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Robust $\mathcal{D}$-stability via discrete controllers for continuous-time uncertain systems with multiple delays ⋆

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Abstract: This work addresses the allocation of closed-loop poles of a discretized system from a continuous-time one with multiple input delays, aiming at its control through a computer. In order to handle a practical challenge presented in Network Control System (NCS) approaches, uncertain sampling period, distinct input time delays and parametric uncertainties in polytopic form can be propagated from the original state space representation to the discretized state model. The resulting discrete-time time-delay system has a very specific feature, so that it can be converted into an augmented linear system without time-delay. In this context, the main contribution of the present paper consists of a Linear Matrix Inequality (LMI) based control synthesis condition composed of homogeneous polynomial matrices of arbitrary degree, which ensures the continuous-time system stability and simultaneously the allocation of the closed-loop poles of the augmented system in a $\mathcal{D}$-stable region. Numerical simulations illustrate the exposed.

Keywords: robust control; multi-input system; $\mathcal{D}$-stability; discretized linear systems; linear matrix inequalities.

1. INTRODUCTION

An Network Control System (NCS) is a control system in which plants, sensors, controllers, and actuators are connected through communication networks. In design of NCS, one considers important issues, e.g., uncertain sampling period, network-induced delay, communication constraints, and so on. In this context, this paper addresses the modelling of an NCS with different time-delays in each individual input channel, such that a discrete-time system with multiple input delays is transformed into a delay free one by using state augmentation techniques (Lian et al. (2003), Zhou et al. (2010)). In order to handle a practical challenge presented in NCS approaches, in addition to distinct multiple input time delays an uncertain sampling period is also considered.

In the context of multiple input delays, it is worth noting that most research efforts have been devoted to design of continuous-time feedback controllers. To the author's knowledge, there exists a lack of methods to design a digital controller that assures the stability of the closed-loop hybrid system (continuous-time plant and digital controller).

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This work proposes a condition based on a set of Linear Matrix Inequalities (LMIs) defined in terms of homogeneous polynomial parameter-dependent matrices of arbitrary degree (Oliveira and Peres (2007)) and the main aim is to guarantee the stability of the continuous-time plant with multiple input delays and closed-loop poles allocation of the discretized system in a $\mathcal{D}$-stable region.

In order to handle performance requirements, this work uses a cardioid (region formed by the geometric locus with the same damping ratio) approximation, defined as a disc region $\mathcal{D}(\delta, \varrho)$, centered in $\delta + j0$, with radius ϱ, where $|\delta| + \varrho < 1$ (Furuta and Kim (1987)). In addition, both δ and ϱ were parameterized from a given region in the left half-plane (Leandro and Kienitz (2019)).

Polytopic uncertainties are considered, and additional challenges are imposed to circumvent the difficulty of dealing with exponential of matrices in the presence of polytopic algebraic structures, namely in cases where sampling period is not sufficiently small so that the quadratic and higher-order terms in the Taylor series expansion can't be neglected in the uncertainty representation. Therefore, this work extends the results present by Braga et al. (2014), who addressed the problem of uncertain sampling discretization of uncertain continuous-time systems without considering multiple input delays and closed-loop poles allocation to handle performance requirements.

The remainder of this paper is organized as follows. Section 2 introduces definitions and preliminary lemmas. Section 3 brings definitions for a systematic discretization through Taylor series expansion. Section 4 describes the state feedback control design condition for augmented model derived from that proposed in Braga et al. (2014). Section 5 presents numerical examples. Finally, concluding remarks are shown in Section 6.

1.1 Notation

$\mathbb{N}^*$: denotes the set of non-zero natural numbers. $I(0)$: identity (null) matrix of appropriate dimension; diag: a diagonal matrix of appropriate dimension; M^T : matrix M transpose; $He\{M\}$: denotes $M^\mathsf{T}+M$; $\star$: denotes the elements or symmetrical blocks with respect to the diagonal of a symmetric matrix; T : denotes the sampling period.

2. PRELIMINARIES

Consider the following continuous-time uncertain system with distinct multiple input delays:

$$\dot{x}(t) = E(\alpha_1)x(t) + \sum_{i=1}^{r} F_i(\alpha_1)u_i(t-\tau_i), \tag{1}$$

where $E(\alpha_1) \in \mathbb{R}^{n_x \times n_x}$, $F_i(\alpha_1) \in \mathbb{R}^{n_x \times n_u}$ and the ith control channel $u_i(t) \in \mathbb{R}^{n_u \times 1}$ is delayed by $\tau_i > 0$, $i \in \{1, \cdots, r\}$. Suppose that matrices $E(\alpha_1)$ and $F_i(\alpha_1)$ belong to polytope Ω with N vertices defined by:

$$\Omega = \left\{ (E,F)(\alpha_1) | (E,F)(\alpha_1) = \sum_{j=1}^{N} \alpha_{1j}(E,F)_j \right\}, \tag{2}$$

where $\alpha_1 = (\alpha_{11}, \ldots, \alpha_{1N})$ is a time-invariant parameter vector, taking values in a unit simplex Λ_N:

$$\Lambda_N = \left\{ \zeta \in \mathbb{R}^N | \sum_{j=1}^{N} \zeta_j = 1, \zeta_j \geq 0, \forall j \in \{1, ..., N\} \right\}. \tag{3}$$

Moreover, consider the delayed input signal to be sampled with uncertain sampling period, such that $T(\alpha_2)$ lies inside the interval $[T_{min}, T_{max}]$ and can be written as a convex combination of $N = 2$ vertices:

$$T(\alpha_2) = \sum_{i=1}^{2} \alpha_{2i}T_i, \; \alpha_2 \in \Lambda_2, \tag{4}$$

such that Λ_2 is defined in (3). For simplicity, the multiple input delays τ_i, $i \in \{1, \cdots, r\}$, are supposed to be constant and known, in accordance with the exposed by Braga et al. (2014) and references. In this context, consider an uncertain parameter vector, where $\alpha = (\alpha_1, \alpha_2) \in (\Lambda_{N_1} \times \Lambda_{N_2})$, $N_2 = 2$, which is a so-called multi-simplex domain (see Oliveira et al. (2008)).

Assumption 1. For the multiple network-induced delay τ_i, $i \in \{1, \cdots, r\}$, and $T(\alpha_2) \in [T_{\min}, T_{\max}]$, the following relation holds:

$$0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_r \leq T_{min} \leq T_{max}. \tag{5}$$

Considering a zero order hold, the value of state $x(t)$ at sampling instants $\mathrm{k}T(\alpha_2), \mathrm{k} = 1, 2, \ldots,$ is given by:

$$x(\mathrm{k}{+}1) = A(\alpha)x(\mathrm{k}) + \sum_{i=1}^{r} \{B_{d_i}(\alpha)u_i(\mathrm{k}-1) + B_i(\alpha)u_i(\mathrm{k})\}, \tag{6}$$

where instant $\mathrm{k}T(\alpha_2)$ is denoted by k for simplicity, and the uncertain parameter-dependent matrices $A(\alpha)$, $B_i(\alpha)$ and $B_{d_i}(\alpha)$ are:

$$\begin{aligned} &A(\alpha) = e^{E(\alpha_1)T(\alpha_2)}, \\ &B_{d_i}(\alpha) = e^{E(\alpha_1)(T(\alpha_2)-\tau_i)} \left(\int_0^{\tau_i} e^{E(\alpha_1)\varsigma} d\varsigma \right) F_i(\alpha_1), \\ &B_i(\alpha) = \left(\int_0^{T(\alpha_2)-\tau_i} e^{E(\alpha_1)\varsigma} d\varsigma \right) F_i(\alpha_1), \; i = \{1, \cdots, r\}. \end{aligned} \tag{7}$$

The aim of this work is to propose a condition based on a set of LMIs defined in terms of homogeneous polynomial parameter-dependent matrices of arbitrary degree, which allows to guarantee the stability of (1) and the closed-loop pole allocation in (6). The following lemmas were used in the proof of the proposed condition.

