DOI: 10.48011/asba.v2i1.1562

Discrete-time Left-coprime Factors for LPV/LFR Systems

Luiz Benicio Degli Esposte Rosa * **Renan Lima Pereira** *

* *Instituto Tecnológico de Aeronáutica, Praça Marechal Eduardo Gomes, Vila das Acácias, 12228-900 São José dos Campos, SP, Brazil, E-mails:engbenicio@gmail.com, renanrlp@ita.br*

Abstract: This paper presents novel LMI-based conditions to address the discrete-time left-coprime factorization problem for linear parameter-varying (LPV) systems using linear fractional representation (LFR). The conditions have been derived using a special structure for the output injection approach, which from a given observation law allows to synthesize left-coprime factors via $\mathscr{H}_2$ filtering problem. An important characteristic of the proposed method is the ability to recovery the normalized coprime factorization notion as a particular case and obtain less conservative coprime factorizations. A numerical example demonstrates the effectiveness of the proposed conditions in comparison to similar approaches.

Keywords: Discrete-time LPV systems, Linear fractional representation, Left-coprime factorization, LMIs.

1. INTRODUCTION

Coprime factorization descriptions have been playing a significant role in modern control theory. In a practical application sense, the distinguished book Vidyasagar (1985) and the papers Glover and McFarlane (1989); McFarlane and Glover (1992); Verma and Hunt (1993) present the main contributions, attesting to the relevance of this factorization class and justifying its study. These seminal works have been used as the basis for some fundamental problems, such as model reduction realization (Meyer, 1990; McFarlane et al., 1990), fault detection (Marx et al., 2003) and synthesis of robust stabilizing controllers for linear (Prempain and Postlethwaite, 2005; Pereira et al., 2017) and nonlinear systems (Guanrong Chen and Zhengzhi Han, 1998; Bu and Deng, 2012).

The coprime factorization description has basically two forms: right-coprime factors and left-coprime factors for both continuous- and discrete-time domains. For each structure, an extensive research line can be found. Concerning left-coprime factors, many of the proposed approaches deal with the synthesis of robust controllers (McFarlane and Glover, 1992; Gu et al., 2002). In this sense, it stands out the application in linear parameter-varying (LPV) systems, which has received great attention in recent years (Prempain and Postlethwaite, 2008; Li, 2014). An interesting feature of such class is the ability to represent nonlinear systems using a finite set of linear models on a convex hull. In the LPV structure, system dynamics depend on a time-varying parameter vector measured in real time and may be represented either in a polytopic form or through linear fractional representations (LFR). The contribution that introduced the synthesis of LPV controllers using both the LFR framework and left-coprime factors comes from Prempain (2006), where a special structure was used to obtain the left-coprime factors.

However, so far there has been no reference to a discrete-time version of this approach. Typically, the left-coprime factorization has been calculated and then discretized with some available method (e.g. Tustin) for computer implementation. Nevertheless, it is known that the direct representation ensures better performance than an indirect approach relying on discretization methods. Thus, motivated by the results in (Prempain, 2006), the contribution of this paper consists in extending his procedure to obtain discrete-time left-coprime factors for LPV systems using LFR. The existence conditions are given in an LMI framework, where the quadratic stability concept is used to provide the Lyapunov matrix. The effectiveness of the conditions proposed in this paper is evaluated by means of a numerical example, featuring a comparative study with a similar approach. The numerical results demonstrate that the proposed LMI-based conditions are an efficient alternative to obtain less conservative coprime factorizations for LPV systems.

This paper is organized as follows. In Section 2, the problem statement and some preliminary results concerning the description of discrete-time LPV systems using LFR are derived. Section 3 presents the main results of this paper: LMI-based conditions that allow the synthesis of left-coprime factors for discrete-time LPV/LFR systems. Section 4 is dedicated to show the effectiveness of the proposed approach using a numerical example. Section 5 concludes the paper.

1.1 Notation

The following notation is used throughout the paper. $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices, $\mathcal{M} \succ 0$ (or $\mathcal{M} \prec 0$) means $\mathcal{M}$ is symmetric and positive (or negative)

⋆

definite, $Tr(.)$ represents the trace of a matrix and $\star$ indicates symmetric blocks in the matrices. Moreover, the notation $G := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is used to denote a realization of system G and $\mathcal{F}_l(G,H)$ is the lower linear fractional transformation of matrices G and H.

2. PRELIMINARIES

Consider the following discrete-time LPV system $G(\rho)$

$$\left[\begin{array}{c} x(k+1) \\ \hline z(k) \end{array}\right] = \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & D(\rho) \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline w(k) \end{array}\right] \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^v$ is the exogenous input and $z(k) \in \mathbb{R}^q$ is the exogenous output. The state-space matrices of $G(\rho)$ are fixed functions of a time-varying parameter vector $\rho(k)$ that can be rewritten using the LFR framework as

$$x(k+1) = Ax + B_q q + B_w w \tag{2}$$

$$p = C_p x + D_{pq} q + D_{pw} w \tag{3}$$

$$z = C_z x + D_{zq} q + D_{zw} w \tag{4}$$

$$q = \Delta(\rho)p, \quad \Delta(\rho) = diag(\rho_1 I_{s_1}, ..., \rho_m I_{s_m}), \tag{5}$$

