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A stabilizing gradient-based economic MPC for unstable processes: toward the enlargement of the domain of attraction

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Abstract: This work proposes a stabilizing gradient-based economic MPC with enlargement of the domain of attraction, based on the novel combination of three ingredients: terminal equality constraints solely on open-loop non-stable states, an admissible artificial steady-state, and a terminal cost. A further enlargement of the domain of attraction is achieved by including slack variables to soften the bound constraints of states, without affecting the stabilizing properties or capacity to drive the closed-loop system toward the economic target. Finally, a case study based on an unstable reactor is used to demonstrate the properties of the proposed strategy.

Keywords: model-based control, domain of attraction, terminal equality constraint, optimizing control, economic model predictive control, unstable systems, target calculation

1. INTRODUCTION

The performance of hierarchical control schemes can be improved if the control layer can directly access the economic performance of the system (Hinojosa et al., 2017). Such strategies can be classified as one-layer hierarchical control schemes with economic objectives, whose resulting strategies with stabilizing properties can be formulated as (Santana et al., 2020): (i) nonlinear optimization problems that directly solve an economic objective function, the so-called EMPC (Economic Model Predictive controller); and (ii) MPC+RTO controllers, which include into the cost function an additional term related to the economic performance.

One of the great challenges for synthesizing stabilizing MPC control laws, including ones with economic objectives, is to enlarge their associated domain of attraction with the less possible computational burden. The most heavily used ingredient to design stabilizing MPC controllers is to include end-point (terminal) constraints into the resulting optimization problems (Grüne and Pannek, 2011). Although it is a tricky task already for small-scale systems, mainly for output tracking cases, it is common to seek the design of an invariant positive set like a terminal constraint (Mayne, 2014). An alternative way that has been looked into the literature is to collapse this terminal invariant set in terminal equality constraints, so that the system terminal states must reach a steady-state at the end of the control horizon. However, if additional elements are not used in the terminal equality constraint-based control schemes, such as artificial steady-state (Ferramosca et al., 2009; Krupa et al., 2019) and slack variables (Martins and Odloak, 2016), their control laws result in a drastically reduced domain of attraction.

Recent works indicated that methods based on a suitable set of slacked terminal constraints to cancel the effects related to non-stable modes are already an acceptable stage for practical implementation purposes, including guarantees of feasibility, e.g. Martin et al. (2019); Silva et al. (2020). However, their stability proofs are rigorously achieved through two optimization formulations. On the other hand, the approach based on artificial equilibrium points focuses on an only control optimization formulation, but it has a drawback concerning to enforce unnecessarily open-loop stable modes of the system towards the terminal equilibrium point, which contributes to a reduction of the domain of attraction of the associated control laws. In this sense, one contribution of this work is to combine the above-mentioned approaches aiming to derive a one-layer terminal equality constraints-based stabilizing MPC with an enlarged domain of attraction.

Focusing on MPC+RTO strategies, Alvarez and Odloak (2014) proposed an infinite horizon MPC with zone control that drives open-loop stable systems towards its economic optimal condition, in which stabilizing properties are based on slacked terminal equality constraints, and the economic term is an approximation of the gradient of a convex gradient function. Santana et al. (2020) extended this work for unstable systems, also applying slacked terminal equality constraints to ensure both stability and feasibility. An advantage of these strategies is the lack of need to know the economic optimal steady-state in advance, due to the usage of set-point targets as decision variables in the zone control formulation. On the other hand, the Hessian matrix of the economic function can not be zero, and the state-space model applied by Santana et al. (2020), based on an analytical step response of the system, include more states than the canonical Jordan decomposition. Thus, in order to constrain unstable states it is needed more terminal equality constraints than a formulation based on the Jordan decomposition.

Another stabilizing MPC approach, which is based on an one-layer strategy is presented in Alamo et al. (2014). The related objective function evaluates the gradient of an economic cost, guaranteeing convergence to a steady-state that represents the desired economic objective. One advantage of this formulation is that the Hessian matrix of the economic function is not needed. The authors applied non-slacked terminal equality constraints to derive stabilizing properties, setting the unstable states to be zeroed at the end of the control horizon. They enlarge the domain of attraction because the controller can drive the system to any feasible steady-state with a given control horizon, but if the terminal constraints were slacked there would be an additional increase of the domain of attraction. This investigation is another aim of this proposed work. One disadvantage of Alamo et al. (2014) is that the formulation is based on a parameterized Jordan decomposition in the incremental form, which enforces the number of integrating states to be equal to the number of inputs. As a consequence, if pure integrating states are present, it is necessary to apply a state-space based on the analytical step response of the system following González et al. (2011), increasing the number of terminal equality constraints imposed.