Lemma 1. (Gahinet and Apkarian, 1994). Given a matrix $H = H^\mathsf{T} \in \mathbb{R}^{n \times n}$, and two matrices V and U of column dimension n, consider the problem of finding some matrix X of compatible dimensions such that:

$$H + V^\mathsf{T}X^\mathsf{T}U + U^\mathsf{T}XV < 0. \tag{8}$$

Denote by N_U and N_V any matrices whose columns form bases of the null spaces of U and V, respectively. Then (8) is feasible for X if and only if

$$N_V^\mathsf{T}HN_V < 0 \text{ and } N_U^\mathsf{T}HN_U < 0. \tag{9}$$

In the context of this work, Lemma 1 is used such that the LMIs in (9) certify the existence of a solution of (8). This strategy will be used to demonstrate Theorem 3, presented in section 4.

Lemma 2. (as cited by Boyd et al. (1994)). Given a scalar $\lambda > 0$ and any real matrices $\mathcal{O}$ and $\mathcal{U}$ of compatible dimensions, then

$$\mathcal{O}\mathcal{U} + \mathcal{U}^\mathsf{T}\mathcal{O}^\mathsf{T} \leq \lambda\mathcal{O}\mathcal{O}^\mathsf{T} + \lambda^{-1}\mathcal{U}^\mathsf{T}\mathcal{U}. \tag{10}$$

3. DISCRETIZATION OF UNCERTAIN SYSTEMS WITH UNCERTAIN SAMPLING PERIOD AND MULTIPLE INPUT DELAYS

The parameter-dependent matrices $A(\alpha)$, $B_i(\alpha)$ and $B_{d_i}(\alpha)$ described in (7), can be rewritten in terms of homogeneous polynomial matrices $A^{[g]}(\alpha)$, $B_i^{[g]}(\alpha)$ and $B_{d_i}^{[g]}(\alpha)$, formed by a Taylor series expansion of degree $g \in \mathbb{N}^* \times \mathbb{N}^*$ and the residual discretization error $\Delta A^{[g]}(\alpha)$, $\Delta B_i^{[g]}(\alpha)$ and $\Delta B_{d_i}^{[g]}(\alpha)$. Such reasoning was based on main ideas presented in Braga et al. (2014), as follows:

$$\begin{aligned} A(\alpha) = A^{[g]}(\alpha) + \Delta A^{[g]}(\alpha), B_i(\alpha) = B_i^{[g]}(\alpha) + \Delta B_i^{[g]}(\alpha), \\ B_{d_i}(\alpha) = B_{d_i}^{[g]}(\alpha) + \Delta B_{d_i}^{[g]}(\alpha). \end{aligned} \tag{11}$$

The homogeneous polynomials are

$$A^{[g]}(\alpha) = \sum_{n=0}^{g} \prod_{j=1}^{2} \left(\sum_{z=1}^{N_j} \alpha_{jz} \right)^{g-n} \frac{E(\alpha_1)^n}{n!} T(\alpha_2)^n,$$

$$B_i^{[g]}(\alpha) = \sum_{n=1}^{g} \prod_{j=1}^{2} \left(\sum_{z=1}^{N_j} \alpha_{jz} \right)^{g-n} \frac{E(\alpha_1)^{n-1}}{n!} \times \psi_i(\alpha_2)^n F_i(\alpha_1),$$

$$B_{d_i}^{[g]}(\alpha) = \sum_{n=0}^{g} \sum_{s=1}^{g} \left(\sum_{z=1}^{N_1} \alpha_{1z} \right)^{2g-s-n} \left(\sum_{j=1}^{2} \alpha_{2j} \right)^{g-n} \times \frac{\tau_i^s}{s!} \frac{\psi_i(\alpha_2)^n}{n!} E(\alpha_1)^{n+s-1} F_i(\alpha_1), \tag{12}$$

where $\psi_i(\alpha_2) = T(\alpha_2) - \tau_i$, $i \in \{1, \cdots, r\}$. The residual discretization errors are

$$\Delta A^{[g]}(\alpha) = e^{E(\alpha_1)T(\alpha_2)} - A^{[g]}(\alpha),$$
$$\Delta B_i^{[g]}(\alpha) = \left(\int_0^{\psi_i(\alpha_2)} e^{E(\alpha_1)\varsigma} d\varsigma \right) F_i(\alpha_1) - B_i^{[g]}(\alpha),$$
$$\Delta B_{d_i}^{[g]}(\alpha) = e^{E(\alpha_1)\psi_i(\alpha_2)} \left(\int_0^{\tau_i} e^{E(\alpha_1)\varsigma} d\varsigma \right) F_i(\alpha_1) - B_{d_i}^{[g]}(\alpha). \tag{13}$$

Considering that the matrix product is non commutative, one can note the need for a notation that allows for a convenient generalization to represent the matrix coefficients originated by such representation. Definitions that allow for a systematic representation of Taylor series terms can be found in Appendices A and B.

Bounds on the discretization errors described in (13) are defined as

$$\theta_A \triangleq \sup_{\alpha \in \Lambda_{N_1} \times \Lambda_2} \left\| \Delta A^{[g]}(\alpha) \right\|_2, \theta_{B_i} \triangleq \sup_{\alpha \in \Lambda_{N_1} \times \Lambda_2} \left\| \Delta B_i^{[g]}(\alpha) \right\|_2,$$
$$\theta_{B_{d_i}} \triangleq \sup_{\alpha \in \Lambda_{N_1} \times \Lambda_2} \left\| \Delta B_{d_i}^{[g]}(\alpha) \right\|_2, i = \{1, \cdots, r\}, \tag{14}$$

where $\|.\|_2$ represents the $2-$norm, sup is the supremum and

$$\bar{\alpha} \triangleq \arg \sup_{\alpha \in \Lambda_{N_1} \times \Lambda_2} \|.\|_2, \tag{15}$$

which is defined for each one of the norms in (14). An approximation for $(\theta_A, \theta_{B_i}, \theta_{B_{d_i}})$ can be obtained by a $(N_1 + 2)-$dimensional off line search in a grid of values of $\alpha \in (\Lambda_{N_1} \times \Lambda_2)$.

4. STABILIZATION

In this section, a new design condition is proposed to allocate closed-loop system poles inside a desired disc region in the z-plane.

Assumption 2. Consider $F(\alpha_1) = [\, F_1(\alpha_1), \cdots, F_r(\alpha_1)\,] \in \mathbb{R}^{n_x \times rn_u}$, assume that the pair $(E(\alpha_1), F(\alpha_1))$ is stabilizable and there exists a matrix $K = \left[\, K_1{}^\mathsf{T}, \cdots, K_r{}^\mathsf{T} \,\right]^\mathsf{T} \in \mathbb{R}^{rn_u \times n_x}$, such that $(E(\alpha_1) + F(\alpha_1)K)$ is Hurwitz.