where $A \in \mathbb{R}^{n\times n}$, $B_q \in \mathbb{R}^{n\times n_q}$, $B_w \in \mathbb{R}^{n\times n_w}$, $C_p \in \mathbb{R}^{n_p\times n}$, $D_{pq} \in \mathbb{R}^{n_p\times n_q}$, $D_{pw} \in \mathbb{R}^{n_p\times n_w}$,$C_z \in \mathbb{R}^{n_z\times n}$, $D_{zq} \in \mathbb{R}^{n_z\times n_q}$ and $D_{zw} \in \mathbb{R}^{n_z\times n_w}$. Typically, the time-varying parameter vector $\rho(k)$ is available in real-time and belongs to

$$\mathbf{\Delta} = \Delta(\rho), \quad \rho \in \mathcal{P}. \tag{6}$$

Since $\mathcal{P}$ is a polytope, then $\mathbf{\Delta}$ is also a polytope and can be described from their vertices $\Delta_i, i = 1, ..., r$ such that the quantity $max(s_1, ..., s_m)$ corresponds to the LFR degree of the system. In this representation, p and q are assumed to have the same dimensions $n_p = n_q$ and the system $G(\rho)$ to be well posed, i.e.,

$$det(I - D_{pq}\Delta(\rho)) \neq 0, \quad \forall\ \rho \in \mathcal{P}.$$

For a concise notation, the dependence of the signals on k and the dependence of Δ on ρ will be dropped whenever it is clear.

Based on system (2-5), some definitions and lemmas on quadratic performance with convex solvability conditions for discrete-time LPV systems are presented below.

Definition 1. The discrete-time LPV system (1) is quadratically stable if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the following condition holds

$$A(\rho)^T P A(\rho) - P \prec 0, \quad \forall \rho \in \mathcal{P}. \tag{7}$$

This definition of stability holds for discrete-time LPV systems in general form. However, in this paper we restrict our attention to systems that have linear fractional representation of the form (2-5). In this sense, a quadratic stability condition for this class of systems may be given by the following lemma.

Lemma 2. Consider the LFR system given in (2-5) when $w(k) = 0$. If there exist matrices $P = P^T \succ 0$, $M = M^T \succ 0 \in \mathbb{R}^{n\times n}$ such that the following condition holds

$$\begin{bmatrix} A^T PA - P + C_p^T M C_p & A^T P(B_q\Delta_i) + C_p^T M(D_{pq}\Delta_i) \\ \star & \Gamma \end{bmatrix} \prec 0 \tag{8}$$

where $\Gamma = (B_q\Delta_i)^T P(B_q\Delta_i) + (D_{pq}\Delta_i)^T M(D_{pq}\Delta_i) - M$ for $i = 1, ..., r$. Then the system is said to be quadratically stable over $\mathcal{P}$.

Proof. Using Lyapunov's direct method for $x(k+1) = A(\rho)x$, we can show that

$$[Ax + (B_q\Delta)p]^T P[Ax + (B_q\Delta)p] - x^T Px < 0 \tag{9}$$

or in matrix form

$$\begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} A^T PA - P & A^T P(B_q\Delta) \\ (B_q\Delta)^T PA & (B_q\Delta)^T P(B_q\Delta) \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} < 0. \tag{10}$$

For any real matrices G_Δ and H_Δ of compatible dimensions, we get

$$\begin{aligned} x^T G_\Delta p &= x^T G_\Delta C_p x + x^T G_\Delta(D_{pq}\Delta)p \\ p^T H_\Delta p &= p^T H_\Delta C_p x + p^T H_\Delta(D_{pq}\Delta)p \end{aligned} \tag{11}$$

or, equivalently,

$$\begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} C_p^T M C_p & C_p^T M(D_{pq}\Delta) \\ (D_{pq}\Delta)^T M C_p & (D_{pq}\Delta)^T M(D_{pq}\Delta) - M \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = 0. \tag{12}$$

for $G_\Delta = C_p^T M$ and $H_\Delta = [(D_{pq}\Delta)^T + I]M$. More details about this uncertainty modeling can be found in (Fan Wang and Balakrishnan, 2002). Finally, using the S-procedure to combine (10) and (12) into a single LMI condition, the proof is complete.

Fan Wang and Balakrishnan (2002) also presented relaxing conditions for the case when the LFR degree of (5) is one, i.e., no varying parameter appears more than once in the diagonal of $\Delta(\rho)$. Then, instead of a single matrix M, it suffices to find different scaling matrices M_i for the different vertices $i = 1, ..., r$. This is referred to as vertex scaling.

Another important definition and characterization to obtain left-coprime factors is the detectability concept.

Definition 3. The pair $(A(\rho), C(\rho))$ is said to be quadratically detectable over $\mathcal{P}$ if there exist matrices $P = P^T > 0$ and the observer gain H such that the following condition holds

$$[A(\rho)+HC(\rho)]^T P[A(\rho)+HC(\rho)]-P \prec 0, \quad \forall \rho \in \mathcal{P}. \tag{13}$$

A systematic way to determine such characteristic of this class of systems is given by the following lemma.

Lemma 4. Consider the LFR system given in (2-5). If there exist matrices $S = S^T \succ 0$, $M = M^T \succ 0 \in \mathbb{R}^{n\times n}$ and

$$\begin{bmatrix} -S & 0 & A^T S + C_z^T Y^T & C_p^T M \\ \star & -M & (B_q\Delta_i)^T S & (D_{pq}\Delta_i)^T M \\ \star & \star & -S & 0 \\ \star & \star & \star & -M \end{bmatrix} \prec 0 \tag{14}$$

for $i = 1, ..., r$. Then the pair $(A(\rho), C(\rho))$ is said to be quadratically detectable over $\mathcal{P}$ and the observer gain is given by $H = S^{-1}Y$.

