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Robust state derivative feedback LMI-based designs for discretized systems ⋆

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Abstract: This work addresses novel Linear Matrix Inequality (LMI)-based conditions for the design of discrete-time state derivative feedback controllers. The main contribution of this work consists of an augmented discretized model formulated in terms of the state derivative, such that uncertain sampling periods and parametric uncertainties in polytopic form can be propagated from the original continuous-time state space representation. The resulting discrete-time model is composed of homogeneous polynomial matrices with parameters lying in the Cartesian product of simplexes, plus an additive norm-bounded term representing the residual discretization error. Moreover, the referred condition allows for the closed-loop poles allocation of the augmented system in a $\mathcal{D}$-stable region. Finally, numerical simulations illustrate the effectiveness of the proposed method.

Keywords: State derivative feedback; Robust control; $\mathcal{D}$-stability; Discretized linear systems; Linear matrix inequalities.

1. INTRODUCTION

Accelerometers are one of the most used sensors either in field or in laboratory scale experiments. It turns out that the displacement estimation from this sensors tends to be biased (Rossi et al. (2018) and references). Therefore, in engineering applications instead of using a state feedback control law based on displacements and velocities, it may be more convenient to use a state derivative feedback control law.

Regarding contributions concerning state derivative feedback, it is important to highlight that most research efforts have been devoted to the design of continuous-time controllers (Duan et al. (2005), Faria et al. (2009), Tseng and Hsieh (2013), Beteto et al. (2018), among others). On the other hand, several methods treat the problem of discretization of uncertain systems through of the first order Taylor series expansion to circumvent the difficulty of dealing with the exponential of an uncertain matrix (Cardim et al. (2009), Rossi et al. (2018)). To the author's knowledge, in context of state derivative feedback, there exists a lack of methods that allows to handle systematic treatment of the high order terms in the discretization procedure, specially when the sampling period is not sufficiently small such that the quadratic and higher-order terms in the Taylor series expansion can not be neglected in the uncertain representation.

In this context, the main contribution of this work consists of an augmented discretized model formulated in terms of the state derivative, such that uncertain sampling period and parametric uncertainties in polytopic form can be propagated from the original continuous-time state space representation. The resulting discrete-time model is composed of homogeneous polynomial matrices with parameters lying in the Cartesian product of simplexes, plus an additive norm-bounded term representing the residual discretization error. Moreover, LMIs relaxations that include a scalar parameter search are employed for the design of a convenient robust state derivative feedback gain.

Controllers designs involving LMIs have the advantage of including performance indexes in approaching the problem. This work uses a disc region as regional closed-loop pole placement to obtain better transient response suppressing the oscillations and reducing the settling time.

The remainder of this paper is organized as follows. Section 2 introduces definitions and preliminary results. Section 3 brings definitions for a systematic discretization through Taylor series expansion. Section 4 describes the proposed condition based on state derivative feedback control law. Section 5 presents a numerical example. Finally, concluding remarks are shown in Section 6.

1.1 Notation

$\mathbb{N}^*$: denotes the set of non-zero natural numbers. $\mathbb{Z}$: denotes the set of integer numbers. $\mathbb{R}_+$: denotes the set of all positive real numbers. $I(0)$: identity (null) matrix of appro-

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priate dimension; diag: a diagonal matrix of appropriate dimension; $M^\top$: matrix M transpose; $He\{M\}$: denotes $M^\top + M$; $\star$: denotes the elements or symmetrical blocks with respect to the diagonal of a symmetric matrix; T : denotes the sampling period.

2. PRELIMINARIES

Consider the following uncertain continuous-time invariant model:

$$\dot{x}(t) = E(\alpha_1)x(t) + F(\alpha_1)u(t), \tag{1}$$

where $E(\alpha_1) \in \mathbb{R}^{n_x \times n_x}$, $F(\alpha_1) \in \mathbb{R}^{n_x \times n_u}$. Suppose that matrices $E(\alpha_1)$ and $F(\alpha_1)$ belong to polytope Ω defined by:

$$\Omega = \left\{ (E,F)(\alpha_1) | (E,F)(\alpha_1) = \sum_{j=1}^{N_1} \alpha_{1j}(E,F)_j \right\}, \tag{2}$$

where $\alpha_1 = (\alpha_{11}, \ldots, \alpha_{1N_1})$ is a time-invariant parameter vector, taking values in a unit simplex Λ_{N_1}:

$$\Lambda_{N_1} = \left\{ \zeta \in \mathbb{R}^{N_1} | \sum_{j=1}^{N_1} \zeta_j = 1, \zeta_j \geq 0, \forall j \in \{1, ..., N_1\} \right\}. \tag{3}$$

Moreover, consider the input signal to be sampled with uncertain sampling period, such that $T(\alpha_2)$ lies inside the interval $[T_1, T_2]$ and can be written as a convex combination of $N_2 = 2$ vertices:

$$T(\alpha_2) = \sum_{i=1}^{2} \alpha_{2i} T_i, \;\; \alpha_2 \in \Lambda_2, \tag{4}$$

such that Λ_2 is defined in (3). In this sense, consider an uncertain parameter vector, where $\alpha = (\alpha_1, \alpha_2) \in (\Lambda_{N_1} \times \Lambda_2)$, which is a so-called multi-simplex domain (see Oliveira et al. (2008)).

Assume that the system is to be controlled by using sampled measurements of the state derivative $\dot{x}(\mathrm{k}T(\alpha_2))$, $\mathrm{k} \in \mathbb{Z}$.

Assumption 1. The control is updated immediately after the state derivative $\dot{x}(\mathrm{k}T(\alpha_2))$ is measured at each sampling time (Rossi et al. (2018)).

Moreover, consider that zero order hold is employed to keep the control $u(t)$ constant between sampling times, i.e.:

$$u(t) = K(\mathrm{k}T(\alpha_2))^+, t \in [(\mathrm{k}T(\alpha_2))^+, (\mathrm{k}+1)T(\alpha_2)], \tag{5}$$

where K denotes the gain matrix and the superscript $+$ is employed to denotes Assumption 1.

Additionally, given that the control is supposed constant between $[(\mathrm{k}T(\alpha_2))^+, (\mathrm{k}+1)T(\alpha_2)]$, the system described by (1) can be discretized as

$$x((\mathrm{k}+1)T(\alpha_2)) = A(\alpha)x(\mathrm{k}T(\alpha_2)) + B(\alpha)u(\mathrm{k}T(\alpha_2))^+, \tag{6}$$

and the uncertain parameter-dependent matrices $A(\alpha)$, $B(\alpha)$ are:

$$A(\alpha) = e^{E(\alpha_1)T(\alpha_2)}, \;\; B(\alpha) = \left(\int_0^{T(\alpha_2)} e^{E(\alpha_1)\varsigma} d\varsigma \right) F(\alpha_1). \tag{7}$$

Then following the results presented in Braga et al. (2014) and Rossi et al. (2018), the main aim consists in providing an LMIs-based condition that guarantees the stability of (1) and the closed-loop pole allocation of an augmented system formulated from (1) in terms of the state derivative feedback at the sampling time $t = \mathrm{k}T(\alpha_2)^+$, $\forall \alpha_2 \in \Lambda_2$, immediately after the control is updated. Moreover and in context of state derivative feedback control, the condition proposed in this work was defined in terms of homogeneous polynomial parameter dependent matrices of arbitrary degree, which allows to handle systematic treatment of the high order terms in the discretization procedure, specially when the sampling period is not sufficiently small such that the quadratic and higher-order terms in the Taylor series expansion can not be neglected in the uncertain representation.

