w to the error output z, that is, $z = Gw$. Then, the steady-state variance of the estimation error is given by

$$\begin{aligned} \lim_{k\to\infty} \mathrm{E}\left[z_k^{\mathrm{T}} z_k\right] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\left(GS^{\mathrm{w}}G^*\right) d\omega \\ &= ||G||_{\mathrm{H}_2}^2, \end{aligned} \tag{7}$$

where $(\bullet)^*$ is the complex conjugate transpose, and $||\bullet||_{\mathrm{H}_2}$ is the H_2 norm of system. □

Note that the covariance matrix of the white noise is assumed to be $Q = \mathrm{I}_{b\times b}$ in Lemma 2. In order to consider cases where this assumption is not true, the positive semidefinite matrix $Q \succeq 0_{b\times b}$ can be decomposed as $Q = LL^{\mathrm{T}}$, where L is the square root of Q. After, the matrix L is merged with $\mathcal{B}$, to yield $\bar{\mathcal{B}} = \mathcal{B}L$. Then, without loss of generality, the matrix Q used in Lemma 2 is seen as an identity matrix.

The following lemma allows to find the general solution for linear systems.

Lemma 3. (Wang et al., 2018) Consider the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, where $\mathrm{rank}\left(\mathcal{B}\right) = c$, and $\mathcal{A}$ is unknown. The general solution of $\mathcal{A}\mathcal{B} = \mathcal{C}$ is given by

$$\mathcal{A} = \mathcal{C}\mathcal{B}^\dagger + \mathcal{S}\left(\mathrm{I}_{b\times b} - \mathcal{B}\mathcal{B}^\dagger\right), \tag{8}$$

where $\mathcal{S} \in \mathbb{R}^{a\times b}$ is an arbitrary matrix, and $(\bullet)^\dagger$ is the pseudo-inverse operator, that is,

$$\mathcal{B}^\dagger = \left(\mathcal{B}^{\mathrm{T}}\mathcal{B}\right)^{-1}\mathcal{B}^{\mathrm{T}}. \tag{9}$$

□

3. PROBLEM FORMULATION

Consider the discrete-time LTI system as

$$x_k = Ax_{k-1} + Bu_{k-1} + Ew_{k-1}, \tag{10}$$
$$y_k = Cx_k + Fv_k, \tag{11}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $E \in \mathbb{R}^{n\times n_w}$, $C \in \mathbb{R}^{m\times n}$, and $F \in \mathbb{R}^{m\times n_v}$ are constant matrices, $u_{k-1} \in \mathbb{R}^p$ is the input vector, $y_k \in \mathbb{R}^m$ is the output vector, and $x_k \in \mathbb{R}^n$ is the state vector to be estimated. We assume that the input and output vectors, as well as the constant matrices, are known. In turn, $w_{k-1} \in \mathbb{R}^{n_w}$ and $v_k \in \mathbb{R}^{n_v}$ are the process and measurement noise terms, respectively. We assume that the noise terms w_{k-1} and v_k are characterized by the WSS SPs $\{W_k\}$ and $\{V_k\}$ with autocorrelation matrices $Q \succeq 0_{n_w \times n_w}$ and $R \succ 0_{n_v \times n_v}$, respectively. Also, the initial state x_0 is characterized by the GRV $X_0 \sim \mathcal{N}(\hat{x}_0, P_0^{\mathrm{xx}})$, such that X_0, W_{k-1}, and V_k are uncorrelated.

The main idea of observers is to correct the state forecast $\hat{x}_{k|k-1}$, given by the process model (10), based on measurement error $\left(y_k - \hat{y}_{k|k-1}\right)$, where y_k is the measurement and $\hat{y}_{k|k-1}$ is given by the measurement model (11). Such correction is weighted by a gain matrix K_k. To obtain the state estimates, the unknown noise terms w_{k-1} and v_k are replaced by their uncertainty representation. The KF aims to provide the Kalman gain K_k that minimizes $\operatorname{tr}(P_k^{\mathrm{xx}})$. The $\mathrm{OH_2F}$ aims at providing the constant design matrices T, N, and K (to be defined) such that the steady-state variance of the state estimation error is minimized. Each state estimator provides *a posteriori* mean $\hat{x}_k$ and covariance P_k^{xx} estimates. The GRV $X_k \sim \mathcal{N}(0_{n\times 1}, P_k^{\mathrm{xx}})$ characterizes the estimation error $e_k \triangleq (x_k - \hat{x}_k)$, where x_k is the true state.