From (11), the discrete time model (6) can be rewritten as:

$$x(\mathrm{k}+1) = (A^{[g]}(\alpha) + \Delta A^{[g]}(\alpha))x(\mathrm{k}) + \sum_{i=1}^{r} \Big\{ (B_{d_i}^{[g]}(\alpha) + \Delta B_{d_i}^{[g]}(\alpha))u_i(\mathrm{k}-1) + (B_i^{[g]}(\alpha) + \Delta B_i^{[g]}(\alpha))u_i(\mathrm{k}) \Big\}, \tag{16}$$

which can then be recast into the following augmented form:

$$z(\mathrm{k}+1) = (\hat{A}^{[g]}(\alpha) + \Delta\hat{A}^{[g]}(\alpha))z(\mathrm{k}) + (\hat{B}^{[g]}(\alpha) + \Delta\hat{B}^{[g]}(\alpha))u(\mathrm{k}), \tag{17}$$

where:

$$z(\mathrm{k}) = \begin{bmatrix} x(\mathrm{k}) \\ u_1(\mathrm{k}-1) \\ \vdots \\ u_r(\mathrm{k}-1) \end{bmatrix}, \hat{A}^{[g]}(\alpha) = \begin{bmatrix} A^{[g]}(\alpha) & B_d^{[g]}(\alpha) \\ 0 & 0 \end{bmatrix},$$
$$B_d^{[g]}(\alpha) = [\, B_{d_1}^{[g]}(\alpha) \cdots B_{d_r}^{[g]}(\alpha)\,],$$
$$\Delta\hat{A}^{[g]}(\alpha) = \begin{bmatrix} \Delta A^{[g]}(\alpha) & \Delta B_d^{[g]}(\alpha) \\ 0 & 0 \end{bmatrix},$$
$$\Delta B_d^{[g]}(\alpha) = [\, \Delta B_{d_1}^{[g]}(\alpha) \cdots \Delta B_{d_r}^{[g]}(\alpha)\,],$$
$$\hat{B}^{[g]}(\alpha) = \begin{bmatrix} B^{[g]}(\alpha) \\ I \end{bmatrix}, B^{[g]}(\alpha) = \left[\, B_1^{[g]}(\alpha) \cdots B_r^{[g]}(\alpha) \,\right],$$
$$\Delta\hat{B}^{[g]}(\alpha) = \begin{bmatrix} \Delta B^{[g]}(\alpha) \\ 0 \end{bmatrix},$$
$$\Delta B^{[g]}(\alpha) = [\, \Delta B_1^{[g]}(\alpha) \cdots \Delta B_r^{[g]}(\alpha)\,]. \tag{18}$$

Therefore, the state feedback control law is given by:

$$u(\mathrm{k}) = Kz(\mathrm{k}) = \begin{bmatrix} K_{1x} & K_{1u_1} & \cdots & K_{1u_r} \\ \vdots & \vdots & \cdots & \vdots \\ K_{rx} & K_{ru_1} & \cdots & K_{ru_r} \end{bmatrix} \begin{bmatrix} x(\mathrm{k}) \\ u_1(\mathrm{k}-1) \\ \vdots \\ u_r(\mathrm{k}-1) \end{bmatrix}, \tag{19}$$

where $K \in \mathbb{R}^{rn_u \times (n_x + rn_u)}$.

In face of augmented system matrices, an estimate for the upper bounds of $\left\|\Delta\hat{A}^{[g]}(\alpha)\right\|_2$ and $\left\|\Delta\hat{B}^{[g]}(\alpha)\right\|_2$ in the same way as in (14) can be defined respectively as:

$$\hat{\theta}_A \triangleq \sup_{\alpha \in \Lambda_{N_1} \times \Lambda_2} \left\| \Delta\hat{A}^{[g]}(\alpha) \right\|_2, \hat{\theta}_B \triangleq \sup_{\alpha \in \Lambda_{N_1} \times \Lambda_2} \left\| \Delta\hat{B}^{[g]}(\alpha) \right\|_2. \tag{20}$$

In order to allocate closed-loop system poles in the z-plane, a disc region $\mathcal{D}(\delta, \varrho)$ was adopted as approximation to cardioid (characterized by a constant damping ratio locus). This region $\mathcal{D}(\delta, \varrho)$ is centered in $\delta + j0$, with radius ϱ, i.e., $\mathcal{D} = \left\{ z = (\chi + j\upsilon) | (\chi - \delta)^2 + \upsilon^2 < \varrho^2 \right\}$ (Chilali and Gahinet (1996)). The parametrization of δ and ϱ from a conic sector subregion in the s-plane closely follows that presented in Leandro and Kienitz (2019), such that:

$$\delta = e^{\left(\frac{-\phi}{\tan(\phi)}\right)} \cos(-\phi), \;\; \varrho = e^{\left(\frac{-\phi}{\tan(\phi)}\right)} \sin(-\phi), \tag{21}$$

where ϕ is the internal angle of the cone region in the left half-plane. The allocation of the closed-loop poles of (17) in such a region $\mathcal{D}(\delta, \varrho)$ guarantees a transient response limited by decay rate in the interval $[\delta - |\varrho| \,,\; \delta + |\varrho|]$.

The following theorem guarantees system poles of the augmented system in (17) inside the region $\mathcal{D}(\delta, \varrho)$ and the stabilization of the continuous time system with r input delays described in (1).

Theorem 3. Consider a positive definite symmetric matrices $W_{\Bbbk} \in \mathbb{R}^{(n_x+rn_u)\times(n_x+rn_u)}$, $\Bbbk \in \mathcal{K}_2(N,q)$, $q \in \mathbb{N}^2$, $G \in \mathbb{R}^{(n_x+rn_u)\times(n_x+rn_u)}$, $Z \in \mathbb{R}^{rn_u\times(n_x+rn_u)}$, a discretization degree $g \in \mathbb{N}^* \times \mathbb{N}^*$, a Pólya's relaxation degree $p \in \mathbb{N}^2$, $w \triangleq \max\{q+p, g+p\}$, $w \in \mathbb{N}^2$, the pair $(\lambda_A, \lambda_B) \in \mathbb{R}\times\mathbb{R}$ and $\hat{\Theta} \triangleq \left(\lambda_A\hat{\theta}_A^2 + \lambda_B\hat{\theta}_B^2\right)$, a disc $\mathcal{D}(\delta,\varrho)$ centered in $\delta + j0$ and with radius ϱ, where $|\delta| + \varrho < 1$ and a given scalar parameter $\xi \in (-\varrho,\varrho)$, such that the LMIs in (22) and (23) are feasible.

$$X_{\Bbbk} = \sum_{\substack{\Bbbk' \in \mathcal{K}_2(N,p)\\ \mathbb{L}\in\mathcal{L}(p)}} \frac{\prod_{n=1}^{2} p_n!}{\Bbbk'!} W_{\mathbb{L}} > 0,\ \forall \Bbbk \in \mathcal{K}_2(N,q+p), \tag{22}$$

$$M_{\Bbbk} = \frac{\prod_{n=1}^{2} w_n!}{\Bbbk!} M + \sum_{\substack{\Bbbk' \in \mathcal{K}_2(N,w-g)\\ \mathbb{L}\in\mathcal{L}(w-g)}} \frac{\prod_{n=1}^{2} (w_n-g_n)!}{\Bbbk'!} \tilde{M} + \sum_{\substack{\Bbbk' \in \mathcal{K}_2(N,w-q)\\ \mathbb{L}\in\mathcal{L}(w-q)}} \frac{\prod_{n=1}^{2} (w_n-q_n)!}{\Bbbk'!} \breve{M} < 0,\ \forall \Bbbk \in \mathcal{K}_2(N,w), \tag{23}$$

where

$$M = \begin{bmatrix} \hat{\Theta}I & -\xi G^{\mathsf{T}} & \xi Z^{\mathsf{T}} & \xi G^{\mathsf{T}} \\ \star & -G-G^{\mathsf{T}} & Z^{\mathsf{T}} & G^{\mathsf{T}} \\ \star & \star & -\lambda_B I & 0 \\ \star & \star & \star & -\lambda_A I \end{bmatrix}, \tag{24}$$

$$\tilde{M} = \begin{bmatrix} \xi He\left\{\hat{A}_{\mathbb{L}}G - \delta G + \hat{B}_{\mathbb{L}}Z\right\} & \hat{A}_{\mathbb{L}}G - \delta G + \hat{B}_{\mathbb{L}}Z & 0 & 0 \\ \star & 0 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix}, \tag{25}$$

$$\breve{M} = diag(\,-\varrho^2 W_{\mathbb{L}}\ \ W_{\mathbb{L}}\ \ 0\ \ 0\,). \tag{26}$$

Expressions for $\hat{A}_{\mathbb{L}}$ and $\hat{B}_{\mathbb{L}}$ can be found in Appendix B, namely in (B.1) - (B.6) and (B.7) - (B.10). Under the above assumptions the gain $K = ZG^{-1}$ ensures the allocation of the closed-loop poles of (17) in the region $\mathcal{D}(\delta,\varrho)$ and the stabilization of the continuous time system with multiple delays described in (1).