The following lemmas were used in the proof of the proposed condition.

Lemma 1. (Gahinet and Apkarian, 1994). Given a matrix $H = H^\top \in \mathbb{R}^{n \times n}$, and two matrices V and U of column dimension n, consider the problem of finding some matrix X of compatible dimensions such that:

$$H + V^\top X^\top U + U^\top X V < 0. \tag{8}$$

Denote by N_U and N_V any matrices whose columns form bases of the null spaces of U and V, respectively. Then (8) is feasible for X if and only if

$$N_V^\top H N_V < 0 \text{ and } N_U^\top H N_U < 0. \tag{9}$$

Notice that the Lemma 1 is used such that the LMIs in (9) certify the existence of a solution of (8). This strategy will be used to demonstrate Theorem 3, presented in section 4.

Lemma 2. (as cited by Boyd et al. (1994)). Given a scalar $\lambda > 0$ and any real matrices $\mathcal{O}$ and $\mathcal{U}$ of compatible dimensions, then

$$\mathcal{O}\mathcal{U} + \mathcal{U}^\top \mathcal{O}^\top \leq \lambda \mathcal{O}\mathcal{O}^\top + \lambda^{-1} \mathcal{U}^\top \mathcal{U}. \tag{10}$$

3. DISCRETIZATION OF UNCERTAIN SYSTEMS WITH UNCERTAIN SAMPLING PERIOD

The parameter-dependent matrices $A(\alpha)$ and $B(\alpha)$ described in (7) can be rewritten in terms of homogeneous polynomial matrices $A^{[g]}(\alpha)$ and $B^{[g]}(\alpha)$, formed by a Taylor series expansion of degree $g \in \mathbb{N}^* \times \mathbb{N}^*$ and the residual discretization error $\Delta A^{[g]}(\alpha)$ and $\Delta B^{[g]}(\alpha)$. Such reasoning was based on main ideas presented in Braga et al. (2014), as follows:

$$A(\alpha) = A^{[g]}(\alpha) + \Delta A^{[g]}(\alpha), \;\; B(\alpha) = B^{[g]}(\alpha) + \Delta B^{[g]}(\alpha). \tag{11}$$

The homogeneous polynomials are

$$\begin{aligned} A^{[g]}(\alpha) &= \sum_{n=0}^{g} \prod_{j=1}^{2} \left(\sum_{z=1}^{N_j} \alpha_{jz} \right)^{g-n} \frac{E(\alpha_1)^n}{n!} T(\alpha_2)^n, \\ B^{[g]}(\alpha) &= \sum_{n=1}^{g} \prod_{j=1}^{2} \left(\sum_{z=1}^{N_j} \alpha_{jz} \right)^{g-n} \frac{E(\alpha_1)^{n-1}}{n!} T(\alpha_2)^n F(\alpha_1), \end{aligned} \tag{12}$$

where $N_2 = 2$. Then the residual discretization errors can be written as

$$\begin{aligned} \Delta A^{[g]}(\alpha) &= e^{E(\alpha_1)T(\alpha_2)} - A^{[g]}(\alpha), \\ \Delta B^{[g]}(\alpha) &= \left(\int_0^{T(\alpha_2)} e^{E(\alpha_1)\varsigma} d\varsigma \right) F(\alpha_1) - B^{[g]}(\alpha). \end{aligned} \tag{13}$$

Definitions that allow for a systematic representation of Taylor series terms are presented in Appendixes A and B for more clarity.

Bounds on the discretization errors described in (13) are defined as

$$\theta_A \triangleq \sup_{\alpha\in\Lambda_{N_1}\times\Lambda_2} \left\|\Delta A^{[g]}(\alpha)\right\|_2, \theta_B \triangleq \sup_{\alpha\in\Lambda_{N_1}\times\Lambda_2} \left\|\Delta B^{[g]}(\alpha)\right\|_2, \tag{14}$$

where $\|.\|_2$ represents the 2−norm. An approximation for (θ_A, θ_B) can be obtained by a $(N_1 + 2)$−dimensional off line search in a grid of values of $\alpha \in (\Lambda_{N_1} \times \Lambda_2)$.

4. STABILIZATION

In this section, a new design LMI-based condition is proposed to allocate closed-loop system poles inside a desired disc region in the z-plane. The system covered is an augmented discretized model formulated in terms of the state derivative, such that uncertain sampling period and parametric uncertainties in polytopic form can be propagated from the original continuous-time state space representation. To this end, consider:

Assumption 2. Matrix $E(\alpha_1)$, $\forall\alpha_1 \in \Lambda_{N_1}$, is non-singular.

This assumption has been considered in the linear state derivative designs, as can be seen for instance in Rossi et al. (2018), Beteto et al. (2018) and references.

From (11), the discrete-time model (6) can be rewritten as:

$$\begin{aligned} x((\mathrm{k}+1)T(\alpha_2)) = (A^{[g]}(\alpha) + \Delta A^{[g]}(\alpha))x(\mathrm{k}T(\alpha_2))+ \\ (B^{[g]}(\alpha) + \Delta B^{[g]}(\alpha))u(\mathrm{k}T(\alpha_2))^+. \end{aligned} \tag{15}$$

From Theorem 1 in Rossi et al. (2018), the model (15) can be recast in terms of state derivative feedback $\dot{x}(\mathrm{k}T(\alpha_2))$ as follows

$$\begin{aligned} \dot{x}((\mathrm{k}+1)T(\alpha_2)) = A(\alpha)\dot{x}(\mathrm{k}T(\alpha_2))- \\ A(\alpha)F(\alpha_1)u((\mathrm{k}-1)T(\alpha_2))^+ + A(\alpha)F(\alpha_1)u(\mathrm{k}T(\alpha_2))^+, \end{aligned} \tag{16}$$

which can be formulated as augmented model given by:

$$z(\mathrm{k}+1) = (\hat{A}(\alpha))z(\mathrm{k}) + (\hat{B}(\alpha))u(\mathrm{k})^+, \tag{17}$$

such that $\hat{A}(\alpha) = \hat{A}^{[g]}(\alpha)+\Delta\hat{A}^{[g]}(\alpha)$ and $\hat{B}(\alpha) = \hat{B}^{[g]}(\alpha)+ \Delta\hat{B}^{[g]}(\alpha)$ and

$$\begin{aligned} &z(\mathrm{k}) = \begin{bmatrix} \dot{x}(\mathrm{k}) \\ u(\mathrm{k}-1)^+ \end{bmatrix}, \hat{A}^{[g]}(\alpha) = \begin{bmatrix} A^{[g]}(\alpha) & -A^{[g]}(\alpha)F(\alpha_1) \\ 0 & 0 \end{bmatrix}, \\ &\Delta\hat{A}^{[g]}(\alpha) = \begin{bmatrix} \Delta A^{[g]}(\alpha) & -\Delta A^{[g]}(\alpha)F(\alpha_1) \\ 0 & 0 \end{bmatrix}, \\ &\hat{B}^{[g]}(\alpha) = \begin{bmatrix} A^{[g]}(\alpha)F(\alpha_1) \\ I \end{bmatrix}, \Delta\hat{B}^{[g]}(\alpha) = \begin{bmatrix} \Delta A^{[g]}(\alpha)F(\alpha_1) \\ 0 \end{bmatrix}, \end{aligned} \tag{18}$$

where instant $\mathrm{k}T(\alpha_2)$ is denoted by k for simplicity. Therefore, the state feedback control law is given by:

$$u(\mathrm{k}) = Kz(\mathrm{k}) = [\, K_{\dot{x}} \;\; K_u \,] \begin{bmatrix} \dot{x}(\mathrm{k}) \\ u(\mathrm{k}-1)^+ \end{bmatrix}. \tag{19}$$

The homogenization of $A^{[g]}(\alpha)$ and $A^{[g]}(\alpha)F(\alpha_1)$, as well as of the augmented matrices $\hat{A}^{[g]}(\alpha)$ and $\hat{B}^{[g]}(\alpha)$ can be found in Appendix B.