For the KP, the following observer is considered (Kalman, 1960):

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1} + Bu_{k-1}, \tag{12}$$

$$\hat{x}_k = \hat{x}_{k|k-1} + K_{k-1}\left(y_{k-1} - C\hat{x}_{k-1}\right), \tag{13}$$

whose estimation error is given by

$$e_k = Ae_{k-1} + Ew_{k-1} - K_{k-1}\left(Ce_{k-1} + Fv_{k-1}\right). \tag{14}$$

For the KF, the following observer is considered (Kalman, 1960):

$$\hat{x}_k = \hat{x}_{k|k-1} + K_k\left(y_k - C\hat{x}_{k|k-1}\right), \tag{15}$$

whose estimation error is given by

$$\begin{aligned} e_k &= Ae_{k-1} + Ew_{k-1} \\ &\quad - K_k\left(C\left(Ae_{k-1} + Ew_{k-1}\right) + Fv_k\right). \end{aligned} \tag{16}$$

The $\mathrm{OH_2F}$ structure is formulated by further steps as in Tang et al. (2019b). Consider a full-rank matrix $[T\ N]$ such that

$$T + NC = \mathrm{I}_{n\times n}, \tag{17}$$

where $T \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times m}$.

Combining (10) and (11) with (17), we can rewrite the process model (10) as follows:

$$\begin{aligned} x_k &= (T + NC)x_k \\ &= Tx_k + N\left(y_k - Fv_k\right) \\ &= TAx_{k-1} + TBu_{k-1} + TEw_{k-1} - NFv_k + Ny_k. \end{aligned} \tag{18}$$

Applying the usual observer idea to (18) and (11), we obtain the following observer:

$$\hat{x}_k = TA\hat{x}_{k-1} + TBu_{k-1} + K\left(y_{k-1} - C\hat{x}_{k-1}\right) + Ny_k, \tag{19}$$

whose estimation error is given by

$$e_k = (TA - KC)e_{k-1} + TEw_{k-1} - KFv_{k-1} - NFv_k \tag{20}$$

and $K \in \mathbb{R}^{n\times m}$ is defined as the constant observer gain matrix. Note that both past and current measurements are incorporated, in order to combine all acquired data.

For the special case in which $T = \mathrm{I}_{n\times n}$ and $N = 0_{n\times m}$, the $\mathrm{OH_2F}$ is reduced to the $\mathrm{H_2P}$. This fixed-gain predictor is equivalently formulated from (13) at steady state, that is, it represents the stationary gain KP. Analogously, the $\mathrm{H_2F}$ is a fixed-gain filter formulated from (15) at steady state. We highlight that the $\mathrm{H_2F}$ is not a special case of the $\mathrm{OH_2F}$. However, as the $\mathrm{OH_2F}$ is also a filter, and it offers additional degrees of freedom to yield the design matrices T, N, and K, we may reach a smaller $\mathrm{H_2}$ norm.

4. MINIMUM-VARIANCE ESTIMATORS

In this section, both KP and KF algorithms are reviewed. Without loss of generality, the Kalman estimators can be derived for correlated SPs with non-null mean. Here, we consider SPs $\{W_k\}$ and $\{V_k\}$ whose random variables W_k and V_k are uncorrelated and with zero mean.

4.1 The KP for Discrete-Time Linear Systems

Retake the error parcel (14). The term e_k is characterized by the GRV $X_k \sim \mathcal{N}(0_{n\times 1}, P_k^{\mathrm{xx}})$ as

$$X_k = \left(A - K_{k-1}C\right)X_{k-1} + EW_{k-1} - K_{k-1}FV_{k-1}. \tag{21}$$

Applying operations (1)-(2) to X_k, we obtain

$$\begin{aligned} P_k^{\mathrm{xx}} &= \left(A - K_{k-1}C\right)P_{k-1}^{\mathrm{xx}}\left(A - K_{k-1}C\right)^{\mathrm{T}} + EQE^{\mathrm{T}} \\ &\quad + K_{k-1}FRF^{\mathrm{T}}K_{k-1}^{\mathrm{T}}. \end{aligned} \tag{22}$$

To find the Kalman gain K_{k-1}, we solve

$$\partial \operatorname{tr}\left(P_k^{\mathrm{xx}}\left(K_{k-1}\right)\right)/\partial K_{k-1} = 0.$$

In so doing, we obtain

$$K_{k-1} = AP_{k-1}^{\mathrm{xx}}C^{\mathrm{T}}\left(CP_{k-1}^{\mathrm{xx}}C^{\mathrm{T}} + FRF^{\mathrm{T}}\right)^{-1}. \tag{23}$$

Next, we present the KP algorithm to estimate $\hat{x}_k$ and X_k, which characterize the state vector x_k.