Proof. Herein, a generalization of the stability condition given in (8) will be used adopting $Q = P^{-1}$. Taking into account that

$$\begin{bmatrix} -Q & 0 & QA^T & QC_p^T \\ 0 & -N & N(B_q\Delta)^T & N(D_{pq}\Delta)^T \\ AQ & (B_q\Delta)N & -Q & 0 \\ C_pQ & (D_{pq}\Delta)N & 0 & -N \end{bmatrix} \prec 0 \tag{15}$$

ensures stability of (2-5). Then, substituting $A + HC_z$ in condition above, we have

$$\begin{bmatrix} -Q & 0 & Q(A+HC_z)^T & QC_p^T \\ 0 & -N & N(B_q\Delta)^T & N(D_{pq}\Delta)^T \\ (A+HC_z)Q & (B_q\Delta)N & -Q & 0 \\ C_pQ & (D_{pq}\Delta)N & 0 & -N \end{bmatrix} \prec 0. \tag{16}$$

This inequality has nonconvex terms, since the unknown variables Q and H are coupled. In order to solve this problem, a congruence transformation may be used. Multiplying (16) by $\mathbf{diag}\{Q^{-1}, I, Q^{-1}, I\}$ on the left and its transpose on the right, results in

$$\begin{bmatrix} -Q^{-1} & \star & \star & \star \\ 0 & -N & \star & \star \\ Q^{-1}A + Q^{-1}HC_z & Q^{-1}(B_q\Delta)N & -Q^{-1} & \star \\ C_p & (D_{pq}\Delta)N & 0 & -N \end{bmatrix} \prec 0 \tag{17}$$

Again, multiplying (17) by $\mathbf{diag}\{I, N^{-1}, I, N^{-1}\}$ on the right and its transpose on the left, yields

$$\begin{bmatrix} -Q^{-1} & \star & \star & \star \\ 0 & -N^{-1} & \star & \star \\ Q^{-1}A + Q^{-1}HC_z & Q^{-1}(B_q\Delta) & -Q^{-1} & \star \\ N^{-1}C_p & N^{-1}(D_{pq}\Delta) & 0 & -N^{-1} \end{bmatrix} \prec 0 \tag{18}$$

Finally, applying an appropriate change of variables $M = M^T = N^{-1}$, $S = S^T = Q^{-1}$ and $Y = SH$, we obtain

$$\begin{bmatrix} -S & 0 & A^TS + C_z^TY^T & C_p^TM \\ 0 & -M & (B_q\Delta)^TS & (D_{pq}\Delta)^TM \\ SA + YC_z & S(B_q\Delta) & -S & 0 \\ MC_p & M(D_{pq}\Delta) & 0 & -M \end{bmatrix} \prec 0 \tag{19}$$

where $S = S^T \succ 0$ and $M = M^T \succ 0$, concluding the proof of Lemma 4.

Given these preliminaries results, the problem of obtaining left-coprime factorizations for discrete-time LPV systems in the LFR framework can be addressed.

3. DISCRETE-TIME LEFT-COPRIME FACTORS FOR LPV/LFR SYSTEMS

Extending the procedure provided by Prempain (2006) for discrete-time LPV systems, the left-coprime factorization may be obtained from a particular description of the output injection problem depicted in Figure 1. This problem

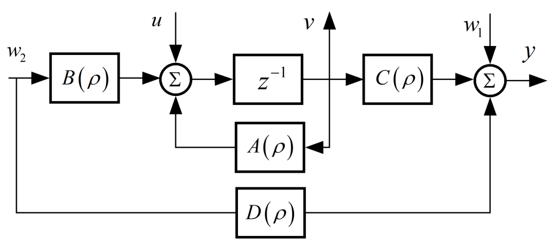

Figure 1. Open-loop interconnection for discrete-time LPV systems.

can be cast from a given observation law $u(k) = Hy(k)$ for the closed-loop system defined by the lower linear fractional transformation $\mathcal{F}_l(G_{OI}(\rho), H)$, where

$$G_{OI}(\rho) := \left[\begin{array}{c|cc|c} A(\rho) & 0 & B(\rho) & I \\ \hline I & 0 & 0 & 0 \\ \hline C(\rho) & I & D(\rho) & 0 \end{array}\right] \tag{20}$$

represents a particular output structure. By doing so, we can obtain the observer gain H from $\|\mathcal{F}_l(G_{OI}(\rho), H)\|_2$ that corresponds to an $\mathscr{H}_2$ filtering problem. To the best of the authors' knowledge, so far there has been no reference in the literature to a discrete-time version of this approach. Therefore, the contribution of this paper is to provide new conditions to obtain left-coprime factorizations for discrete-time LPV/LFR systems.

Notice that the first step to determine the left-coprime factorization consists in solving an $\mathscr{H}_2$ problem for discrete-time LFR systems. In this sense, a quadratic $\mathscr{H}_2$ performance condition for this class of systems may be given by Theorem 5.