Proof: The matrix $W_{\mathbb{L}}$ in (22) can then be recast into the following form:

$$\left(\sum_{n=1}^{N}\alpha_{1n}\right)^{p}\left(\sum_{\bar{n}=1}^{2}\alpha_{2\bar{n}}\right)^{p} W^{[g]}(\alpha) = \sum_{\Bbbk\in\mathcal{K}_2(N,q+p)} \alpha^{\Bbbk}X_{\Bbbk}. \tag{27}$$

Given that $\alpha_1 \in \Lambda_N$ and $\alpha_2 \in \Lambda_2$, Λ_N defined in (3), if $X_{\Bbbk} > 0\ \forall \Bbbk \in \mathcal{K}_2(N,q+p)$, then $W^{[q]}(\alpha) > 0$ holds $\forall \alpha \in (\Lambda_{N_1}\times\Lambda_2)$.

Now, define $\bar{A}(\alpha) = \hat{A}^{[g]}(\alpha) - \delta I + \hat{B}^{[g]}(\alpha)K$ and use Lemma 1 for the following choice of matrices:

$$N_U = \begin{bmatrix} I & 0 & 0 \\ \bar{A}^{\mathsf{T}}(\alpha) & K^{\mathsf{T}} & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},\ N_V = \begin{bmatrix} I & 0 & 0 \\ -\xi I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{28}$$
$$U = \left[\,\bar{A}^{\mathsf{T}}(\alpha)\ \ -I\ \ K^{\mathsf{T}}\ \ I\,\right],\ V = [\,\xi I\ \ I\ \ 0\ \ 0\,].$$

One observes that (8) is equivalent to multiplying (23) by $\alpha^{\Bbbk}$ and to sum up for $\Bbbk \in \mathcal{K}_2(N,w)$. In order to do that, In order to do that, assume $H = diag\left(-\varrho^2 W(\alpha)\ \ W(\alpha)\ \ 0\ \ 0\right)$, $X = G$ in (8) and consider $KG = Z$.

From matrices in (28) and considering $N_V^{\mathsf{T}} H N_V < 0$ in (9), one has

$$\begin{bmatrix} (\hat{\Theta}I - \varrho^2 W^{[q]}(\alpha) + \xi^2 W^{[q]}(\alpha)) & 0 & 0 \\ 0 & -\lambda_B I & 0 \\ 0 & 0 & -\lambda_A I \end{bmatrix} < 0. \tag{29}$$

For $W^{[q]}(\alpha) > 0$ and $\hat{\Theta} > 0$, then $|\xi| < \varrho$.

Additionally from (28) the condition $N_U^{\mathsf{T}} H N_U < 0$ in (9) can be written as

$$\begin{bmatrix} \hat{\Theta}I - \varrho^2 W^{[q]}(\alpha) & 0 & 0 \\ 0 & -\lambda_B I & 0 \\ 0 & 0 & -\lambda_A I \end{bmatrix} + \Xi^{\mathsf{T}} W^{[q]}(\alpha)^{-1}\Xi < 0,$$
$$\Xi = \left[\,W^{[q]}(\alpha)\bar{A}^{\mathsf{T}}(\alpha)\ \ W^{[q]}(\alpha)K^{\mathsf{T}}\ \ W^{[q]}(\alpha)\,\right]. \tag{30}$$

For $W^{[q]}(\alpha) > 0$, using Schur's complement and changing the second and fourth columns, and doing the same for the second and fourth lines, yields:

$$\begin{bmatrix} \hat{\Theta}I - \varrho^2 W^{[q]}(\alpha) & \bar{A}(\alpha)W^{[q]}(\alpha) & 0 & 0 \\ W^{[q]}(\alpha)\bar{A}^{\mathsf{T}}(\alpha) & -W^{[q]}(\alpha) & W^{[q]}(\alpha) & W^{[q]}(\alpha)K^{\mathsf{T}} \\ 0 & W^{[q]}(\alpha) & -\lambda_A I & 0 \\ 0 & KW^{[q]}(\alpha) & 0 & -\lambda_B I \end{bmatrix} < 0. \tag{31}$$

Multiplying by $-I$, applying Schur's complement with respect to $\lambda_B I$ and given that $\hat{\Theta} = (\lambda_A\hat{\theta}_A^2 + \lambda_B\hat{\theta}_B^2)$, inequality (31) results in:

$$\begin{bmatrix} \varrho^2 W^{[q]}(\alpha) - \lambda_A\hat{\theta}_A^2 I & -\bar{A}(\alpha)W^{[q]}(\alpha) & 0 \\ \star & W^{[q]}(\alpha) & -W^{[q]}(\alpha) \\ \star & \star & \lambda_A I \end{bmatrix} - \lambda_B \begin{bmatrix} -\hat{\theta}_B I \\ 0 \\ 0 \end{bmatrix} \left[\,-\hat{\theta}_B I\ \ 0\ \ 0\,\right] - \begin{bmatrix} 0 \\ -W^{[q]}(\alpha)K^{\mathsf{T}} \\ 0 \end{bmatrix} \lambda_B^{-1} \left[\,0\ \ -KW^{[q]}(\alpha)\ \ 0\,\right] > 0. \tag{32}$$

Considering Lemma 2 with

$$\mathcal{O}^{\mathsf{T}} = \left[\,-\hat{\theta}_B I\ \ 0\ \ 0\,\right], \mathcal{U} = \left[\,0\ \ -KW^{[q]}(\alpha)\ \ 0\,\right], \lambda = \lambda_B, \tag{33}$$

and the upper bounds defined in (20), then inequality (32) can be modified replacing $\hat{\theta}_B I$ by $\Delta\hat{B}^{[g]}(\alpha)$ to obtain the more stringent, but more useful condition

$$\begin{bmatrix} \varrho^2 W^{[q]}(\alpha) - \lambda_A\hat{\theta}_A^2 I & -\bar{A}(\alpha)W^{[q]}(\alpha) & 0 \\ \star & W^{[q]}(\alpha) & -W^{[q]}(\alpha) \\ \star & \star & \lambda_A I \end{bmatrix} - \begin{bmatrix} 0 & \Delta B^{[g]}(\alpha)KW^{[q]}(\alpha) & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix} > 0. \tag{34}$$