Algorithm 1. (Kalman, 1960) The KP algorithm is summarized as

$$\begin{aligned} [\hat{x}_k, P_k^{\mathrm{xx}}] = \mathrm{KP}\Big(&\hat{x}_{k-1}, P_{k-1}^{\mathrm{xx}}, A, B, E, \\ &Q, C, F, R, u_{k-1}, y_{k-1}\Big). \end{aligned} \tag{24}$$

1: Compute the Kalman gain K_{k-1} (23).
2: Compute the *a posteriori* estimates $\hat{x}_k$ (13) and P_k^{xx} (22). □

4.2 The KF for Discrete-Time Linear Systems

Retake the error parcel (16). The term e_k is characterized by the GRV $X_k \sim \mathcal{N}(0_{n\times 1}, P_k^{\mathrm{xx}})$ as

$$X_k = \left(\mathrm{I}_{n\times n} - K_kC\right)\left(AX_{k-1} + EW_{k-1}\right) - K_kFV_k. \tag{25}$$

Applying operations (1)-(2) to X_k, we obtain

$$\begin{aligned} P_k^{\mathrm{xx}} &= \left(\mathrm{I}_{n\times n} - K_kC\right)\left(AP_{k-1}^{\mathrm{xx}}A^{\mathrm{T}} + EQE^{\mathrm{T}}\right) \\ &\quad \times \left(\mathrm{I}_{n\times n} - K_kC\right)^{\mathrm{T}} + K_kFRF^{\mathrm{T}}K_k^{\mathrm{T}}. \end{aligned} \tag{26}$$

To find the Kalman gain K_k, we carry out

$$\partial \operatorname{tr}\left(P_k^{\mathrm{xx}}\left(K_k\right)\right)/\partial K_k = 0,$$

obtaining

$$\begin{aligned} K_k &= \left(AP_{k-1}^{\mathrm{xx}}A^{\mathrm{T}} + EQE^{\mathrm{T}}\right)C^{\mathrm{T}} \\ &\quad \times \left(C\left(AP_{k-1}^{\mathrm{xx}}A^{\mathrm{T}} + EQE^{\mathrm{T}}\right)C^{\mathrm{T}} + FRF^{\mathrm{T}}\right)^{-1}. \end{aligned} \tag{27}$$

Next, we present the KF algorithm to estimate $\hat{x}_k$ and X_k, which characterize the state vector x_k.

Algorithm 2. (Kalman, 1960) The KF algorithm is summarized as

$$[\hat{x}_k, P_k^{\mathrm{xx}}] = \mathrm{KF}\Big(\hat{x}_{k-1}, P_{k-1}^{\mathrm{xx}}, A, B, E, \\ Q, C, F, R, u_{k-1}, y_k\Big). \tag{28}$$

1: Compute the Kalman gain K_k (27).
2: Compute the *a posteriori* estimates $\hat{x}_k$ (15) and P_k^{xx} (26). □

5. STEADY-STATE MINIMUM-VARIANCE ESTIMATORS

To derive the $\mathrm{OH_2F}$, the SPs are assumed to be white noise. This is necessary to use Lemma 2. In order to cover the cases in which the noise is colored, we augment the state vector x_k with some noise terms, such that the remaining noise is seen as white noise.

5.1 *The* $\mathrm{OH_2F}$ *for Discrete-Time Linear Systems*

Consider the error e_k given by (20). The term e_k is characterized by the GRV $X_k \sim \mathcal{N}(0_{n\times 1}, P_k^{\mathrm{xx}})$ as

$$X_k = (TA - KC)\,X_{k-1} + TEW - KFV - NFV. \tag{29}$$

Applying operations (1)-(2) to X_k, we obtain

$$P_k^{\mathrm{xx}} = (TA-KC)\,P_{k-1}^{\mathrm{xx}}\,(TA-KC)^{\mathrm{T}} + TEQ(TE)^{\mathrm{T}} \\ + KFR(KF)^{\mathrm{T}} + NFR(NF)^{\mathrm{T}}. \tag{30}$$

By convenience, we rewrite (20) as

$$e_k = \tilde{A}e_{k-1} + \tilde{B}d_{k-1}, \tag{31}$$

where

$$\tilde{A} = TA - KC, \tag{32}$$

$$\tilde{B} = [TE \;\; -KF \;\; -NF], \tag{33}$$

$$d_{k-1} = \left[w_{k-1}^{\mathrm{T}} \;\; v_{k-1}^{\mathrm{T}} \;\; v_k^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{34}$$