Theorem 5. (Quadratic $\mathscr{H}_2$ performance). Consider the LFR system (21-24),

$$x(k+1) = Ax + B_q q + B_w w \tag{21}$$
$$p = C_p x + D_{pq} q + D_{pw} w \tag{22}$$
$$z = C_z x \tag{23}$$
$$q = \Delta(\rho)p, \quad \Delta(\rho) = diag(\rho_1 I_{s_1}, ..., \rho_m I_{s_m}). \tag{24}$$

If there exist matrices $W = W^T \succ 0$, $N = N^T \succ 0 \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}$ such that

$$\nu_2^2 := \min Tr(Y), \tag{25}$$

$$\begin{bmatrix} Y & C_zW \\ WC_z^T & W \end{bmatrix} \succ 0, \tag{26}$$

$$\begin{bmatrix} -W & 0 & 0 & WA^T & WC_p^T \\ \star & -N & \dfrac{D_{pw}}{2} & N(B_q\Delta_i)^T & N(D_{pq}\Delta_i)^T \\ \star & \star & -I & B_w^T & \dfrac{D_{pw}^T}{2} \\ \star & \star & \star & -W & 0 \\ \star & \star & \star & \star & -N \end{bmatrix} \prec 0, \tag{27}$$

for $i = 1, ..., r$. Then, the quadratic $\mathscr{H}_2$ norm of the system (21-24) exists for all values of the parameter $\rho \in \mathcal{P}$ and can be calculated as $\nu_2 < \sqrt{Tr(Y)}$.

Proof. Taking into account a discrete-time version of the generalized quadratic $\mathscr{H}_2$ norm condition provided by Scherer et al. (1997),

$$\begin{bmatrix} Y & C_z \\ C_z^T & P \end{bmatrix} \succ 0, \;\; \Delta V(x) - w^Tw \prec 0 \tag{28}$$

and developing the second part of (28), we have

$$(Ax + B_q\Delta p + B_w w)^T P (Ax + B_q\Delta p + B_w w) - x^TPx < w^Tw \tag{29}$$

or in matrix form,

$$\begin{bmatrix} x \\ p \\ w \end{bmatrix}^T \begin{bmatrix} A^TPA - P & A^TP(B_q\Delta) & A^TPB_w \\ (B_q\Delta)^TPA & (B_q\Delta)^TP(B_q\Delta) & (B_q\Delta)^TPB_w \\ B_w^TPA & B_w^TP(B_q\Delta) & B_w^TPB_w - I \end{bmatrix} \begin{bmatrix} x \\ p \\ w \end{bmatrix} < 0. \tag{30}$$

Next, Schur complement and the change of variable $P = W^{-1}$ are applied, so that (30) can be recast as

$$\begin{bmatrix} -W^{-1} & 0 & 0 & A^T \\ 0 & 0 & 0 & (B_q\Delta)^T \\ 0 & 0 & -I & B_w^T \\ A & B_q\Delta & B_w & -W \end{bmatrix} \prec 0 \quad (31)$$

Now, using the same procedure done in the previous section to describe the parametric uncertainties, one obtains

$$\begin{bmatrix} x \\ p \\ w \end{bmatrix}^T \begin{bmatrix} C_p^T M C_p & \star & \star \\ (D_{pq}\Delta)^T M C_p & (2,2) & \star \\ D_{pw}^T(\frac{C_p^T M}{2})^T & (3,2) & 0 \end{bmatrix} \begin{bmatrix} x \\ p \\ w \end{bmatrix} = 0,$$

where

$$(2,2) = -M + (D_{pq}\Delta)^T M (D_{pq}\Delta) \quad (32)$$

$$(3,2) = D_{pw}^T(\frac{(MD_{pq}\Delta + M)^T}{2})^T. \quad (33)$$

Again, application of the S-procedure to conditions (31) and (32) results in

$$\begin{bmatrix} C_p^T M C_p - W^{-1} & \star & \star & \star \\ (D_{pq}\Delta)^T M C_p & -M + (D_{pq}\Delta)^T M (D_{pq}\Delta) & \star & \star \\ D_{pw}^T(\frac{C_p^T M}{2})^T & D_{pw}^T(\frac{(MD_{pq}\Delta + M)^T}{2})^T & -I & \star \\ A & B_q\Delta & B_w & -W \end{bmatrix} \prec 0. \quad (34)$$

As such inequality has nonconvex terms, an appropriate congruence transformation is also used. Multiplying (33) by $\mathbf{diag}\{W, I, I, I\}$ on the left and its transpose on the right yields

$$\begin{bmatrix} W C_p^T M C_p W - W & \star & \star & \star \\ (D_{pq}\Delta)^T M C_p W & -M + (D_{pq}\Delta)^T M (D_{pq}\Delta) & \star & \star \\ D_{pw}^T(\frac{C_p^T M}{2})^T W & D_{pw}^T(\frac{(MD_{pq}\Delta + M)^T}{2})^T & -I & \star \\ AW & B_q\Delta & B_w & -W \end{bmatrix} \prec 0. \quad (35)$$

Using some mathematical manipulations, (34) can be rewritten as

$$\begin{bmatrix} -W & 0 & 0 & WA^T & WC_p^T \\ 0 & -M & \frac{MD_{pw}}{2} & (B_q\Delta)^T & (D_{pq}\Delta)^T \\ 0 & \frac{D_{pw}^T M}{2} & -I - \frac{D_{pw}^T M D_{pw}}{4} & B_w^T & \frac{D_{pw}^T}{2} \\ AW & B_q\Delta & B_w & -W & 0 \\ C_p W & D_{pq}\Delta & \frac{D_{pw}}{2} & 0 & -M^{-1} \end{bmatrix} \prec 0 \quad (36)$$

Applying the congruence transformation $\mathbf{diag}\{I, M^{-1}, I, I, I\}$ and the change of variable $M^{-1} = N = N^T$, we have

$$\begin{bmatrix} -W & \star & \star & \star & \star \\ 0 & -N & \star & \star & \star \\ 0 & \frac{D_{pw}^T}{2} & -I - \frac{D_{pw}^T M D_{pw}}{4} & \star & \star \\ AW & (B_q\Delta)N & B_w & -W & \star \\ C_p W & (D_{pq}\Delta)N & \frac{D_{pw}}{2} & 0 & -N \end{bmatrix} \prec 0 \quad (37)$$