Now, consider the same procedure from (32), apply Lemma 2, proceed in the same way for $\hat{\theta}_A$ and $\Delta\bar{A}^{[g]}(\alpha)$, replace $\bar{A}(\alpha)$ by $\hat{A}^{[g]}(\alpha) - \delta I + \hat{B}^{[g]}(\alpha)K$, then one finds:

$$\begin{bmatrix} \varrho^2 W^{[q]}(\alpha) & (\hat{A}(\alpha) - \delta I + \hat{B}(\alpha)K)W^{[q]}(\alpha) \\ \star & W^{[q]}(\alpha) \end{bmatrix} > 0, \quad (35)$$

where $\hat{A}(\alpha)$ and $\hat{B}(\alpha)$ are defined in (11) and (18). Inequality (35) can then be recast into the following form:

$$\begin{bmatrix} -\varrho^2 W^{[q]}(\alpha) & \hat{\mathbb{A}}(\alpha)W^{[q]}(\alpha) - \delta W^{[q]}(\alpha) \\ \star & -W^{[q]}(\alpha) \end{bmatrix} < 0, \quad (36)$$

where $\hat{\mathbb{A}}(\alpha) = \hat{A}^{[g]}(\alpha) + \hat{B}^{[g]}(\alpha)K$. As show in Chilali and Gahinet (1996), the augmented matrix $\hat{\mathbb{A}}(\alpha)$ is $\mathcal{D}$-stable if and only if there exist $W^{[q]}(\alpha) > 0$ such that

$$\gamma \otimes W^{[q]}(\alpha) + \beta \otimes (\hat{\mathbb{A}}(\alpha)W^{[q]}(\alpha)) + \beta^{\mathsf{T}} \otimes (\hat{\mathbb{A}}(\alpha)W^{[q]}(\alpha))^{\mathsf{T}} < 0. \quad (37)$$

(the symbol $\otimes$ denotes the Kronecker product of matrices). For a particular case where region $\mathcal{D}$ is a circle of radius ϱ and centre $(\delta, 0)$, then γ and β are

$$\gamma = \begin{bmatrix} -\rho & -\delta \\ -\delta & -\rho \end{bmatrix}, \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (38)$$

Thus (37) is equivalent to (36), which ensures the closed-loop pole allocation of the augmented matrix $\hat{\mathbb{A}}$ in a region $\mathcal{D}(\delta, \varrho)$ $\forall \alpha \in (\Lambda_{N_1} \times \Lambda_2)$. □

Remark 1. It is import to emphasize that for high values of $\hat{\Theta}$ and values of ξ very close to ϱ, it is unlikely to find $W^{[q]}(\alpha)$ satisfying condition (29).

Remark 2. Considering the Theorem 3 and Polya's relaxation, as degree p increases, the condition converges towards a solution whenever it exits, exploiting a generalization of the Pólya's theorem for the case of positive polynomials with matrix-valued coefficients (Oliveira and Peres (2007)).

Remark 3. The proposed approach can be generalized to inputs with different dimensions.

5. NUMERICAL EXPERIMENTS

Numerical experiments were implemented in MATLAB® Software, version 7 (R2010b), using YALMIP (Lofberg (2004)) and SeDuMi (Sturm (1999)).

Example 1. Consider a mass-spring continuous-time system given by Iwasaki (1996) and described by (1). The system matrices are:

$$E(\beta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta/2 & \beta/2 & 0 & 0 \\ \beta/3 & -\beta/3 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \quad (39)$$

where $\beta \in [\,3.6,\ 5.4\,]$. Consider both an uncertain sampling period, such that T varies inside the interval $[\,0.4,\ 0.6\,]$ s and a network-induced time-delay $\tau_i = 0.2$ s, $i \in \{1\}$. This example addresses the flexibility of Theorem 3 to handle performance requirements. Thus, a damping factor of 0.2 is proposed.

In order to apply Theorem 3, the following parameters were adopted: $g = (\,6,\ 6\,)$, $q = (\,1,\ 1\,)$, $p = (\,0,\ 0\,)$, $\xi = 0$. Considering $g = (\,5,\ 5\,)$, Theorem 3 provides no feasible solution for $\xi = 0$ and $\phi = \arccos(0.2)$. Considering $g = (\,6,\ 6\,)$, the bounds on the discretization errors according to (20) are:

$$\hat{\theta}_A = 0.0023,\ \hat{\theta}_B = 1.0588 \times 10^{-5}. \quad (40)$$

The result obtained for $\phi = \arccos(0.2)$ is presented in (41):

$$K = [\,-3.9290\ 0.0247\ -6.2386\ -3.6061\ -0.5676\,]. \quad (41)$$

Now, considering the same parameters and $\phi = 90°$, the following gain was obtained:

$$K = [\,0.0864\ -1.2266\ -2.3358\ -1.790\ -0.2415\,]. \quad (42)$$

Figure 1 shows the closed-loop pole allocation with two diferents gains (41) and (42), assuming $\phi = \arccos(0.2)$ and $\phi = 90°$, respectively. As can be seen, all the closed-loop poles associated with gain in (41) are inside the disc region of radius $\varrho = 0.7409$ and centered at $(0, 0.1512)$, which aproximates the cardioid related to damping factor 0.2.

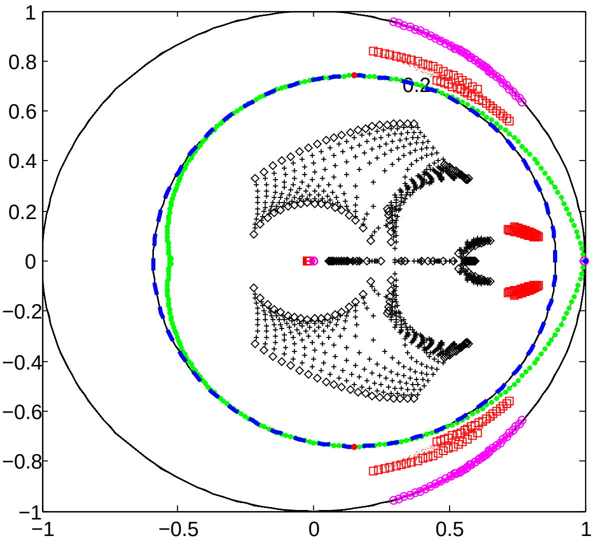

Figure 1 $*$: Open-loop system eigenvalues. $\bullet$: Closed-loop system eigenvalues of (17) settings $\phi = 90°$ in (21). $+$: Closed-loop system eigenvalues of (17) settings $\phi = \arccos(0.2)$ in (21). - -: Region $\mathcal{D}(0.1512, 0.7409)$. . .: Cardioid corresponding to damping factor 0.2. The eigenvalues corresponding to polytope vertices are represented by $\circ$, $\diamond$ and $\square$.

Example 2. Consider the continuous-time system with multiple input delays described by (1) and borrowed from Tsubakino et al. (2016), whose system matrices are

$$E(\eta) = \begin{bmatrix} 0 & 1 & 0 \\ -3 & \eta & 0 \\ -6 & 2 & 3 \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, F_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad (43)$$

where $\eta \in [\,2,\ 4\,]$. The system is sampled with a period belonging to the interval $[\,0.1,\ 0.4\,]$ s and with input delays $\tau_1 = 0.01$ s and $\tau_2 = 0.09$ s, $i \in \{1, 2\}$. The aim is to design a robust state feedback digital controller in face of the uncertain sampling period, such that this controller ensures the stability of the uncertain continuous-time system with distinct input delays and guarantees the closed-loop poles of the correspondent discretized system in a arbitrary region $\mathcal{D}(\delta, \varrho)$.

In order to obtain $\hat{\theta}_A$ and $\hat{\theta}_B$ sufficiently small $g = (\,7,\ 7\,)$ was adopted such that condition given by Theorem 3 results in a feasible solution. Bounds on discretization errors are

$$\hat{\theta}_A = 4.7541 \times 10^{-4},\ \hat{\theta}_B = 6.2660 \times 10^{-5}. \quad (44)$$

Applying Theorem 3 with $g = (\,7,\,7\,)$, $q = (\,1,\,1\,)$, $p = (\,0,\,0\,)$, $\xi = 0$ and $\phi = 90°$ the following gain matrix K was obtained:

$$K = \begin{bmatrix} 12.6578 & -4.7023 & -4.8667 & -0.0056 & -0.0199 \\ 13.8717 & 0.1297 & -6.3435 & -0.0016 & -0.4928 \end{bmatrix}. \tag{45}$$

Additionally for $\phi = 86.5°$ one obtains:

$$K = \begin{bmatrix} 2.5992 & -8.5972 & -2.8334 & -0.0711 & 0.3912 \\ 6.1343 & -3.0540 & -4.7234 & -0.0176 & -0.1851 \end{bmatrix}. \tag{46}$$

As an illustration, Figures 2 and 3 show the closed-loop pole allocation in a region defined by $\phi = 86.5°$ and transient response for (45) and (46), respectively. As can be seen, the closed-loop responses for (46) presents better transient responses with reduction in the settling time.