In order to compute the design matrices T, N, and K, we formulate a constrained optimization problem from Lemma 1. Since the matrices T and N are related by (17), we use Lemma 3 to make explicit such relation in terms of $S \in \mathbb{R}^{n\times(n+m)}$. Thereby, we note two nonlinear products involving PS and PK, then we define the linearizing transformations $Y_1 = PS$ and $Y_2 = PK$. As $\mathrm{tr}\left(\mathcal{B}^{\mathrm{T}}P\mathcal{B}\right)$ in Lemma 1 yields the nonlinear products $S^{\mathrm{T}}PS$ and $K^{\mathrm{T}}PK$, we define the upper bounds

$$S^{\mathrm{T}}PS = Y_1^{\mathrm{T}}P^{-1}Y_1 \prec Y_3, \tag{35}$$

$$K^{\mathrm{T}}PK = Y_2^{\mathrm{T}}P^{-1}Y_2 \prec Y_4. \tag{36}$$

The prior procedures transform the original optimization problem in a constrained optimization problem based on LMIs, whose outputs are the desired matrices T, N, and K. This result is presented in the following theorem.

Theorem 1. Consider a parameter $\psi > 0$ and matrices $P \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{n\times(n+m)}$, $Y_2 \in \mathbb{R}^{n\times m}$, $Y_3 \in \mathbb{R}^{(n+m)\times(n+m)}$, and $Y_4 \in \mathbb{R}^{m\times m}$, where $P = P^{\mathrm{T}} \succ 0_{n\times n}$, $Y_3 = Y_3^{\mathrm{T}} \succ 0_{(n+m)\times(n+m)}$, and $Y_4 = Y_4^{\mathrm{T}} \succ 0_{m\times m}$. Given the LMIs

$$\mathrm{tr}\,(\Omega_1) + \mathrm{tr}\,(\Omega_2) + \mathrm{tr}\left(F^{\mathrm{T}}Y_4F\right) < \psi, \tag{37}$$

$$\begin{bmatrix} \mathrm{I}_{n\times n} - P & \Omega_3^{\mathrm{T}} \\ \Omega_3 & -P \end{bmatrix} \prec 0_{2n\times 2n}, \tag{38}$$

$$\begin{bmatrix} -Y_3 & Y_1^{\mathrm{T}} \\ Y_1 & -P \end{bmatrix} \prec 0_{(2n+m)\times(2n+m)}, \tag{39}$$

$$\begin{bmatrix} -Y_4 & Y_2^{\mathrm{T}} \\ Y_2 & -P \end{bmatrix} \prec 0_{(n+m)\times(n+m)}, \tag{40}$$

where

$$\Omega_1 \triangleq E^{\mathrm{T}}\alpha_1^{\mathrm{T}}(\Theta^{\dagger})^{\mathrm{T}}\left(P\Theta^{\dagger}\alpha_1 + Y_1\Psi\alpha_1\right)E \\ + E^{\mathrm{T}}\alpha_1^{\mathrm{T}}\Psi^{\mathrm{T}}\left(Y_1^{\mathrm{T}}\Theta^{\dagger}\alpha_1 + Y_3\Psi\alpha_1\right)E, \tag{41}$$

$$\Omega_2 \triangleq F^{\mathrm{T}}\alpha_2^{\mathrm{T}}(\Theta^{\dagger})^{\mathrm{T}}\left(P\Theta^{\dagger}\alpha_2 + Y_1\Psi\alpha_2\right)F \\ + F^{\mathrm{T}}\alpha_2^{\mathrm{T}}\Psi^{\mathrm{T}}\left(Y_1^{\mathrm{T}}\Theta^{\dagger}\alpha_2 + Y_3\Psi\alpha_2\right)F, \tag{42}$$

$$\Omega_3 \triangleq P\Theta^{\dagger}\alpha_1 A + Y_1\Psi\alpha_1 A - Y_2 C, \tag{43}$$

$$\Theta \triangleq \begin{bmatrix} \mathrm{I}_{n\times n} \\ C \end{bmatrix},\; \alpha_1 \triangleq \begin{bmatrix} \mathrm{I}_{n\times n} \\ 0_{m\times n} \end{bmatrix},\; \alpha_2 \triangleq \begin{bmatrix} 0_{n\times m} \\ \mathrm{I}_{m\times m} \end{bmatrix}, \tag{44}$$

$$\Psi \triangleq \mathrm{I}_{(n+m)\times(n+m)} - \Theta\Theta^{\dagger}, \tag{45}$$

by solving the constrained optimization problem

$$\begin{array}{ll} \min & \psi \\ \text{s.t.} & (37)\text{–}(40), \end{array} \tag{46}$$