In face of augmented system matrices, an estimate for the upper bounds of $\left\|\Delta\hat{A}^{[g]}(\alpha)\right\|_2$ and $\left\|\Delta\hat{B}^{[g]}(\alpha)\right\|_2$ in the same way as in (14) can be defined respectively as:

$$\hat{\theta}_A \triangleq \sup_{\alpha\in\Lambda_{N_1}\times\Lambda_2} \left\|\Delta\hat{A}^{[g]}(\alpha)\right\|_2, \hat{\theta}_B \triangleq \sup_{\alpha\in\Lambda_{N_1}\times\Lambda_2} \left\|\Delta\hat{B}^{[g]}(\alpha)\right\|_2. \tag{20}$$

In order to allocate closed-loop system poles in the z-plane, a disc region $\mathcal{D}(\delta, \varrho)$, centered in $\delta + j0$, with radius ϱ, was adopted. This region approximates cardioids formed by the geometric locus with the same damping ratio. The parametrization of δ and ϱ from a conic sector subregion in the s-plane closely follows that presented in Leandro and Kienitz (2019), such that:

$$\delta = e^{\left(\frac{-\phi}{\tan(\phi)}\right)}\cos(-\phi),\;\; \varrho = e^{\left(\frac{-\phi}{\tan(\phi)}\right)}\sin(-\phi), \tag{21}$$

where ϕ is the internal angle of the left half-plane cone region. Moreover, the allocation of the closed-loop poles of (17) in a region $\mathcal{D}(\delta, \varrho)$ guarantees a transient response limited by decay rate in the interval $[\delta - |\varrho|, \delta + |\varrho|]$.

Considering state derivative feedback, the following theorem proposes robust conditions for the stabilization of (17).

Theorem 3. Consider a positive definite symmetric matrices $W_{\mathbb{k}} \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$, $\mathbb{k} \in \mathcal{K}_2(N, q)$, $q \in \mathbb{N}^2$, $G \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$, $Z \in \mathbb{R}^{n_u\times(n_x+n_u)}$, a discretization degree $g \in \mathbb{N}^* \times \mathbb{N}^*$, a Pólya's relaxation degree $p \in \mathbb{N}^2$, $w \triangleq \max\{q + p, g + p\}$, the pair $(\lambda_A, \lambda_B) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $\hat{\Theta} \triangleq \left(\lambda_A\hat{\theta}_A^2 + \lambda_B\hat{\theta}_B^2\right)$, a disc $\mathcal{D}(\delta, \varrho)$ centered in $\delta + j0$ and with radius ϱ, where $|\delta| + \varrho < 1$ and a given scalar parameter $\xi \in (-\varrho, \varrho)$, such that the LMIs in (22) and (23) are feasible.

$$X_{\mathbb{k}} = \sum_{\substack{\mathbb{k}'\in\mathcal{K}_2(N,p)\\ \mathbb{L}\in\mathcal{L}(p)}} \frac{\prod_{n=1}^{2} p_n!}{\mathbb{k}'!} W_{\mathbb{L}} > 0,\; \forall\mathbb{k} \in \mathcal{K}_2(N, q + p), \tag{22}$$

$$\begin{aligned} M_{\mathbb{k}} = \frac{\prod_{n=1}^{2} w_n!}{\mathbb{k}!} M + \sum_{\substack{\mathbb{k}'\in\mathcal{K}_2(N,w-g)\\ \mathbb{L}\in\mathcal{L}(w-g)}} \frac{\prod_{n=1}^{2} (w_n - g_n)!}{\mathbb{k}'!} \tilde{M} + \\ \sum_{\substack{\mathbb{k}'\in\mathcal{K}_2(N,w-q)\\ \mathbb{L}\in\mathcal{L}(w-q)}} \frac{\prod_{n=1}^{2} (w_n - q_n)!}{\mathbb{k}'!} \breve{M} < 0,\; \forall\mathbb{k} \in \mathcal{K}_2(N, w), \end{aligned} \tag{23}$$

where

$$M = \begin{bmatrix} \hat{\Theta}I & -\xi G^{\mathsf{T}} & \xi Z^{\mathsf{T}} & \xi G^{\mathsf{T}} \\ \star & -G - G^{\mathsf{T}} & Z^{\mathsf{T}} & G^{\mathsf{T}} \\ \star & \star & -\lambda_B I & 0 \\ \star & \star & \star & -\lambda_A I \end{bmatrix}, \tag{24}$$

$$\tilde{M} = \begin{bmatrix} \xi He\left\{\hat{A}_{\mathbb{L}}G - \delta G + \hat{B}_{\mathbb{L}}Z\right\} & \hat{A}_{\mathbb{L}}G - \delta G + \hat{B}_{\mathbb{L}}Z & 0 & 0 \\ \star & 0 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix}, \tag{25}$$

$$\breve{M} = diag(\, -\varrho^2 W_{\mathbb{L}} \;\; W_{\mathbb{L}} \;\; 0 \;\; 0 \,). \tag{26}$$

Expressions for $\hat{A}_{\mathbb{L}}$, $\hat{B}_{\mathbb{L}}$ and $W_{\mathbb{L}}$ can be found in Appendixes A and B, namely in (A.8), (B.1), (B.4) and (B.6) - (B.9). Under the above assumptions the gain $K = ZG^{-1}$ ensures the allocation of the closed-loop poles of (17) in the region $\mathcal{D}(\delta, \varrho)$ and the stabilization of the continuous-time system described in (1).

Proof: Considering (A.8) and $\mathbb{L} \in \mathcal{L}(.)$ in Appendix A, the matrix $W_{\mathbb{L}}$ in (22) can then be recast into the following form:

$$\left(\sum_{n=1}^{N_1} \alpha_{1n}\right)^p \left(\sum_{\bar{n}=1}^{2} \alpha_{2\bar{n}}\right)^p W^{[g]}(\alpha) = \sum_{\mathbb{k} \in \mathcal{K}_2(N,q+p)} \alpha^{\mathbb{k}} X_{\mathbb{k}}. \quad (27)$$

Given that $\alpha_1 \in \Lambda_{N_1}$ and $\alpha_2 \in \Lambda_2$, Λ_{N_1} defined in (3), if $X_{\mathbb{k}} > 0$ $\forall \mathbb{k} \in \mathcal{K}_2(N, q+p)$, then $W^{[q]}(\alpha) > 0$ holds $\forall \alpha \in (\Lambda_{N_1} \times \Lambda_2)$.

Now, define $\bar{A}(\alpha) = \hat{A}^{[g]}(\alpha) - \delta I + \hat{B}^{[g]}(\alpha) K$ and use Lemma 1 for the following choice of matrices:

$$N_U = \begin{bmatrix} I & 0 & 0 \\ \bar{A}^{\mathsf{T}}(\alpha) & K^{\mathsf{T}} & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \; N_V = \begin{bmatrix} I & 0 & 0 \\ -\xi I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad (28)$$
$$U = \begin{bmatrix} \bar{A}^{\mathsf{T}}(\alpha) & -I & K^{\mathsf{T}} & I \end{bmatrix}, \; V = \begin{bmatrix} \xi I & I & 0 & 0 \end{bmatrix}.$$

One observes that (8) is equivalent to multiplying (23) by $\alpha^{\mathbb{k}}$ and to sum up for $\mathbb{k} \in \mathcal{K}_2(N, w)$. In order to do that, assume $H = diag\left(-\varrho^2 W(\alpha) \; W(\alpha) \; 0 \; 0\right)$, $X = G$ in (8) and consider $KG = Z$.