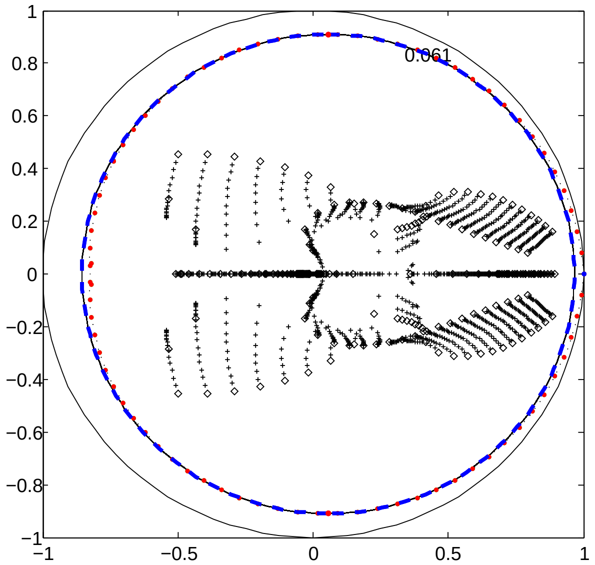

Figure 2 +: Closed-loop system eigenvalues of (17) settings $\phi = 86.5°$ in (21). - -: Region $\mathcal{D}(0.0557, 0.9101)$. . .: Cardioid related to damping factor 0.06. The polytope vertices are represented by $\diamond$.

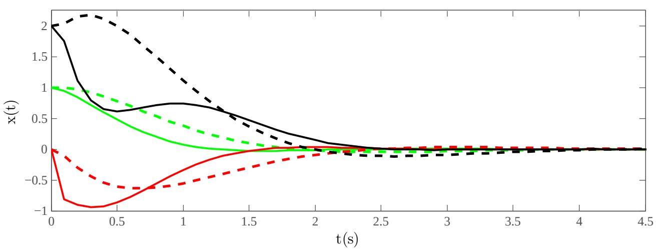

Figure 3 Evolution of the system states in Example 2 for $\alpha_{11} = 0.2$, $\alpha_{12} = 0.8$, $\alpha_{21} = 0.9$, $\alpha_{22} = 0.1$ and $x_0 = [1\ 0\ 2]^{\mathsf{T}}$. Dotted line: transient response for $\phi = 90°$. Continuous line: transient response for $\phi = 86.5°$

6. CONCLUSION

This work addressed robust $\mathcal{D}$-stability via discrete controllers for continuous-time uncertain systems. In this context, multiple input delays and polytopic parameter uncertainties in the original state space representation were propagated to the discretized model, and LMI based control synthesis condition composed of homogeneous polynomial matrices of arbitrary degree was used.

For illustration, two numerical examples were presented. The first example highlighted the use of the proposed condition in order to handle performance requirements. In the second example, the closed-loop poles of the augmented model were placed inside a desired disc region in the presence of input delays.

Future work could be concerned with the design of robust state derivative feedback control law taking into account discretization employed herein.

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Appendix A. NOTATIONS AND DEFINITIONS

The following definitions allow the representation of a Taylor series terms of an arbitrary degree.

Consider a vector of parameters $\zeta_s = (\zeta_{s1}, ..., \zeta_{sN_s}) \in \Lambda_{N_s}$, $s \in \{1, ..., \omega\}$ and $\zeta = (\zeta_{11}, ..., \zeta_{1N_1}, ..., \zeta_{\omega 1}, ..., \zeta_{\omega N_\omega}) \in (\Lambda_{N_1} \times ... \times \Lambda_{N_\omega})$, such that $\zeta! = \zeta_{11}! \times \cdots \times \zeta_{1N_1}! \times \cdots \times \zeta_{\omega 1}! \times \cdots \times \zeta_{\omega N_\omega}!$. For the sake of generality, a degree $g = (g_1, ..., g_\omega)$, $g_s \in \mathbb{N}^*$ was adopted.

- Set $\mathcal{K}(N_s, g_s)$ is given by

$$\mathcal{K}(N_s, g_s) \triangleq \{k_s = (k_{s1} \cdots k_{sN_s}) \in \mathbb{N}^{N_s} | k_{s1} + \cdots + k_{sN_s} = g_s\}. \tag{A.1}$$

To exemplify, for $\omega = 1$ consider a dynamic matrix $E(\alpha_1)$ in (1), such that

$$E(\alpha_1) = \sum_{i=1}^{N_1} \alpha_{1i} E_i, \ \alpha_1 \in \Lambda_{N_1}, \tag{A.2}$$

and a case with $N_1 = 2$ and $g_1 = 3$, then

$$E(\alpha_1)E(\alpha_1)E(\alpha_1) = \alpha_{11}^0 \alpha_{12}^3 E_2E_2E_2 + \alpha_{11}^1 \alpha_{12}^2 (E_1E_2E_2 + E_2E_1E_2 + E_2E_2E_1) + \alpha_{11}^2 \alpha_{12}^1 (E_1E_1E_2 + E_1E_2E_1 + E_2E_1E_1) + \alpha_{11}^3 \alpha_{12}^0 E_1E_1E_1. \tag{A.3}$$

The exponents of the parameters $\alpha_1 \in \Lambda_{N_1}$ are represented as follows:

$$k_1 \in \mathcal{K}(2, 3) = \{(03), (12), (21), (30)\}. \tag{A.4}$$

- Set $\mathcal{K}_\omega(N, g)$ is formed by the Cartesian product of all N_s-tuples of the ω unit simplexes and is defined by

$$\mathcal{K}_\omega(N, g) \triangleq \{N = (N_1, \cdots, N_\omega), g = (g_1, \cdots, g_\omega), \Bbbk = (k_{11} \cdots k_{1N_1} \cdots k_{\omega 1} \cdots k_{\omega N \omega}) \in \mathbb{N}^{N_1 \times \cdots \times N_\omega} | k_{s1} + \cdots + k_{sN_s} = g_s\}. \tag{A.5}$$

To exemplify, consider $T(\alpha_2)$ in (4) and (A.2). Then, for $N_1 = 2$ the following product can be written

$$E(\alpha_1)T(\alpha_2) = \alpha_{11}^1 \alpha_{12}^0 \alpha_{21}^1 \alpha_{22}^0 E_1T_1 + \alpha_{11}^1 \alpha_{12}^0 \alpha_{21}^0 \alpha_{22}^1 E_1T_2 + \alpha_{11}^0 \alpha_{12}^1 \alpha_{21}^1 \alpha_{22}^0 E_2T_1 + \alpha_{21}^0 \alpha_{22}^1 \alpha_{21}^0 \alpha_{22}^1 E_2T_2, \tag{A.6}$$

where

$$\Bbbk \in \{(1010), (1001), (0110), (0101)\}. \tag{A.7}$$

- Set $\mathcal{R}(k_{s1}...k_{sN_s})$ is formed by

$$\mathcal{R}(k_{s1} \cdots k_{sN_s}) \triangleq \{\mathbb{r} = (\mathfrak{r}_1 \cdots \mathfrak{r}_{g_s}) \in \mathbb{N}^{g_s} | \forall i \in \{1, \cdots, N_s\}, k_i = \# \{j \in \{1, \cdots, g_s\} | \mathfrak{r}_j = i\}\}, \tag{A.8}$$

where # denotes the number of elements of a set.