the matrices T, N, and K are obtained from

$$T = \Theta^{\dagger}\alpha_1 + P^{-1}Y_1\Psi\alpha_1, \tag{47}$$

$$N = \Theta^{\dagger}\alpha_2 + P^{-1}Y_1\Psi\alpha_2, \tag{48}$$

$$K = P^{-1}Y_2. \tag{49}$$

□

Proof. First, we relate (31) to Lemma 1. Doing $\mathcal{A} = \tilde{A}$, $\mathcal{B} = \tilde{B}$, and $\mathcal{C} = \mathrm{I}_{n\times n}$, we obtain

$$\tilde{B}^{\mathrm{T}}P\tilde{B} = \\ \begin{bmatrix} E^{\mathrm{T}}T^{\mathrm{T}}PTE & -E^{\mathrm{T}}T^{\mathrm{T}}PKF & -E^{\mathrm{T}}T^{\mathrm{T}}PNF \\ -F^{\mathrm{T}}K^{\mathrm{T}}PTE & F^{\mathrm{T}}K^{\mathrm{T}}PKF & F^{\mathrm{T}}K^{\mathrm{T}}PNF \\ -F^{\mathrm{T}}N^{\mathrm{T}}PTE & F^{\mathrm{T}}N^{\mathrm{T}}PKF & F^{\mathrm{T}}N^{\mathrm{T}}PNF \end{bmatrix}, \tag{50}$$

$$(TA-KC)^{\mathrm{T}}P\,(TA-KC) - P + \mathrm{I}_{n\times n} \prec 0_{n\times n}. \tag{51}$$

Next, Schur's complement is applied to obtain

$$\begin{bmatrix} \mathrm{I}_{n\times n} - P & (TA-KC)^{\mathrm{T}}P \\ P\,(TA-KC) & -P \end{bmatrix} \prec 0_{2n\times 2n}. \tag{52}$$

Now, we apply Lemma 3 to (17) to relate T and N by means of a same variable S. Doing $\mathcal{A} = [T\;N]$, $\mathcal{B} = [\mathrm{I}_{n\times n}\;C^{\mathrm{T}}]^{\mathrm{T}}$, and $\mathcal{C} = \mathrm{I}_{n\times n}$, we obtain

$$[T\;\;N] = \\ \begin{bmatrix} \mathrm{I}_{n\times n} \\ C \end{bmatrix}^{\dagger} + S\left(\mathrm{I}_{(n+m)\times(n+m)} - \begin{bmatrix} \mathrm{I}_{n\times n} \\ C \end{bmatrix}\begin{bmatrix} \mathrm{I}_{n\times n} \\ C \end{bmatrix}^{\dagger}\right) \\ = \Theta^{\dagger} + S\Psi. \tag{53}$$

From (53), the matrices T and N can be written as

$$T = \Theta^{\dagger}\alpha_1 + S\Psi\alpha_1, \tag{54}$$

$$N = \Theta^{\dagger}\alpha_2 + S\Psi\alpha_2, \tag{55}$$

where α_1 and α_2 are given by (44).

Given the linearizing transformations $Y_1 = PS$ and $Y_2 = PK$, (54) substituted in $P(TA - KC)$ yields (43). Then, (52) becomes the LMI (38). Now, retake the upper bounds Y_3 (35) and Y_4 (36). Applying Schur's complement, LMIs

(39) and (40) are obtained. Also, substituting (54) and (55) in (50) yields a block-diagonal matrix given by

$$H = \left\{\Omega_1,\ F^{\mathrm{T}}Y_4F,\ \Omega_2\right\}.$$

Take the trace of each matrix within H and bound their summation by ψ. By minimizing $\psi > 0$, the constrained optimization problem is finished. □

Since the constrained optimization problem (46) is solved only once, and it may be complex, we consider that it will be solved offline. Next, we present the $\mathrm{OH_2F}$ algorithm to estimate $\hat{x}_k$ and X_k, which characterize the state vector x_k.