From matrices in (28) and considering $N_V^{\mathsf{T}} H N_V < 0$ in (9), one has

$$\begin{bmatrix} \hat{\Theta} I - \varrho^2 W^{[q]}(\alpha) + \xi^2 W^{[q]}(\alpha) & 0 & 0 \\ 0 & -\lambda_B I & 0 \\ 0 & 0 & -\lambda_A I \end{bmatrix} < 0. \quad (29)$$

For $W^{[q]}(\alpha) > 0$ and $\hat{\Theta} \to 0_+$, then $|\xi| < \varrho$.

Additionally from (28) the condition $N_U^{\mathsf{T}} H N_U < 0$ in (9) can be written as

$$\begin{bmatrix} \hat{\Theta} I - \varrho^2 W^{[q]}(\alpha) & 0 & 0 \\ 0 & -\lambda_B I & 0 \\ 0 & 0 & -\lambda_A I \end{bmatrix} + \Xi^{\mathsf{T}} W^{[q]}(\alpha)^{-1} \Xi < 0, \quad (30)$$

where $\Xi = \begin{bmatrix} W^{[q]}(\alpha)\bar{A}^{\mathsf{T}}(\alpha) & W^{[q]}(\alpha) K^{\mathsf{T}} & W^{[q]}(\alpha) \end{bmatrix}$. Using Schur's complement and changing the second and fourth columns, and doing the same for the second and fourth lines, yields:

$$\begin{bmatrix} \hat{\Theta} I - \varrho^2 W^{[q]}(\alpha) & \bar{A}(\alpha) W^{[q]}(\alpha) & 0 & 0 \\ W^{[q]}(\alpha)\bar{A}^{\mathsf{T}}(\alpha) & -W^{[q]}(\alpha) & W^{[q]}(\alpha) & W^{[q]}(\alpha) K^{\mathsf{T}} \\ 0 & W^{[q]}(\alpha) & -\lambda_A I & 0 \\ 0 & K W^{[q]}(\alpha) & 0 & -\lambda_B I \end{bmatrix} < 0. \quad (31)$$

Multiplying by $-I$, applying Schur's complement with respect to $\lambda_B I$ and given that $\hat{\Theta} = (\lambda_A \hat{\theta}_A^2 + \lambda_B \hat{\theta}_B^2)$, inequality (31) results in:

$$\begin{bmatrix} \varrho^2 W^{[q]}(\alpha) - \lambda_A \hat{\theta}_A^2 I & -\bar{A}(\alpha) W^{[q]}(\alpha) & 0 \\ \star & W^{[q]}(\alpha) & -W^{[q]}(\alpha) \\ \star & \star & \lambda_A I \end{bmatrix} - \lambda_B \begin{bmatrix} -\hat{\theta}_B I \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -\hat{\theta}_B I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -W^{[q]}(\alpha) K^{\mathsf{T}} \\ 0 \end{bmatrix} \lambda_B^{-1} \begin{bmatrix} 0 & -K W^{[q]}(\alpha) & 0 \end{bmatrix} > 0. \quad (32)$$

Considering Lemma 2 with $\mathcal{O}^{\mathsf{T}} = \begin{bmatrix} -\hat{\theta}_B I & 0 & 0 \end{bmatrix}$, $\mathcal{U} = \begin{bmatrix} 0 & -K W^{[q]}(\alpha) & 0 \end{bmatrix}$, $\lambda = \lambda_B$ and the upper bounds defined in (20), then inequality (32) can be modified replacing $\hat{\theta}_B I$ by $\Delta \hat{B}^{[g]}(\alpha)$ to obtain the more stringent, but more useful condition

$$\begin{bmatrix} \varrho^2 W^{[q]}(\alpha) - \lambda_A \hat{\theta}_A^2 I & -\bar{A}(\alpha) W^{[q]}(\alpha) & 0 \\ \star & W^{[q]}(\alpha) & -W^{[q]}(\alpha) \\ \star & \star & \lambda_A I \end{bmatrix} - \begin{bmatrix} 0 & \Delta B^{[g]}(\alpha) K W^{[q]}(\alpha) & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix} > 0. \quad (33)$$

Now, consider the same procedure from (32), apply Lemma 2, proceed in the same way for $\hat{\theta}_A$ and $\Delta \bar{A}^{[g]}(\alpha)$, replace $\bar{A}(\alpha)$ by $\hat{A}^{[g]}(\alpha) - \delta I + \hat{B}^{[g]}(\alpha) K$, then one finds:

$$\begin{bmatrix} \varrho^2 W^{[q]}(\alpha) & (\hat{A}(\alpha) - \delta I + \hat{B}(\alpha) K) W^{[q]}(\alpha) \\ \star & W^{[q]}(\alpha) \end{bmatrix} > 0, \quad (34)$$

where $\hat{A}(\alpha)$ and $\hat{B}(\alpha)$ are defined in (11) and (18). Inequality (34) can then be recast into the following form:

$$\begin{bmatrix} -\varrho^2 W^{[q]}(\alpha) & \hat{\mathbb{A}}(\alpha) W^{[q]}(\alpha) - \delta W^{[q]}(\alpha) \\ \star & -W^{[q]}(\alpha) \end{bmatrix} < 0, \quad (35)$$

where $\hat{\mathbb{A}}(\alpha) = \hat{A}^{[g]}(\alpha) + \hat{B}^{[g]}(\alpha) K$. As shown in Chilali and Gahinet (1996), the augmented matrix $\hat{\mathbb{A}}(\alpha)$ is $\mathcal{D}$-stable if and only if there exist $W^{[q]}(\alpha) > 0$, $\forall \alpha \in \Lambda_{N_1} \times \Lambda_2$, such that

$$\gamma \otimes W^{[q]}(\alpha) + \beta \otimes (\hat{\mathbb{A}}(\alpha) W^{[q]}(\alpha)) + \beta^{\mathsf{T}} \otimes (\hat{\mathbb{A}}(\alpha) W^{[q]}(\alpha))^{\mathsf{T}} < 0. \quad (36)$$

Herein the symbol $\otimes$ denotes the Kronecker product of matrices. For a particular case where region $\mathcal{D}$ is a circle of radius ϱ and centre $(\delta, 0)$, then γ and β are

$$\gamma = \begin{bmatrix} -\varrho & -\delta \\ -\delta & -\varrho \end{bmatrix}, \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (37)$$

Thus (36) is equivalent to (35), which ensures the closed-loop pole allocation of the augmented matrix $\hat{\mathbb{A}}$ in a region $\mathcal{D}(\delta, \varrho)$ $\forall \alpha \in (\Lambda_{N_1} \times \Lambda_2)$. □

Remark 1. The proof of the stabilization of the continuous-time system described in (1) can be found in Appendix C.

Remark 2. In the context of continuous-time case, the pole regions defined by the geometric locus with the same decay rate, a minimum damping ratio and a maximum damped natural frequency, are convex (Chilali and Gahinet, 1996). The discrete-time regions defined from this regions are not necessarily convex. An alternative could be derived taking into account inner-approximations by well known LMI regions, as proposed in Wisniewski et al. (2019). However, the approach adopted herein leads to simpler design and can be appropriate to handle closed-loop specifications, as will be illustrated in Section 5.