Finally, expanding condition (36), we note that the second term is positive semidefinite, resulting in

$$\begin{bmatrix} -W & 0 & 0 & WA^T & WC_p^T \\ 0 & -N & \frac{D_{pw}}{2} & N(B_q\Delta)^T & N(D_{pq}\Delta)^T \\ 0 & \frac{D_{pw}^T}{2} & -I & B_w^T & \frac{D_{pw}^T}{2} \\ AW & (B_q\Delta)N & B_w & -W & 0 \\ C_p W & (D_{pq}\Delta)N & \frac{D_{pw}}{2} & 0 & -N \end{bmatrix} \prec 0, \quad (38)$$

thus concluding the first part of Theorem 5. The last part of the proof consists in submitting the inequality

$$\begin{bmatrix} Y & C_z \\ C_z^T & P \end{bmatrix} \succ 0 \quad (39)$$

to the congruence transformation $\mathbf{diag}\{I, P^{-1}\}$ and the change of variable $P^{-1} = W$, resulting in (26). Hence, the proof is complete.

3.1 Left-coprime factorization synthesis

Definition 6. Suppose that $(C(\rho), A(\rho))$ is detectable in a quadratic sense. Then, there exists the left-coprime factorization $G(\rho) = \tilde{M}(\rho)^{-1}\tilde{N}(\rho)$

$$\begin{bmatrix} \tilde{M}(\rho) & \tilde{N}(\rho) \end{bmatrix} := \left[\begin{array}{c|cc} A(\rho) + HC(\rho) & H & B(\rho) \\ \hline C(\rho) & I & 0 \end{array}\right] \quad (40)$$

Note that the main problem of obtaining left-coprime factorization for parameter-dependent systems consists in determining a stabilizing gain H. As mentioned before, such factorization may be described using the output injection representation. In the LFR framework, $G_{OI}(\rho)$ is given by

$$x(k+1) = Ax + B_q q + [0 \quad B_u]w + u \quad (41)$$
$$p = C_p x + D_{pq} q + [0 \quad D_{pu}]w \quad (42)$$
$$z = x \quad (43)$$
$$y = C_y x + D_{yq} q + [I \quad D_{yu}]w \quad (44)$$
$$q = \Delta(\rho)p, \quad \Delta(\rho) = diag(\rho_1 I_{s_1}, ..., \rho_m I_{s_m}). \quad (45)$$

Since $q = \Delta(\rho)p$ and adopting the observation law $u = Hy$, the closed-loop system becomes

$$x(k+1) = (A + HC_y)x + (B_q\Delta)p + [H \quad B_u]w \quad (46)$$
$$p = C_p x + (D_{pq}\Delta)p + [0 \quad D_{pu}]w \quad (47)$$
$$z = x \quad (48)$$

for $D_{yq} = D_{yu} = 0$. Then, the observer gain H that composes the left-coprime factorization can be determined from $\|\mathcal{F}_l(G_{OI}(\rho), H)\|_2$. Such LMI-based condition may be cast by the following theorem.

Theorem 7. Consider the LFR system given in (45-47). If there exist matrices $P = P^T \succ 0$ and $M = M^T \succ 0 \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{n\times n_y}$ and $X \in \mathbb{R}^{n\times n}$, such that

$$\nu_2^2 = \min Tr(X), \quad (49)$$

$$\begin{bmatrix} X & I \\ I & P \end{bmatrix} \succ 0, \quad (50)$$

$$\begin{bmatrix} -P & \star & \star & \star & \star & \star \\ 0 & -M & \star & \star & \star & \star \\ 0 & 0 & -I & \star & \star & \star \\ 0 & \frac{D_{pu}^T}{2}M & 0 & -I & \star & \star \\ PA+ZC_y & P\left(B_q\Delta_i\right) & Z & PB_u & -P & \star \\ MC_p & M\left(D_{pq}\Delta_i\right) & 0 & M\frac{D_{pu}}{2} & 0 & -M \end{bmatrix} \prec 0, \tag{51}$$

for $i = 1, ..., r$. Then the observer gain $H = P^{-1}Z$ for all values of the parameter $\rho \in \mathcal{P}$ and the $\mathscr{H}_2$ norm can be calculated as $\nu_2 < \sqrt{Tr\,(X)}$.

Proof. Comparing systems (21-24) and (40-44), and substituting the matrices in (45-47) to determine the quadratic $\mathscr{H}_2$ performance, (27) becomes

$$\begin{bmatrix} -W & \star & \star & \star & \star & \star \\ 0 & -N & \star & \star & \star & \star \\ 0 & 0 & -I & \star & \star & \star \\ 0 & \frac{D_{pu}^T}{2} & 0 & -I & \star & \star \\ AW+HC_yW & \left(B_q\Delta\right)N^T & H & B_u & -W & \star \\ C_pW & \left(D_{pq}\Delta\right)N^T & 0 & \frac{D_{pu}}{2} & 0 & -N \end{bmatrix} \prec 0 \tag{52}$$