To exemplify, consider (A.3) and from the tuples in (A.4), one has the sets

$$\begin{aligned} &\mathcal{R}(03) = \{(222)\}, \ \mathcal{R}(12) = \{(122), (212), (221)\}, \\ &\mathcal{R}(21) = \{(112), (121), (211)\}, \ \mathcal{R}(30) = \{(111)\}, \end{aligned} \tag{A.9}$$

which correspond to the matrices subindex in (A.3).

Now, consider the set $\mathcal{K}_\omega(N, \bar{g})$, $\bar{g} = (\bar{g}_1, ..., \bar{g}_\omega)$, formed by the following $(N_1 + ... + N_\omega)$-tuples: $(k'_{11}...k'_{1N_1}...k'_{\omega 1}...k'_{\omega N_\omega})$, such that $\bar{g}_s \in \mathbb{N}$, $\bar{g}_s \le g_s$. For each $\Bbbk \in \mathcal{K}_\omega(.)$, define:

- Set $\mathcal{L}(\bar{g})$ is given by:

$$\mathcal{L}(\bar{g}) \triangleq \{\mathbb{L} = (l_{11} \cdots l_{1N_1} \cdots l_{\omega 1} \cdots l_{\omega N_\omega}) \in \mathbb{N}^{N_1 \times \cdots \times N_\omega} | \forall \Bbbk' \in \mathcal{K}_\omega(N, \bar{g}), \mathbb{L} = \Bbbk - \Bbbk', l_{si} \ge 0, i \in \{1, \cdots, N_s\}\}. \tag{A.10}$$

  The set $\mathcal{L}(\bar{g})$ also depends on $\Bbbk$, which was omitted to its notation for clarity.

To exemplify, from (A.4) consider $k_1 = (12)$. Assume $\bar{g}_1 = 1$, yielding $k' \in \mathcal{K}(2, 1) = \{(01), (10)\}$. Thus one has

$$\mathbb{L} \in \mathcal{L}(\bar{g}_1) = \{(11), (02)\}. \tag{A.11}$$

- Set $\mathcal{T}(k_{s1}...k_{sN_s})$ is given by

$$\mathcal{T}(k_{s1} \cdots k_{sN_s}) \triangleq \{\mathbb{i}_s = (\mathbb{i}_{s1} \cdots \mathbb{i}_{sN_s}) \in \mathbb{N}^{N_s} | \forall k_{sj} > 0, j \in \{1, \cdots, N_s\}, \mathbb{i}_{sj} = 1, \mathbb{i}_{sN_s - j} = 0\} \tag{A.12}$$

To exemplify, from (A.11) set $\mathcal{T}$ is:

$$\mathbb{i}_1 \in \mathcal{T}(11) = \{(10), (01)\}, \mathbb{i}_1 \in \mathcal{T}(02) = \{(01)\}. \tag{A.13}$$

Appendix B. HOMOGENEOUS POLYNOMIAL DISCRETIZED MATRICES

Using what was described above, one can present expressions for homogeneous polynomial matrices of degree $g \in \mathbb{N}^* \times \mathbb{N}^*$. Factorizing the homogeneous polynomials in (12) by $\alpha^{\Bbbk}$, one has

$$A^{[g]}(\alpha) = \sum_{\Bbbk \in \mathcal{K}_2(N,g)} \alpha^{\Bbbk} \underbrace{\left(\sum_{n=0}^{g} \sum_{\substack{\Bbbk' \in \mathcal{K}_2(N, g-n) \\ \mathbb{L} \in \mathcal{L}(g-n)}} \frac{T_1^{l_{21}} T_2^{l_{22}}}{l_{21}! l_{22}!} \sum_{(\mathfrak{r}_{11}...\mathfrak{r}_{1g_1}) \in \mathcal{R}(l_{11}...l_{1N_1})} \frac{((g-n)!)^2}{\Bbbk'!} E_{\mathfrak{r}_{11}}...E_{\mathfrak{r}_{1g_1}} \right)}_{\mathcal{A}_{\Bbbk}}, \tag{B.1}$$

then, in a compact form

$$A^{[g]}(\alpha) = \sum_{\Bbbk \in \mathcal{K}_2(N,g)} \alpha^{\Bbbk} \mathcal{A}_{\Bbbk}. \tag{B.2}$$

From (12) and (18), $B^{[g]}(\alpha)$ can be written as

$$B_i^{[g]}(\alpha) = \sum_{\Bbbk \in \mathcal{K}_2(N,g)} \alpha^{\Bbbk} \underbrace{\left(\sum_{n=1}^{g} \sum_{\substack{\Bbbk' \in \mathcal{K}_2(N, g-n) \\ \mathbb{L} \in \mathcal{L}(g-n)}} \frac{\psi_{i_1}^{l_{21}} \psi_{i_2}^{l_{22}}}{l_{21}! l_{22}!} \sum_{\mathbb{i}_{1j} \in \mathcal{T}(l_{11}...l_{1N_1})} \sum_{(\mathfrak{r}_{11}...\mathfrak{r}_{1g_1}) \in \mathcal{R}(l_{11}...l_{1N_1} - \mathbb{i}_1)} \frac{((g-n)!)^2}{\Bbbk'!} E_{\mathbb{r}} F_{ij} \right)}_{\mathcal{B}_{i\Bbbk}}, \tag{B.3}$$

then, in a compact form

$$B^{[g]}(\alpha) = \sum_{\Bbbk_2 \in \mathcal{K}(N,g)} \alpha^{\Bbbk} \mathcal{B}_{\Bbbk}, \tag{B.4}$$

where $\mathcal{B}_{\Bbbk} = \left[\mathcal{B}_{1_{\Bbbk}} \; \dots \; \mathcal{B}_{r_{\Bbbk}} \right]$. From (12) and (18), $B_d^{[g]}(\alpha)$ can be written as

$$B_{d_i}^{[g]}(\alpha) = \underbrace{\sum_{\Bbbk \in \mathcal{K}_2((N_1,2),(2g,g))} \alpha^{\Bbbk} \Bigg(\underbrace{\sum_{n=0}^{g} \sum_{\bar{n}=1}^{g} \frac{\tau_i{}^{\bar{n}}}{\bar{n}!}}\; \underbrace{\sum_{\substack{\Bbbk' \in \mathcal{K}_2((N_1,2),(2g-n-\bar{n},g-n)) \\ \mathbb{L} \in \mathcal{L}(2g-n-\bar{n},g-n)}} \sum_{\mathbb{i}_{1j} \in \mathcal{T}(I_{11} \dots l_{1N_1})} \frac{\psi_{i1}^{l_{21}} \psi_{i2}^{l_{22}}}{l_{21}! l_{22}!}} \sum_{(\mathfrak{r}_{11} \dots \mathfrak{r}_{1g_1}) \in \mathcal{R}(l_{11} \dots l_{1N_1} - \mathbb{i}_1)} \frac{(2g-n-\bar{n})!\,(g-n)!}{\Bbbk'!} E_{\mathfrak{r}} F_{ij},}_{\mathcal{B}_{d\,i\,\Bbbk}} \tag{B.5}$$

then in a compact form

$$B_d^{[g]}(\alpha) = \sum_{\Bbbk \in \mathcal{K}_2((N_1,2),(2g,g))} \alpha^{\Bbbk} \mathcal{B}_{d\Bbbk}, \tag{B.6}$$

where $\mathcal{B}_{d\Bbbk} = \left[\mathcal{B}_{d_1\,\Bbbk} \; \dots \; \mathcal{B}_{d_r\,\Bbbk} \right]$. Finally, polynomial augmented matrices $\hat{A}^{[g]}(\alpha)$ and $\hat{B}^{[g]}(\alpha)$ have different degrees, thus should be homogenized as