5. NUMERICAL EXPERIMENTS

The numerical experiment in what follows was implemented in MATLAB® Software, version 7 (R2010b), using YALMIP (Lofberg (2004)) and SeDuMi (Sturm (1999)).

Consider an active suspension system for a car seat plant described by Faria et al. (2009), whose model is described by (1). The corresponding system matrices are:

$$E(\mu) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-c_1 - c_2}{M} & \frac{c_2}{M} & \frac{-b_1 - b_2}{M} & \frac{b_2}{M} \\ \frac{c_2}{\mu} & -\frac{c_2}{\mu} & \frac{b_2}{\mu} & -\frac{b_2}{\mu} \end{bmatrix}, F(\mu) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{M} & -\frac{1}{M} \\ 0 & \frac{1}{\mu} \end{bmatrix} \tag{38}$$

and the state vector $x(t) = [\, x_1(t)\ x_2(t)\ \dot{x}_1(t)\ \dot{x}_2(t) \,]^{\mathsf{T}}$. The states x_1 and x_2 denote the vertical displacements of masses M and μ, respectively, and $\dot{x}_1$ and $\dot{x}_2$ represent the corresponding velocities. The model consists of a car mass M and driver-plus-seat mass μ. Vertical vibrations caused by a street may be partially attenuated by shock absorbers (stiffness c_1 and damping b_1). Undesirable vibrations can also be reduced by appropriately mounted car seat suspension elements (stiffness c_2 and damping b_2). Finally, damping of vibration of the masses M and μ can be increased by changing the control inputs $u_1(t)$ and $u_2(t)$. The parameters values are: $M = 1500$ kg, $c_1 = 4 \times 10^4$ N/m, $c_2 = 5 \times 10^3$ N/m, $b_1 = 4 \times 10^3$ Ns/m and $b_2 = 5 \times 10^2$ Ns/m. Consider both an uncertain seat-plus-driver mass, such that $\mu \in [70, 120]$ kg, and an uncertain sampling period, such that T, defined in (4), varies inside the interval $[\,0.1,\ 0.25\,]$ s. In this quarter car model, the acceleration $\ddot{x}_2(\mathrm{k}T(\alpha_2))$ is used to measured the driver comfort.

Notice that the system whose matrices are described in (38) is stable. However, the augmented matrix $\hat{A}(\alpha)$ presented in (17) becomes unstable for $g \leq (3, 3)$. Figure 1 (a) illustrates the open-loop transient response in accordance with different g. The discretization degrees $g \geq (4, 4)$ are shown in Figure 1 (b) with more details for better clarity.

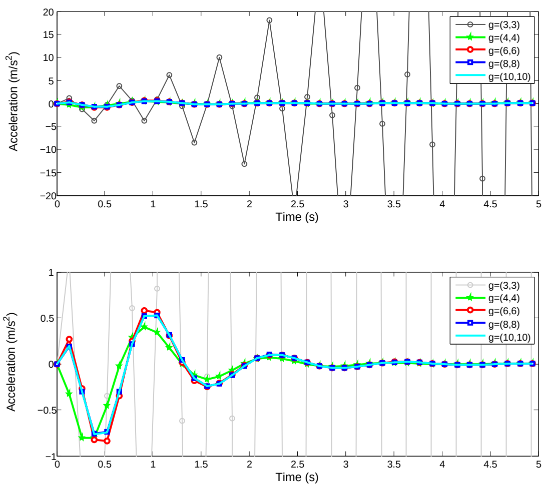

Figure 1 Open-loop transient response for $\alpha_{11} = 0.4$, $\alpha_{12} = 0.6$, $\alpha_{21} = 0.8$, $\alpha_{22} = 0.2$ and $x_0 = [\ 0.2\ 0.1\ 0\ 0\ 0\ ]^{\mathsf{T}}$.

Although this is not the point of this work, the Figure 2 illustrates the use of the Pólya's relaxation, according to which as degree p increases, the condition converges towards a solution whenever it exits.

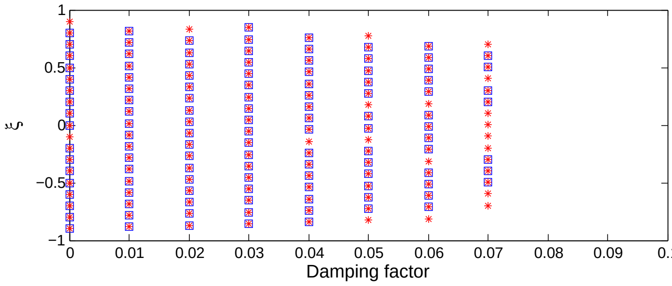

Figure 2 $*$: Feasible cases associated with $q = (4, 4)$ and $p = (4, 4)$. $\square$: Feasible cases associated with $q = (1, 1)$ and $p = (0, 0)$. In both $g = (10, 10)$ was considered.

The main objective of this example is to design a state derivative feedback control law as described in (19), in order to place the closed-loop poles of the system (38) in a desired region $\mathcal{D}$-stable for any values of the uncertain parameters μ and T. This approach dispenses with the need for a preliminary synthesis of a state feedback control law, so that the resulting controller can be readily implemented to assure the stability of the continuous-time plant by means of the digital controller.

In order to apply Theorem 3, a damping factor of 0.2 is chosen. Additionally, the following parameters were adopted: $q = (\,1,\ 1\,)$, $p = (\,1,\ 1\,)$, $\xi = 0.6$ and $\xi = -0.6$. The choice $q = (\,1,\ 1\,)$ implies in a homogeneous polynomially parameter-dependent Lyapunov matrix, whose monomials are presented in Appendix A, (A.9). From (20) and using $g = (\,13,\ 13\,)$, the bounds on the discretization errors are:

$$\hat{\theta}_{\hat{A}} = 2.2142 \times 10^{-6},\ \hat{\theta}_{\hat{B}} = 7.6391 \times 10^{-9}. \tag{39}$$

Considering $\xi = 0.6$ and $\xi = -0.5$, two gain matrices $K_{\xi=0.6}$ and $K_{\xi=-0.5}$, respectively, were obtained. The result obtained for $\xi = 0.6$ is presented in (40):

$$K_{\xi=0.6} = \begin{bmatrix} 6181.9 & 391.4 & -437.9 & -75.1 & -0.3 & 0.4 \\ -502.7 & -391.1 & -118.2 & -39.9 & -0.1 & 0.4 \end{bmatrix}. \tag{40}$$

Considering the same parameters above and $\xi = -0.5$, the following gain matrix was obtained:

$$K_{\xi=-0.5} = \begin{bmatrix} 4207.9 & 331.2 & 72.5 & -7.5 & 0.2 & 0.0 \\ -755.8 & -288.7 & -137.6 & -31.7 & -0.1 & 0.3 \end{bmatrix}. \tag{41}$$

Figure 3 shows the closed-loop pole allocation with $K_{\xi=-0.5}$ in (41). As can seen, all the closed-loop poles associated with gain matrix in (41) are inside the disc region of radius $\varrho = 0.7409$ and centered at $(0, 0.1512)$, which approximates the cardioid related to damping factor 0.2.

Figures 4 shows the closed-loop transient response for $K_{\xi=-0.5}$ and $K_{\xi=0.6}$, respectively, compared with open-loop transient response. As can be seen both closed-loop systems present better transient response when compared with open-loop system.

Figure 5 illustrates the control input correspondent to respective gain matrix, $K_{\xi=-0.5}$ and $K_{\xi=0.6}$. It is worth mentioning the degree of freedom given by the search in scalar parameter ξ, contributing to feasible conditions, as well as a more convenient transient responses.