Notice that such condition has nonconvex terms, so an appropriate congruence transformation should be used. Multiplying (51) by $\mathbf{diag}\{W^{-1}, I, I, I, W^{-1}, I\}$ on the left and its transpose on the right, and applying the change of variables $W^{-1} = P$ and $Z = PH$, yields

$$\begin{bmatrix} -P & \star & \star & \star & \star & \star \\ 0 & -N & \star & \star & \star & \star \\ 0 & 0 & -I & \star & \star & \star \\ 0 & \frac{D_{pu}^T}{2} & 0 & -I & \star & \star \\ PA+ZC_y & P\left(B_q\Delta\right)N^T & Z & PB_u & -P & \star \\ C_p & \left(D_{pq}\Delta\right)N^T & 0 & \frac{D_{pu}}{2} & 0 & -N \end{bmatrix} \prec 0 \tag{53}$$

Nonetheless, the condition still has nonconvex terms. To solve this problem, another congruence transformation is required. Multiplying (52) by $\mathbf{diag}\{I, N^{-1}, I, I, I, N^{-1}\}$ on the right and its transpose on the left and applying the change of variable $N^{-1} = M$, we have

$$\begin{bmatrix} -P & \star & \star & \star & \star & \star \\ 0 & -M & \star & \star & \star & \star \\ 0 & 0 & -I & \star & \star & \star \\ 0 & \frac{D_{pu}^T}{2}M & 0 & -I & \star & \star \\ PA+ZC_y & P\left(B_q\Delta_i\right) & Z & PB_u & -P & \star \\ MC_p & M\left(D_{pq}\Delta_i\right) & 0 & M\frac{D_{pu}}{2} & 0 & -M \end{bmatrix} \prec 0, \tag{54}$$

for $i = 1, ..., r$, concluding the first part of Theorem 7. Using a similar procedure as done previously, condition (26) becomes (49). Hence, the proof is complete.

Remark 1: If the LFR degree in (44) is one, the vertex scaling method in (Fan Wang and Balakrishnan, 2002) can be applied to relax the conditions on Theorem 7. In this particular case, the single scaling matrix $M \succ 0$ can be replaced with matrices M_i, where $i = 1, ..., r$ are the vertices of the polytope $\mathcal{P}$. Note that $r-1$ decision variables are added to the problem but conservatism is reduced.

Corollary 8. Consider the LFR system $G(\rho)$ given in (40), which is by assumption quadratically detectable. Let P and Y be the solutions of the optimization problem of Theorem 7; let $H = P^{-1}Z$ and define $Z_2^TZ_2 := (I + C_y^TPC_y)^{-1}$ and $R_1 := I + D_{yu}D_{yu}^T$. Then, there exists a contractive left-coprime factorization

$$G(\rho) = \tilde{M}^{-1}(\rho)\tilde{N}(\rho) \tag{55}$$

where $\eta = \tilde{M}(\rho)y$, for $u = 0$, and $\eta = \tilde{N}(\rho)u$, for $y = 0$, given by

$$\xi(k+1) = (A + HC_y)\xi + (B_q + HD_{yq})q + (B_u + HD_{yu})u \tag{56}$$

$$p = C_p\xi + D_{pq}q + D_{pu}u \tag{57}$$

$$\eta = Z_2(C_y\xi + D_{yq} + y + D_{yu}u) \tag{58}$$

$$q = \Delta(\rho)p, \quad \Delta(\rho) = diag(\rho_1 I_{s_1}, ..., \rho_m I_{s_m}). \tag{59}$$

Proof. Following the proof for the continuous-time version in Prempain (2006), the generalized Riccati inequality associated with the $\mathscr{H}_2$ norm minimization problem for the regularized $G_{OI}(\rho)$ plant is given by $H_2 = -(A(\rho)PC^T + BD^T)Z_2^T$. $H = H_2Z_2$ solves the original $\mathscr{H}_2$ minimization problem based on $G_{OI}(\rho)$. This optimal observer gain H is also the gain required to form the contractive left-coprime factors of the plant, solving the inequality given in Gu et al. (2002)

$$\Psi P\Psi^T - P - \Psi PC^TZ_2^TZ_2CP\Psi + B(\rho)R_2^{-1}B(\rho)^T < 0, \tag{60}$$

where

$$\Psi = A(\rho) - B(\rho)R_2^{-1}D^TC \tag{61}$$

$$R_2 = I + D^TD \tag{62}$$

$$R_1 = I + DD^T \tag{63}$$

$$Z_2^TZ_2 = (R_1 + CPC^T)^{-1}. \tag{64}$$

This completes the proof.

4. NUMERICAL EXAMPLE

In order to evaluate the effectiveness of the proposed method, consider the following LPV/LFR system borrowed from (Prempain, 2006). The model was adapted to address the present problem using forward Euler method (Toth, 2010), being described by

$$x(k+1) = \begin{bmatrix} 1 & -T \\ T & 1-T \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ T & T \end{bmatrix} q + \begin{bmatrix} T \\ 0 \end{bmatrix} u \tag{65}$$

$$p = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} q + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u \tag{66}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \end{bmatrix} q + \begin{bmatrix} 0 \end{bmatrix}, \tag{67}$$

where $q = \begin{bmatrix} \rho_1(kT) & 0 \\ 0 & \rho_2(kT) \end{bmatrix} p$ and ρ_1 and ρ_2 are time-varying parameters constrained to the set

$$\mathcal{P} = \Big((r(kT)cos\phi, r(kT)sin\phi) \quad \phi \in \big[0, \frac{\pi}{4}\big] \cup \big[\pi, \frac{5\pi}{4}\big] \Big). \tag{68}$$

Note that the system features varying parameters in both $A(\rho)$ and $B(\rho)$ matrices. Following the same procedure done in (Prempain, 2006), let us define the parameter