Algorithm 3. The $\mathrm{OH_2F}$ algorithm is summarized as

$$[\hat{x}_k, P_k^{\mathrm{xx}}] = \mathrm{OH_2F}\Big(\hat{x}_{k-1}, P_{k-1}^{\mathrm{xx}}, A, B, E, Q, C, F, R, T, N, K, u_{k-1}, y_{k-1}, y_k\Big). \quad (56)$$

1: Compute the *a posteriori* estimates $\hat{x}_k$ (19) and P_k^{xx} (30). □

Remark 1. Here, the optimization problems based on LMIs are solved via semidefinite programming with the CVX toolbox (Grant and Boyd, 2014). □

5.2 *The* $\mathrm{H_2P}$ *for Discrete-Time Linear Systems*

According to the observer (19), consider that $T = \mathrm{I}_{n\times n}$ and $N = 0_{n\times m}$. Therefore, the $\mathrm{OH_2F}$ is reduced to the $\mathrm{H_2P}$, whose mean and error estimates are given by (13) and (14), respectively, with the constant gain K to be found. The $\mathrm{H_2P}$ has the advantage of computing the Kalman gain K only once. By relating the error system to Lemma 1, we have $\mathcal{A} = (A - KC)$, $\mathcal{B} = [E \ \ -KF]$, and $\mathcal{C} = \mathrm{I}_{n\times n}$. In so doing, the following result is derived.

Corollary 1. Consider a parameter $\psi > 0$ and matrices $P = P^{\mathrm{T}} \succ 0_{n\times n}$, $Y_2 \in \mathbb{R}^{n\times m}$, and $Y_4 = Y_4^{\mathrm{T}} \succ 0_{m\times m}$. Consider also the LMIs

$$\mathrm{tr}\left(E^{\mathrm{T}}PE\right) + \mathrm{tr}\left(F^{\mathrm{T}}Y_4F\right) < \psi, \quad (57)$$

$$\begin{bmatrix} -Y_4 & Y_2^{\mathrm{T}} \\ Y_2 & -P \end{bmatrix} \prec 0_{(n+m)\times(n+m)}, \quad (58)$$

$$\begin{bmatrix} \mathrm{I}_{n\times n} - P & A^{\mathrm{T}}P - C^{\mathrm{T}}Y_2^{\mathrm{T}} \\ PA - Y_2C & -P \end{bmatrix} \prec 0_{2n\times 2n}. \quad (59)$$

Bu solving the constrained optimization problem

$$\min \ \psi \quad \text{s.t.} \ (57)\text{–}(59), \quad (60)$$

we obtain the matrix $K = P^{-1}Y_2$. □

Proof. This proof is similar to the prior proof with $T = \mathrm{I}_{n\times n}$ and $N = 0_{n\times m}$. □

5.3 *The* $\mathrm{H_2F}$ *for Discrete-Time Linear Systems*

Let the instant k be enough large to assume the steady state. Therefore, the KF is reduced to the $\mathrm{H_2F}$, whose mean and error estimates are given by (15) and (16), respectively, with the constant gain K to be found. The $\mathrm{H_2F}$ has the advantage of computing the Kalman gain K only once. By relating the error system to Lemma 1, we have $\mathcal{A} = (A - KCA)$, $\mathcal{B} = [(E - KCE) \ \ -KF]$, and $\mathcal{C} = \mathrm{I}_{n\times n}$. In so doing, the following result is derived.

Corollary 2. Consider a parameter $\psi > 0$ and matrices $P = P^{\mathrm{T}} \succ 0_{n\times n}$, $Y_2 \in \mathbb{R}^{n\times m}$, and $Y_4 = Y_4^{\mathrm{T}} \succ 0_{m\times m}$. Consider also the LMIs

$$\mathrm{tr}\left(E^{\mathrm{T}}ME\right) + \mathrm{tr}\left(F^{\mathrm{T}}Y_4F\right) < \psi, \quad (61)$$

$$\begin{bmatrix} -Y_4 & Y_2^{\mathrm{T}} \\ Y_2 & -P \end{bmatrix} \prec 0_{(n+m)\times(n+m)}, \quad (62)$$

$$\begin{bmatrix} \mathrm{I}_{n\times n} - P & A^{\mathrm{T}}P - A^{\mathrm{T}}C^{\mathrm{T}}Y_2^{\mathrm{T}} \\ PA - Y_2CA & -P \end{bmatrix} \prec 0_{2n\times 2n}, \quad (63)$$

where

$$M = P - Y_2C - C^{\mathrm{T}}Y_2^{\mathrm{T}} + C^{\mathrm{T}}Y_4C. \quad (64)$$

By solving the constrained optimization problem

$$\min \ \psi \quad \text{s.t.} \ (61)\text{–}(63), \quad (65)$$

we obtain the matrix $K = P^{-1}Y_2$. □

Proof. This proof is similar to the prior proof with $T = \mathrm{I}_{n\times n}$ and $N = 0_{n\times m}$. □