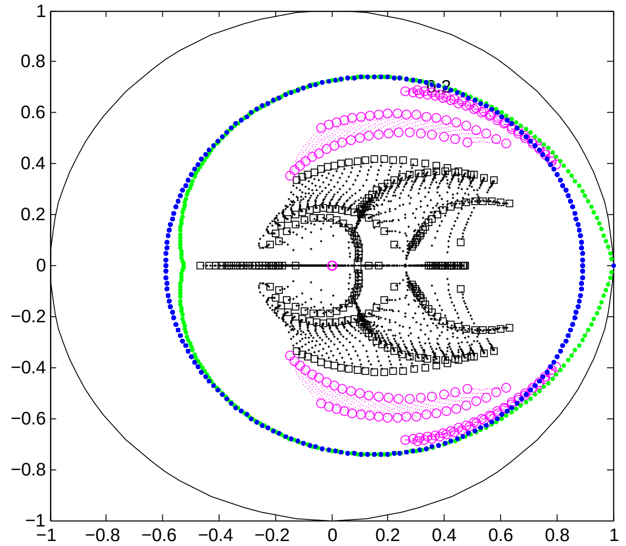

Figure 3 •: Open-loop system eigenvalues. +: Closed-loop system eigenvalues. ..: Region $\mathcal{D}(0.1512, 0.7409)$. ..: Cardioid corresponding to damping factor 0.2. The eigenvalues corresponding to polytope vertices are represented by ○ and □.

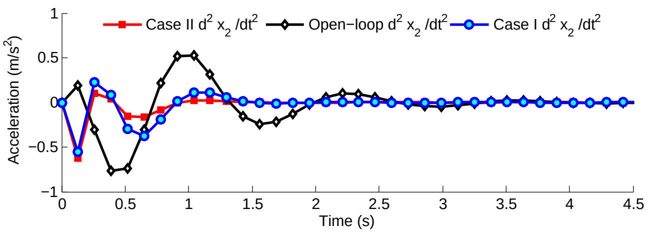

Figure 4 Open-loop and closed-loop transient responses. Case I: $(\hat{A}(\alpha) + \hat{B}(\alpha)K_{\xi=-0.5})$ and Case II: $(\hat{A}(\alpha) + \hat{B}(\alpha)K_{\xi=0.6})$ Parameters: $\alpha_{11} = 0.4$, $\alpha_{12} = 0.6$, $\alpha_{21} = 0.8$, $\alpha_{22} = 0.2$ and $x_0 = [\,0.2\ 0.1\ 0\ 0\ 0]^{\mathsf{T}}$.

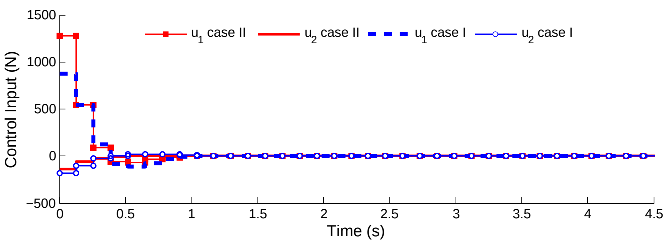

Figure 5 Case I: Control input related to $(\hat{A}(\alpha) + \hat{B}(\alpha)K_{\xi=-0.5})$. Case II: Control input related to $(\hat{A}(\alpha) + \hat{B}(\alpha)K_{\xi=0.6})$.

6. CONCLUSION

This work addressed robust $\mathcal{D}$-stability via state derivative feedback controllers for continuous-time uncertain systems. Polytopic parameter uncertainties in the original state space representation and uncertain sampling period were propagated to the discretized model. An LMI-based control synthesis condition composed of homogeneous polynomial matrices of arbitrary degree can be employed by means of a state derivative augmented model.

For illustration, a numerical example was presented. The state derivative feedback was employed in order to improve closed-loop transient response bringing more driver comfort, illustrated by means of the state $\ddot{x}(\mathrm{k}T(\alpha_2))$ transient response. Moreover, the numerical example illustrated that from the same region $\mathcal{D}$-stable, different input controls can be chosen by means of different values of the scalar parameter ξ with better damping and settling time when compared each other.

Future work is concerned with the design of robust state derivative feedback control law taking into account uncertain input delay and the validation of the proposed condition in experimental settings.

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Appendix A. NOTATIONS AND DEFINITIONS

The following definitions allow the representation of a Taylor series terms of an arbitrary degree.

Consider a vector of parameters $\zeta_s = (\zeta_{s1}, ..., \zeta_{sN_s}) \in \Lambda_{N_s}$, $s \in \{1, ..., \omega\}$ and $\zeta = (\zeta_{11}, ..., \zeta_{1N_1}, ..., \zeta_{\omega 1}, ..., \zeta_{\omega N_\omega}) \in (\Lambda_{N_1} \times ... \times \Lambda_{N_\omega})$. For the sake of generality, a degree $g = (g_1, ..., g_\omega)$, $g_s \in \mathbb{N}^*$ was adopted.

- Set $\mathcal{K}(N_s, g_s)$ is given by

$$\mathcal{K}(N_s, g_s) \triangleq \left\{k_s = (k_{s1} \cdots k_{sN_s}) \in \mathbb{N}^{N_s} | k_{s1} + \cdots \right. \\ \left. \cdots + k_{sN_s} = g_s\right\}. \tag{A.1}$$

To exemplify, for $\omega = 1$ consider a dynamic matrix $E(\alpha_1)$ in (1), such that

$$E(\alpha_1) = \sum_{i=1}^{N_1} \alpha_{1i} E_i, \ \alpha_1 \in \Lambda_{N_1}, \tag{A.2}$$

and a case with $N_1 = 2$ and $g_1 = 3$, then

$$E(\alpha_1)E(\alpha_1)E(\alpha_1) = \alpha_{11}^0\alpha_{12}^3 E_2E_2E_2 + \alpha_{11}^1\alpha_{12}^2(E_1E_2E_2 + E_2E_1E_2 + E_2E_2E_1) + \alpha_{11}^2\alpha_{12}^1(E_1E_1E_2 + E_1E_2E_1 + E_2E_1E_1) + \alpha_{11}^3\alpha_{12}^0 E_1E_1E_1. \tag{A.3}$$

The exponents of the parameters $\alpha_1 \in \Lambda_{N_1}$ are represented as follows:

$$k_1 \in \mathcal{K}(2,3) = \{(03), (12), (21), (30)\}. \tag{A.4}$$

- Set $\mathcal{K}_\omega(N, g)$ is formed by the Cartesian product of all N_s-tuples of the ω unit simplexes and is defined by

$$\mathcal{K}_\omega(N, g) \triangleq \{N = (N_1, \cdots, N_\omega), g = (g_1, \cdots, g_\omega), \\ \Bbbk = (k_{11} \cdots k_{1N_1} \cdots k_{\omega 1} \cdots k_{\omega N\omega}) \in \mathbb{N}^{N_1 \times \cdots \times N_\omega} | k_{s1} + \\ \cdots + k_{sN_s} = g_s\}. \tag{A.5}$$