β as the maximum value of $|r(kT)|$. Thus, $\mathbf{\Delta}$ can be described from six vertices that compose the Cartesian coordinates for points $AEICGJA$ given by $A = (-\beta, 0)$, $E = \left(-\beta, -\beta\frac{\sqrt{2}}{2}\right)$, $I = \left(-\beta\frac{\sqrt{2}}{2}, -\beta\frac{\sqrt{2}}{2}\right)$, $C = (\beta, 0)$, $G = \left(\beta, \beta\frac{\sqrt{2}}{2}\right)$ and $J = \left(\beta\frac{\sqrt{2}}{2}, \beta\frac{\sqrt{2}}{2}\right)$. Adopting a sampling time $T = 100[ms]$ in (42) and a maximum value $\beta = 0$, we obtain the normalized left-coprime factorization from Theorem 7, given by

$$\left[\, \tilde{M} \;\; \tilde{N} \,\right] := \left[\begin{array}{cc|cc} 0.9336 & -0.1 & -0.0664 & 0.1 \\ 0.0700 & 0.9 & -0.0300 & 0 \\ \hline 0.9649 & 0 & 0.9649 & 0 \end{array}\right] \tag{69}$$

with $\|\mathcal{F}_l\,(G_{OI}\,(\rho)\,, H)\|_2 = 0.3148$. Now, for $\beta = 1$, using Theorem 7 again with Remark 1 to obtain H and then Corollary 8, one obtains the contractive left-coprime factorization

$$\left[\, \tilde{M}\,(\rho) \;\; \tilde{N}\,(\rho) \,\right] = \left[\begin{array}{cc|cc} 0.8368 & -0.1 & -0.1632 & 0.1 \\ 0.1383 & (2,2) & 0.0383 & (2,4) \\ \hline 0.9177 & 0 & 0.9177 & 0 \end{array}\right], \tag{70}$$

where $(2,2) = 0.9 + 0.1\rho_1 - 0.1\rho_2 - 0.1\,(\rho_1\rho_2)$ and $(2,4) = 0.1\rho_1 - 0.2\rho_2 - 0.1\,(\rho_1\rho_2)$ with a determined value for $\|\mathcal{F}_l\,(G_{OI}\,(\rho)\,, H)\|_2 = 0.7296$.

The contractiveness of the left-coprime factorization in (70) may be evaluated using the results presented in (Wood et al., 1996),

$$||\left[\tilde{M}(\rho) \;\; \tilde{N}(\rho)\right]||_{i,2} = \sup_{\substack{\omega \in \mathscr{L}_2 \\ ||\omega||_2 \le 1}} \; \sup_{\rho \in \Omega_N} ||\left[\tilde{M}(\rho) \;\; \tilde{N}(\rho)\right]\omega||_{i,2}, \tag{71}$$

where $||(\cdot)||_{i,2}$ is the maximum singular value over all significant frequencies. As expected, $||\left[\tilde{M}(\rho) \;\; \tilde{N}(\rho)\right]||_{i,2} \approx 1$. This means that for all values of ρ in the parameter set, the norm is constrained to be less than or equal to one. Figure 2 shows the results for 1000 random values of ρ.

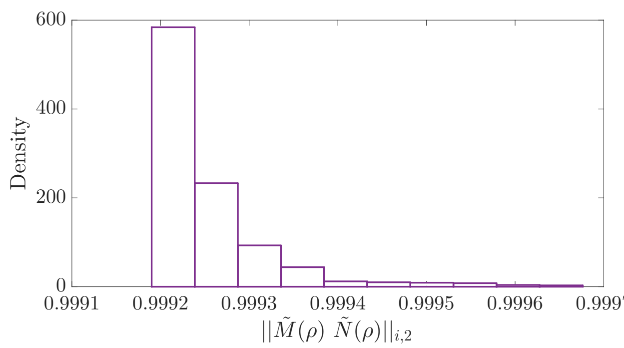

Figure 2. $||\left[\tilde{M}(\rho) \;\; \tilde{N}(\rho)\right]||_{i,2}$ for random values of ρ in the parameter space.

The norm values are shown to be considerably close to one over the whole parameter space. This proximity between the normalized and contractive factorizations allows us to use the latter as an approximation of the former for LPV systems.

It is noteworthy that the left-coprime factorization problem for polytopic description was solved by Pereira et al. (2017). In this sense, an interesting analysis consists in comparing the performance and the contractiveness of the left-coprime factors obtained. Figure 3 shows the relation between $\|\mathcal{F}_l\,(G_{OI}\,(\rho)\,, H)\|_2$ values and their respective β parameters such that the compared methods are feasible. Note that in the region of interest, $0 \le \beta \le 1$, the LFR approach provides less conservative results than its polytopic counterpart. As it is well-documented in the literature, there is a strong relation between the $\mathscr{H}_2$ norm of the coprime factors and the contractiveness, such analysis will be omitted here. Thus, we note that the proposed method obtain less conservative left-coprime factorizations than a traditional approach.