6. NUMERICAL EXAMPLE

In this section, we use the five-state numerical example from Tang et al. (2019b). In order to compare the performance of the KP, KF, $\mathrm{H_2P}$, $\mathrm{H_2F}$, and $\mathrm{OH_2F}$, three indexes are employed, namely: (i) the *mean processing time* T^{CPU}; (ii) the *root mean square error* of the mean estimate of the j-th state (RMSE_j)

$$\mathrm{RMSE}_j = \frac{1}{\mathrm{MC}}\sum_{i=1}^{\mathrm{MC}}\sqrt{\frac{1}{\zeta}\sum_{k=1}^{\zeta}\left(x_{j,k} - \hat{x}_{j,k,i}\right)^2}, \quad (66)$$

for $j = 1, \ldots, n$, where n is the state vector dimension, ζ is the time horizon, and MC is the number of Monte Carlo simulations; and (iii) the *mean trace* (MT)

$$\mathrm{MT} = \frac{1}{\mathrm{MC}}\frac{1}{\zeta}\sum_{i=1}^{\mathrm{MC}}\sum_{k=1}^{\zeta}\mathrm{tr}\left(P_{k,i}^{\mathrm{xx}}\right). \quad (67)$$

The simulations are run with the parameters $\zeta = 60$ and $\mathrm{MC} = 1000$. However, to compute the indexes RMSE and MT, only the first 10 time steps ($\zeta = 10$) are used. This choice is made to compare all algorithms during the transient, where there exist significant differences among them. In (23) and (27), the trace of the *a posteriori* covariance matrix P_k^{xx} is minimized, at each time step, by the KP and KF, respectively; in (46), (60), and (65), the steady-state variance of the error e_k (31) is minimized by the $\mathrm{OH_2F}$, $\mathrm{H_2P}$, and $\mathrm{H_2F}$, respectively. Then, we expect that the KP and KF reach the smallest indexes RMSE and MT with respect to the predictors and filters, respectively, and that the $\mathrm{OH_2F}$ reaches the smallest MT with respect to the fixed-gain algorithms. The computer configuration is: 4 GB RAM, Windows 7 Ultimate, and Intel Core 2 Quad CPU Q6700 2.66 GHz.

6.1 *Problem Description and Setup*

Consider the system with the form (10)-(11), defined by the following parameters:

$$A = \begin{bmatrix} -0.54 & 0.45 & 0.36 & 0 & 0 \\ 0.63 & 0.45 & 0.18 & 0.36 & 0 \\ 0.09 & 0.45 & 0.27 & 0.09 & 0.18 \\ 0 & 0 & 0.25 & 0.25\sqrt{2} & -0.25\sqrt{2} \\ 0 & 0 & 0 & 0.25\sqrt{2} & 0.25\sqrt{2} \end{bmatrix}, \quad (68)$$

$$B = 0_{5\times1},\ E = [-1\ 0\ 0\ 0\ 1]^{\mathrm{T}}, \quad (69)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ F = 0.1 \times \mathrm{I}_{2\times2}. \quad (70)$$

It is simulated with initial state $x_0 = [4\ {-4}\ 4\ {-4}\ 4]^{\mathrm{T}}$. The noise realizations w_{k-1} and v_k take values from the GRVs $W \sim \mathcal{N}(0, 1)$ and $V \sim \mathcal{N}(0_{2\times1}, \mathrm{I}_{2\times2})$, respectively.

To estimate states, the tuning is set as $\hat{x}_0 = 0_{5\times1}$, $P_0^{\mathrm{xx}} = \frac{25}{\varsigma}\mathrm{I}_{5\times5}$, $Q = 1$, and $R = \mathrm{I}_{2\times2}$, where $\varsigma = 18.2051$ is the greatest realization of the chi-square variable with five degrees of freedom that implies 99.73% confidence level. The state estimates are initialized such that the initial interval contains the true states with 99.73% confidence level. After solving (46), we obtain the matrices

$$T = \begin{bmatrix} 0.0099 & 0 & 0 & 0.0005 & 0 \\ 0.0006 & 1 & 0 & -0.1588 & 0 \\ -0.0012 & 0 & 1 & 0.0036 & 0 \\ 0.0005 & 0 & 0 & 0.8691 & 0 \\ 0.9867 & 0 & 0 & 0.1384 & 1 \end{bmatrix}, \quad (71)$$

$$N = \begin{bmatrix} 0.9901 & -0.0005 \\ -0.0006 & 0.1588 \\ 0.0012 & -0.0036 \\ -0.0005 & 0.1309 \\ -0.9867 & -0.1384 \end{bmatrix}, \quad (72)$$