$$\hat{A}^{[g]}(\alpha) = \sum_{\Bbbk \in \mathcal{K}_2((N_1,2),(2g,g))} \alpha^{\Bbbk} \hat{A}_{\Bbbk}, \tag{B.7}$$

where

$$\hat{A}_{\Bbbk} = \begin{bmatrix} \displaystyle\sum_{\substack{\Bbbk' \in \mathcal{K}_2((N_1,2),(g,0)) \\ \mathbb{L} \in \mathcal{L}(g,0)}} \frac{g!0!}{\Bbbk'!} \mathcal{A}_{\mathbb{L}} & \mathcal{B}_{d_{\Bbbk}} \\ 0 & 0 \end{bmatrix}; \tag{B.8}$$

and

$$\hat{B}^{[g]}(\alpha) = \sum_{\Bbbk \in \mathcal{K}_2((N_1,2),(2g,g))} \alpha^{\Bbbk} \hat{B}_{\Bbbk}, \tag{B.9}$$

where

$$\hat{B}_{\Bbbk} = \begin{bmatrix} \displaystyle\sum_{\substack{\Bbbk' \in \mathcal{K}_2((N_1,2),(g,0)) \\ \mathbb{L} \in \mathcal{L}(g,0)}} \frac{g!0!}{\Bbbk'!} \mathcal{B}_{\mathbb{L}} \\ \displaystyle\sum_{\Bbbk' \in \mathcal{K}_2((N_1,2),(2g,g))} \frac{(2g)!g!}{\Bbbk'!} I \end{bmatrix}. \tag{B.10}$$

Appendix C. CLOSED-LOOP UNCERTAIN CONTINUOUS-TIME SYSTEM STABILITY

Following the exposed in Braga et al. (2014), for any $\alpha \in (\Lambda_{N_1} \times \Lambda_2)$ and a given sampling period $T(\alpha_2)$, the solution of (1) over the interval $t \in [\,\mathrm{k}T(\alpha_2),\, \mathrm{k}T(\alpha_2) + T(\alpha_2)\,]$ is given by

$$x(t) = e^{E(\alpha_1)((\mathrm{k}+1)T(\alpha_2) - \mathrm{k}T(\alpha_2))} x(\mathrm{k}T(\alpha_2)) + \sum_{i=1}^{r} \Bigg(\int_{\mathrm{k}T(\alpha_2)}^{\mathrm{k}T(\alpha_2)+T(\alpha_2)} e^{E(\alpha_1)(\mathrm{k}T(\alpha_2)+T(\alpha_2)-\varsigma)} F_i(\alpha_1) u_i(\varsigma - \tau_i) d\varsigma \Bigg). \tag{C.1}$$

Signal $u_i(t)$ is piecewise constant over the sampling interval, then the delayed signal $u_i(t-\tau_i)$ is also piecewise constant. Considering that $u_i(t-\tau_i)$ varies between sampling instants, to evaluate (C.1) it is then convenient to split the integration limits into two parts such that

$$x(t) = e^{E(\alpha_1)((\mathrm{k}+1)T(\alpha_2) - \mathrm{k}T(\alpha_2))} x(\mathrm{k}T(\alpha_2)) \sum_{i=1}^{r} \Bigg(\int_{\mathrm{k}T(\alpha_2)}^{\mathrm{k}T(\alpha_2)+\tau_i} e^{E(\alpha_1)(\mathrm{k}T(\alpha_2)+T(\alpha_2)-\varsigma)} d\varsigma F_i(\alpha_1) u_i((\mathrm{k}-1)T(\alpha_2)) + \int_{\mathrm{k}T(\alpha_2)+\tau_i}^{\mathrm{k}T(\alpha_2)+T(\alpha_2)} e^{E(\alpha_1)(\mathrm{k}T(\alpha_2)+T(\alpha_2)-\varsigma)} d\varsigma F_i(\alpha_1) u_i(\mathrm{k}T(\alpha_2)) \Bigg), \tag{C.2}$$

where $u_i(.)$ is constant in each part. By means of a change in variables, such that $\varsigma = \mathrm{k}T(\alpha_2) + \tau_i - s$ and $\varsigma = \mathrm{k}T(\alpha_2) + T(\alpha_2) - s$, equation (C.2) can be rewritten as

$$x(t) = e^{E(\alpha_1)T(\alpha_2)} x(\mathrm{k}T(\alpha_2)) + \sum_{i=1}^{r} \Big(e^{E(\alpha_1)(T(\alpha_2)-\tau_i)} \int_0^{\tau_i} e^{E(\alpha_1)s} ds F_i(\alpha_1) u_i((\mathrm{k}-1)T(\alpha_2)) + \int_0^{T(\alpha_2)-\tau_i} e^{E(\alpha_1)s} ds\, F_i(\alpha_1) u_i(\mathrm{k}T(\alpha_2)). \tag{C.3}$$

Taking the supremum of (C.3) and using triangle inequality, one has:

$$\sup_{t \in [\,\mathrm{k}T(\alpha_2),\, (\mathrm{k}+1)T(\alpha_2)\,]} \|x(t)\| \leq \sup_{\alpha \in \Lambda_N} \left\| e^{E(\alpha_1)T(\alpha_2)} \right\| \|x(\mathrm{k}T(\alpha_2))\| + \sum_{i=1}^{r} \sup_{\alpha \in \Lambda_N} \left\| e^{E(\alpha_1)(T(\alpha_2)-\tau_i)} \int_0^{\tau_i} e^{E(\alpha_1)s} ds F_i(\alpha_1) \right\| \|u_i((\mathrm{k}-1)T(\alpha_2))\| + \sum_{i=1}^{r} \sup_{\alpha \in \Lambda_N} \left\| \int_0^{T(\alpha_2)-\tau_i} e^{E(\alpha_1)s} ds F_i(\alpha_1) \right\| \|u_i(\mathrm{k}T(\alpha_2))\|, \tag{C.4}$$

where $\Lambda_N = \Lambda_{N_1} \times \Lambda_2$. From (C.4) and using (13)-(15) is it possible to write

$$\sup_{t \in [\,\mathrm{k}T(\alpha_2),\, (\mathrm{k}+1)T(\alpha_2)\,]} \|x(t)\| \leq \underbrace{\left\| A^{[g]}(\alpha) + \Delta A^{[g]}(\bar{\alpha}) \right\|}_{I} \|x(\mathrm{k}T(\alpha_2))\| + \sum_{i=1}^{r} \underbrace{\left\| B_{d_i}^{[g]}(\alpha) + \Delta B_{d_i}^{[g]}(\bar{\alpha}) \right\|}_{II} \|u_i(\mathrm{k}T(\alpha_2) - T(\alpha_2))\| + \sum_{i=1}^{r} \underbrace{\left\| B_i^{[g]}(\alpha) + \Delta B_i^{[g]}(\bar{\alpha}) \right\|}_{III} \|u_i(\mathrm{k}T(\alpha_2))\|. \tag{C.5}$$

Given that $z(\mathrm{k}T(\alpha_2))$ in (17) converge to zero, $x(\mathrm{k}T(\alpha_2))$, $u_i(\mathrm{k}T(\alpha_2))$ and $u_i(\mathrm{k}T(\alpha_2) - T(\alpha_2))$, $i \in \{1, \cdots, r\}$, also converges to zero when $\mathrm{k} \to \infty$. For any $\alpha \in (\lambda_{N_1} \times \Lambda_2)$, I, II and III will be always be bounded. In this way, from (C.5), $x(t) \to 0$ when $t \to \infty$ and asyntotic closed-loop stability of the uncertain system (1) is ensured by control law in (19).