To exemplify, consider $T(\alpha_2)$ in (4) and (A.2). Then, for $N_1 = 2$ the following product can be written

$$E(\alpha_1)T(\alpha_2) = \alpha_{11}^1\alpha_{12}^0\alpha_{21}^1\alpha_{22}^0 E_1T_1 + \alpha_{11}^1\alpha_{12}^0\alpha_{21}^0\alpha_{22}^1 E_1T_2 + \\ \alpha_{11}^0\alpha_{12}^1\alpha_{21}^1\alpha_{22}^0 E_2T_1 + \alpha_{21}^0\alpha_{22}^1\alpha_{21}^0\alpha_{22}^1 E_2T_2, \tag{A.6}$$

where

$$\Bbbk \in \{(1010), (1001), (0110), (0101)\}. \tag{A.7}$$

Herein, inspired by Oliveira and Peres (2007), the homogeneous polynomially parameter-dependent Lyapunov matrix of arbitrary degree, $q \in \mathbb{N} \times \mathbb{N}$, adopted in this work is defined as

$$W^{[q]}(\alpha) = \sum_{\Bbbk \in \mathcal{K}_2(N,q)} \alpha^{\Bbbk} W_{\Bbbk}, \ \alpha \in \Lambda_N \times \Lambda_2. \tag{A.8}$$

As an illustration, given $N_1 = 2$, $q = (1, 1)$, one has

$$W^{[q]}(\alpha) = \alpha_{12}^1\alpha_{22}^1 W_{(0101)} + \alpha_{12}^1\alpha_{21}^1 W_{(0110)} + \\ \alpha_{11}^1\alpha_{22}^1 W_{(1001)} + \alpha_{11}^1\alpha_{21}^1 W_{(1010)}. \tag{A.9}$$

- Set $\mathcal{R}(k_{s1}...k_{sN_s})$ is formed by

$$\mathcal{R}(k_{s1} \cdots k_{sN_s}) \triangleq \{\mathbb{r} = (\mathfrak{r}_1 \cdots \mathfrak{r}_{g_s}) \in \mathbb{N}^{g_s} | \forall i \in \{1, \cdots \\ \cdots, N_s\}, k_i = \#\{j \in \{1, \cdots, g_s\} | \mathfrak{r}_j = i\}\}, \tag{A.10}$$

where # denotes the number of elements of a set.

To exemplify, consider (A.3) and from the tuples in (A.4), one has the sets

$$\mathcal{R}(03) = \{(222)\}, \ \mathcal{R}(12) = \{(122), (212), (221)\}, \\ \mathcal{R}(21) = \{(112), (121), (211)\}, \ \mathcal{R}(30) = \{(111)\}, \tag{A.11}$$

which correspond to the matrices subindex in (A.3).

Now, consider the set $\mathcal{K}_\omega(N, \bar{g})$, $\bar{g} = (\bar{g}_1, ..., \bar{g}_\omega)$, formed by the following $(N_1 + ... + N_\omega)$-tuples: $(k'_{11}...k'_{1N_1}...k'_{\omega 1}...k'_{\omega N_\omega})$, such that $\bar{g}_s \in \mathbb{N}$, $\bar{g}_s \leq g_s$. For each $\Bbbk \in \mathcal{K}_\omega(.)$, define:

- Set $\mathcal{L}(\bar{g})$ is given by:

$$\mathcal{L}(\bar{g}) \triangleq \\ \{\mathbb{L} = (l_{11} \cdots l_{1N_1} \cdots l_{\omega 1} \cdots l_{\omega N_\omega}) \in \mathbb{N}^{N_1 \times \cdots \times N_\omega} \\ |\forall \Bbbk' \in \mathcal{K}_\omega(N, \bar{g}), \mathbb{L} = \Bbbk - \Bbbk', l_{si} \geq 0, i \in \{1, \cdots, N_s\}\}. \tag{A.12}$$

The set $\mathcal{L}(\bar{g})$ also depends on $\Bbbk$, which was omitted to its notation for clarity.

To exemplify, from (A.4) consider $k_1 = (12)$. Assume $\bar{g}_1 = 1$, yielding $k' \in \mathcal{K}(2, 1) = \{(01), (10)\}$. Thus one has

$$\mathbb{L} \in \mathcal{L}(\bar{g}_1) = \{(11), (02)\}. \tag{A.13}$$

- Set $\mathcal{T}(k_{s1}...k_{sN_s})$ is given by

$$\mathcal{T}(k_{s1} \cdots k_{sN_s}) \triangleq \{\mathbb{i}_s = (\mathbb{i}_{s1} \cdots \mathbb{i}_{sN_s}) \in \mathbb{N}^{N_s} | \forall k_{sj} > 0, \\ j \in \{1, \cdots, N_s\}, \mathbb{i}_{sj} = 1, \mathbb{i}_{sN_s - j} = 0\} \tag{A.14}$$

To exemplify, from (A.13) set $\mathcal{T}$ is:

$$\mathbb{i}_1 \in \mathcal{T}(11) = \{(10), (01)\}, \mathbb{i}_1 \in \mathcal{T}(02) = \{(01)\}. \tag{A.15}$$

Appendix B. HOMOGENEOUS POLYNOMIAL DISCRETIZED MATRICES

Using what was described above, one can present expressions for homogeneous polynomial matrices of degree $g \in \mathbb{N}^* \times \mathbb{N}^*$. Factorizing the homogeneous polynomials in (12) by $\alpha^{\Bbbk}$, one has

$$A^{[g]}(\alpha)=\sum_{\mathbb{k}\in\mathcal{K}_2(N,g)}\alpha^{\mathbb{k}}\Bigg(\underbrace{\sum_{n=0}^{g}\sum_{\substack{\mathbb{k}'\in\mathcal{K}_2(N,g-n)\\ \mathbb{L}\in\mathcal{L}(g-n)}}\frac{T_1^{l_{21}}T_2^{l_{22}}}{l_{21}!l_{22}!}}\underbrace{\sum_{(\mathfrak{r}_{11}\ldots\mathfrak{r}_{1g_1})\in\mathcal{R}(l_{11}\ldots l_{1N_1})}\frac{((g-n)!)^2}{\mathbb{k}'!}E_{\mathfrak{r}_{11}}\ldots E_{\mathfrak{r}_{1g_1}}}_{\mathcal{A}_{\mathbb{k}}}\Bigg), \tag{B.1}$$

then, in a compact form

$$A^{[g]}(\alpha)=\sum_{\mathbb{k}\in\mathcal{K}_2(N,g)}\alpha^{\mathbb{k}}\mathcal{A}_{\mathbb{k}}. \tag{B.2}$$

Using the definitions above, $A^{[g]}(\alpha)F(\alpha_1)$ can be written as

$$\begin{aligned}
A^{[g]}(\alpha)F(\alpha_1) &= F(\alpha_1)+T(\alpha_2)E(\alpha_1)F(\alpha_1)+\\
&\frac{T^2(\alpha_2)}{2!}E^2(\alpha_1)F(\alpha_1)+\ldots+\frac{T^g(\alpha_2)}{g!}E^g(\alpha_1)F(\alpha_1),\\
&=\prod_{j=1}^{2}\left(\sum_{n=1}^{N_j}\alpha_{jn}\right)^{g}F(\alpha_1)+\prod_{j=1}^{2}\left(\sum_{n=1}^{N_j}\alpha_{jn}\right)^{g-1}\times\\
&T(\alpha_2)E(\alpha_1)F(\alpha_1)+\frac{T^g(\alpha_2)}{g!}E^g(\alpha_1)F(\alpha_1),\\
&=\sum_{\mathbb{k}\in\mathcal{K}_2((N,2),(g+1,g))}\alpha^{\mathbb{k}}\Bigg(\sum_{\substack{\mathbb{k}'\in\mathcal{K}_2(N,g)\\ \mathbb{L}\in\mathcal{L}(g)}}\sum_{\mathbb{i}_{1j}\in\mathcal{T}(l_{11}\ldots l_{1N_1})}\frac{(g!)^2}{\mathbb{k}'!}\times\\
&F_j+\ldots+\sum_{\substack{\mathbb{k}'\in\mathcal{K}_2(N,g-n)\\ \mathbb{L}\in\mathcal{L}(g-n)}}\sum_{\mathbb{i}_{1j}\in\mathcal{T}(l_{11}\ldots l_{1N_1})}\sum_{\mathbb{r}\in\mathcal{R}(l_{11}\ldots l_{1N_1}-i_1)}\times\\
&\frac{((g-n)!)^2}{\mathbb{k}'!}\frac{T^{l_{21}}T^{l_{22}}}{l_{21}!l_{22}!}E_{\mathbb{r}}F_j+\ldots+\sum_{\substack{\mathbb{k}'\in\mathcal{K}_2(N,0)\\ \mathbb{L}\in\mathcal{L}(0)}}\sum_{\mathbb{i}_{1j}\in\mathcal{T}(l_{11}\ldots l_{1N_1})}\times\\
&\sum_{\mathbb{r}\in\mathcal{R}(l_{11}\ldots l_{1N_1}-\mathbb{i}_1)}\frac{(0!)^2}{\mathbb{k}'!}\frac{T^{l_{21}}T^{l_{22}}}{l_{21}!l_{22}!}E_{\mathbb{r}}F_j\Bigg)
\end{aligned} \tag{B.3}$$