$$K^{\mathrm{OH_2F}} = \begin{bmatrix} -0.0026 & 0.0007 \\ 0.2854 & 0.0909 \\ -0.0435 & 0.0509 \\ 0.1523 & 0.0726 \\ -0.4161 & 0.0705 \end{bmatrix}, \quad (73)$$

with the minimal H_2 norm equal to 0.1755. Also, by solving (60) and (65), the corresponding matrices are found:

$$K^{\mathrm{H_2P}} = \begin{bmatrix} -0.5343 & 0.1241 \\ 0.6234 & 0.2068 \\ -0.0878 & 0.1011 \\ 0.3479 & 0.1655 \\ -0.3480 & -0.0045 \end{bmatrix}, \quad (74)$$

$$K^{\mathrm{H_2F}} = \begin{bmatrix} 0.9901 & -0.0009 \\ -0.0010 & 0.2793 \\ 0.0023 & -0.0058 \\ -0.0009 & 0.2298 \\ -0.9834 & -0.2424 \end{bmatrix}, \quad (75)$$

whose minimal H_2 norm is equal to 1.4208 and 0.2015, respectively.

6.2 State Estimation

In Table 1 and Figure 1, the computed indexes are presented. First, note that the T^{CPU} of the Kalman estimators is larger than the T^{CPU} of the fixed-gain algorithms. It occurs because both KP and KF compute a matrix inversion at each time step to obtain the Kalman gain, while the other algorithms do not. This is the main advantage of the fixed-gain algorithms. Second, we differ the indexes RMSE and MT. On one hand, the RMSE measures *exactness*, since the true state x_k is compared to the mean estimate $\hat{x}_k$. On the other hand, the MT measures *precision*, since it is related to the uncertainty region spreading, that is, the covariance matrices P_k^{xx}. According to the indexes RMSE and MT, the filters are more precise and exact than the predictors. This is always true, since the filters use both past and current measurements.

Table 1. Numerical results from the state estimation of the system (68)-(70).

Estimators	$T^{\mathrm{CPU}} \times 10^{-5}$(s)	MT
KP	7.5472	2.7581
$\mathrm{H_2P}$	4.3301	2.9439
KF	6.7300	1.0844
$\mathrm{H_2F}$	4.6705	1.4327
$\mathrm{OH_2F}$	6.1300	1.2407

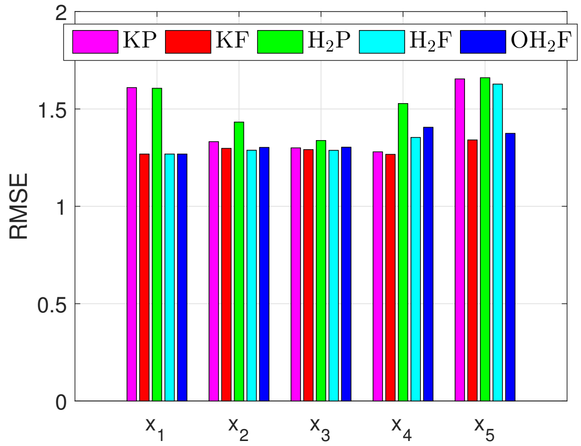

Figure 1. Index RMSE for each estimated state of the system (68)-(70).

Regarding the fixed-gain filters, the $\mathrm{OH_2F}$ returns an H_2 norm smaller than the $\mathrm{H_2F}$. As the H_2 norm is a variance measure, and it influences the precision of the algorithm, the $\mathrm{OH_2F}$ reaches a smaller MT as illustrated in Table 1. About the index RMSE, no significant difference is noted, because some states get worse while others get better.

7. CONCLUSION

This paper proposed a novel state observer based on H_2 norm, called $\mathrm{OH_2F}$. Its novelty is to design a fixed-gain filter based on the steady-state variance. For LTI systems, its observer structure is more general than the KP one. This paper also revisited the regular KP and KF algorithms to highlight similarities and differences to the $\mathrm{OH_2F}$. In terms of numerical comparisons involving both transient and steady-state parcels, no significant difference is noted between Kalman estimators and fixed-gain algorithms. This is expected because the varying gain algorithms also reach the steady state. Therefore, the idea was to compute the performance indexes RMSE and MT during the transient only. In so doing, we showed that the $\mathrm{OH_2F}$ can yield results more precise than the $\mathrm{H_2F}$, when its H_2 norm is smaller. Moreover, we showed that the $\mathrm{H_2P}$ is a special case of the $\mathrm{OH_2F}$, and that it corresponds to the KP at steady state. The proposed algorithm is indicated to applications where both low processing time and good accuracy are requirements to make feasible the usage of state estimators at real time, such as unmanned aerial vehicles that flight in hovering state.

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