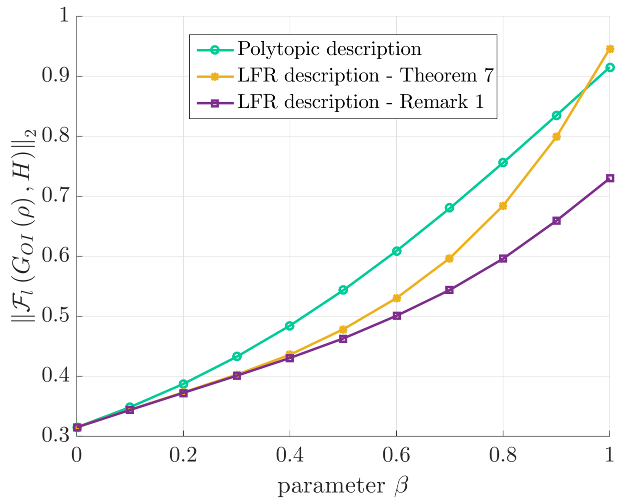

Figure 3. Relation between $\|\mathcal{F}_l\,(G_{OI}\,(\rho)\,, H)\|_2$ values and their respective β such that the compared methods are feasible.

5. CONCLUSION

This paper presented novel LMI-based conditions to address the discrete-time left-coprime factorization problem for LPV/LFR systems. The main drawback of the LMI conditions derived here stems from using a fixed Lyapunov matrix to obtain the coprime factorization description. However, such an assumption remains quite attractive due to its low requirement of computational effort. The numerical example has shown that the proposed conditions provide an efficient and alternative procedure to solve the problem stated and recovers the normalized coprime factorization notion when the system reduces to a single point. In future research, we intend to use such results to provide new control strategies based on Youla-Kucera parameterization.

REFERENCES

Bu, N. and Deng, M. (2012). Operator-Based Robust Right Coprime Factorization and a Nonlinear Control Scheme for Nonlinear Plant with Unknown Perturbations. *Asian Journal of Control*, 14(6), 1655–1661. doi: 10.1002/asjc.498.

Fan Wang and Balakrishnan, V. (2002). Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems. *IEEE Transactions on Automatic Control*, 47(5), 720–734.

Glover, K. and McFarlane, D. (1989). Robust stabilization of normalized coprime factor plant descriptions with H_∞-bounded uncertainty. *IEEE Transactions on Automatic Control*, 34(8), 821–830. doi:10.1109/9.29424.

Gu, D.W., Petkov, P., and Konstantinov, M. (2002). Formulae for discrete H_∞ loop shaping design procedure controllers. *IFAC Proceedings Volumes*, 35(1), 109 – 113. doi:https://doi.org/10.3182/20020721-6-ES-1901.00353. 15th IFAC World Congress.

Guanrong Chen and Zhengzhi Han (1998). Robust right coprime factorization and robust stabilization of nonlinear feedback control systems. *IEEE Transactions on Automatic Control*, 43(10), 1505–1509. doi:10.1109/9.720519.

Li, L. (2014). Coprime factor model reduction for discrete-time uncertain systems. *Systems & Control Letters*, 74, 108–114. doi:10.1016/J.SYSCONLE.2014.08.010.

Marx, B., Koenig, D., and Georges, D. (2003). Robust Fault Diagnosis for Descriptor Systems - A Coprime Factorization Approach. *IFAC Proceedings Volumes*, 36(5), 477–482. doi:10.1016/S1474-6670(17)36537-0.

McFarlane, D. and Glover, K. (1992). A loop-shaping design procedure using H_∞ synthesis. *IEEE Transactions on Automatic Control*, 37(6), 759–769. doi:10.1109/9.256330.

McFarlane, D., Glover, K., and Vidyasagar, M. (1990). Reduced-order controller design using coprime factor model reduction. *IEEE Transactions on Automatic Control*, 35(3), 369–373. doi:10.1109/9.50362.

Meyer, D. (1990). Fractional balanced reduction: model reduction via fractional representation. *IEEE Transactions on Automatic Control*, 35(12), 1341–1345. doi:10.1109/9.61011.

Pereira, R.L., Kienitz, K.H., and Guaracy, F.H.D. (2017). Discrete-time static H_∞ loop shaping control for LPV systems. *25th Mediterranean Conference on Control and Automation (MED)*, 619–624. doi:10.1109/MED.2017.7984186.

Prempain, E. and Postlethwaite, I. (2005). Static H_∞ loop shaping control of a fly-by-wire helicopter. *Automatica*, 41(9), 1517–1528. doi:10.1016/J.AUTOMATICA.2005.04.001.

Prempain, E. (2006). On Coprime factors for Parameter-Dependent Systems. *Proceedings of the 45th IEEE Conference on Decision and Control*, 5796–5800. doi:10.1109/CDC.2006.376773.

Prempain, E. and Postlethwaite, I. (2008). L_2 and H_2 performance analysis and gain-scheduling synthesis for parameter-dependent systems. *Automatica*, 44(8), 2081–2089. doi:10.1016/J.AUTOMATICA.2007.12.008.

Scherer, C., Gahinet, P., and Chilali, M. (1997). Multiobjective output-feedback control via lmi optimization. *IEEE Transactions on Automatic Control*, 42(7), 896–911.

Toth, R. (2010). *Modeling and identification of linear parameter-varying systems*. Lecture notes in control and information sciences. Springer, Germany. doi:10.1007/978-3-642-13812-6.

Verma, M. and Hunt, L. (1993). Right coprime factorizations and stabilization for nonlinear systems. *IEEE Transactions on Automatic Control*, 38(2), 222–231. doi:10.1109/9.250511.

Vidyasagar, M. (1985). *Control System Synthesis: A Factorization Approach*. Prentice-Hall, Inc. doi:10.2200/S00351ED1V01Y201105CRM002.

Wood, G.D., Goddard, P.J., and Glover, K. (1996). Approximation of linear parameter-varying systems. In *Proceedings of 35th IEEE Conference on Decision and Control*, volume 1, 406–411 vol.1.