$$A^{[g]}(\alpha)F(\alpha_1)=\sum_{\mathbb{k}\in\mathcal{K}_2((N_1,2)(g+1,g))}\alpha^{\mathbb{k}}\times\underbrace{\Bigg(\sum_{n=0}^{g}\sum_{\substack{\mathbb{k}'\in\mathcal{K}_2(N,g-n)\\ \mathbb{L}\in\mathcal{L}(g-n)}}\frac{T_1^{l_{21}}T_2^{l_{22}}}{l_{21}!l_{22}!}}\underbrace{\sum_{\mathbb{i}_{1j}\in\mathcal{T}(l_{11}\ldots l_{1N_1})}\sum_{(\mathfrak{r}_{11}\ldots\mathfrak{r}_{1g_1})\in\mathcal{R}(l_{11}\ldots l_{1N_1}-\mathbb{i}_1)}\frac{((g-n)!)^2}{\mathbb{k}'!}E_{\mathbb{r}}F_j\Bigg)}_{\dot{\mathcal{A}}_{\mathbb{k}}} \tag{B.4}$$

then, in a compact form

$$A(\alpha)F(\alpha_1)=\sum_{\mathbb{k}\in\mathcal{K}_2((N_1,2),(g+1,g))}\alpha^{\mathbb{k}}\dot{\mathcal{A}}_{\mathbb{k}}. \tag{B.5}$$

Finally, polynomial augmented matrices $\hat{A}^{[g]}(\alpha)$ and $\hat{B}^{[g]}(\alpha)$ have different degrees, thus should be homogenized as

$$\hat{A}^{[g]}(\alpha)=\sum_{\mathbb{k}\in\mathcal{K}_2((N_1,2),(g+1,g))}\alpha^{\mathbb{k}}\hat{A}_{\mathbb{k}}, \tag{B.6}$$

where

$$\hat{A}_{\mathbb{k}}=\begin{bmatrix}\displaystyle\sum_{\substack{\mathbb{k}'\in\mathcal{K}_2((N_1,2),(1,0))\\ \mathbb{L}\in\mathcal{L}(1,0)}}\frac{1!0!}{\mathbb{k}'!}\mathcal{A}_{\mathbb{L}} & -\dot{\mathcal{A}}_{\mathbb{k}}\\ 0 & 0\end{bmatrix}; \tag{B.7}$$

and

$$\hat{B}^{[g]}(\alpha)=\sum_{\mathbb{k}\in\mathcal{K}_2((N_1,2),(g+1,g))}\alpha^{\mathbb{k}}\hat{B}_{\mathbb{k}}, \tag{B.8}$$

where

$$\hat{B}_{\mathbb{k}}=\begin{bmatrix}\dot{\mathcal{A}}_{\mathbb{L}}\\ \displaystyle\sum_{\mathbb{k}'\in\mathcal{K}_2((N_1,2),(g+1,g))}\frac{(g+1)!g!}{\mathbb{k}'!}I\end{bmatrix}. \tag{B.9}$$

Appendix C. CLOSED-LOOP UNCERTAIN CONTINUOUS TIME SYSTEM STABILITY

Following the exposed in Braga et al. (2014), for any $\alpha\in(\Lambda_{N_1}\times\Lambda_2)$ and a given sampling period $T(\alpha_2)$, the solution of (16) over the interval $t\in\left[\,\mathrm{k}T(\alpha_2)^+,\ (\mathrm{k}+1)T(\alpha_2)\,\right]$ is given by

$$x(t)=e^{E(\alpha_1)(t-\mathrm{k}T(\alpha_2))}\dot{x}(\mathrm{k}T(\alpha_2))-(e^{E(\alpha_1)(t-\mathrm{k}T(\alpha_2))}F(\alpha_1))u((\mathrm{k}-1)T(\alpha_2))^+ +(e^{E(\alpha_1)(t-\mathrm{k}T(\alpha_2))}F(\alpha_1))u(\mathrm{k}T(\alpha_2))^+ \tag{C.1}$$

Taking the supremum of (C.1) and using triangle inequality, one has:

$$\sup_{t\in[\mathrm{k}T(\alpha_2)^+,(\mathrm{k}+1)T(\alpha_2)]}\|x(t)\|\leq\sup_{t\in[\mathrm{k}T(\alpha_2)^+,(\mathrm{k}+1)T(\alpha_2)]}\left\|e^{E(\alpha_1)(t-\mathrm{k}T(\alpha_2))}\right\|\|\dot{x}(\mathrm{k}T(\alpha_2))\|+\sup_{t\in[\mathrm{k}T(\alpha_2)^+,(\mathrm{k}+1)T(\alpha_2)]}\left\|e^{E(\alpha_1)(t-\mathrm{k}T(\alpha_2))}F(\alpha_1)\right\|\left\|u(\mathrm{k}T(\alpha_2))^+-u((\mathrm{k}-1)T(\alpha_2))^+\right\| \tag{C.2}$$

Thus, from (C.2) and using (13), it is possible to write

$$\sup_{t\in[\mathrm{k}T(\alpha_2)^+,(\mathrm{k}+1)T(\alpha_2)]}\|x(t)\|\leq\left\|A^{[g]}(\alpha)+\Delta A^{[g]}(\alpha)\right\|\|\dot{x}(\mathrm{k}T(\alpha_2))\|+\left\|\dot{A}^{[g]}(\alpha)+\Delta\dot{A}^{[g]}(\alpha)\right\|\left\|u(\mathrm{k}T(\alpha_2))^+-u((\mathrm{k}-1)T(\alpha_2))^+\right\| \tag{C.3}$$

Where $\dot{A}^{[g]}(\alpha)$ and $\Delta\dot{A}^{[g]}(\alpha)$ denotes $A(\alpha)^{[g]}F(\alpha_1)$ and $\Delta A(\alpha)^{[g]}F(\alpha_1)$, respectively. From (C.3), as $z(\mathrm{k}T(\alpha_2))$ in (17), and equivalently, $\dot{x}(\mathrm{k}T(\alpha_2))$, $u(\mathrm{k}T(\alpha_2))^+$ and $u((\mathrm{k}-1)T(\alpha_2))^+$ converge to zero as $\mathrm{k}\to\infty$, then $x(t)\to 0$ as $t\to\infty$, and the asymptotic closed-loop stability of the continuous-time model (1) with state derivative feedback control law (19) is ensured.