Finally, the aim of this work is to propose a novel stabilizing gradient-based economic MPC, which combines terminal equality constraints on non-stable states, an admissible economic optimum artificial steady-state, and a terminal cost to obtain both stabilizing properties and enlargement of the domain of attraction. The Jordan decomposition is used without further parametrization, which reduces the number of terminal constraints imposed. Furthermore, another contribution is to investigate the effect of slack variables to tackle unfeasible conditions imposed by the constraints of the problem, which is a further element to increase the domain of attraction. In this approach, there is no need to know in advance the target steady-state or the Hessian matrix of the economic function.

This work is organized as follows. Section 2 presents the proposed infinity horizon MPC formulation, presenting its stabilizing properties, convergence to the economic target and enlargement of the domain of attraction. Section 3 presents a case study that explore the characteristics of the controller. Finally, Section 4 offers some concluding remarks.

2. THE PROPOSED ONE-LAYER GRADIENT-BASED ECONOMIC MPC

In order to reduce the hierarchical control structure into one-layer and allow it to evaluate the economic performance, take f_{e} as a function that describes this performance, such as: profit, energy demand or efficient. Aiming to derive a model predictive controller for linear systems, this function can be approximated by the Taylor series expansion around a reference equilibrium point:

$$f_{\mathrm{e}} \approx f_{\mathrm{e}}\big|_{\mathrm{ss}} + \left.\frac{\mathrm{d}f_{\mathrm{e}}}{\mathrm{d}\hat{\boldsymbol{x}}}\right|_{\mathrm{ss}} \cdot \hat{\boldsymbol{x}}_{\mathrm{s}} + \left.\frac{\mathrm{d}f_{\mathrm{e}}}{\mathrm{d}\boldsymbol{u}}\right|_{\mathrm{ss}} \cdot \boldsymbol{u}_{\mathrm{s}}, \tag{1}$$

where ($\hat{\boldsymbol{x}}_{\mathrm{s}}$,$\boldsymbol{u}_{\mathrm{s}}$) is any equilibrium point pair, taking as deviation variables from the reference equilibrium point, which are related by the linear discrete state-space model:

$$\hat{\boldsymbol{x}}(j+1) = \hat{\boldsymbol{A}} \cdot \hat{\boldsymbol{x}}(j) + \hat{\boldsymbol{B}} \cdot \boldsymbol{u}(j), \tag{2}$$

where $\hat{\boldsymbol{x}}(j)$ is the state vector at time step j, $\boldsymbol{u}(j)$ is the input vector, $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$ are matrices of appropriate dimensions. (2) can be straightforwardly converted to the velocity form (González et al., 2008) by considering $\begin{bmatrix} \hat{\boldsymbol{x}}(j) \\ \boldsymbol{u}(j-1) \end{bmatrix}$ as an augmented state vector:

$$\boldsymbol{x}(j+1) = \boldsymbol{A} \cdot \boldsymbol{x}(j) + \boldsymbol{B} \cdot \Delta\boldsymbol{u}(j), \tag{3}$$

$$\boldsymbol{y}(j) = \boldsymbol{C} \cdot \boldsymbol{x}(j), \tag{4}$$

where $\boldsymbol{x}(j)$ is the augmented state vector at time step j, $\Delta\boldsymbol{u}(j)$ is the vector of input increments, $\boldsymbol{y}(j)$ is the output vector, $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices of appropriate dimensions. Therefore, (5) can be rewritten as:

$$f_{\mathrm{e},k} \approx f_{\mathrm{e}}\big|_{\mathrm{ss}} + \left.\frac{\mathrm{d}f_{\mathrm{e}}}{\mathrm{d}\boldsymbol{x}}\right|_{\mathrm{ss}} \cdot \boldsymbol{x}_{\mathrm{s}}, \tag{5}$$

taking $\frac{\mathrm{d}f_{\mathrm{e}}}{\mathrm{d}\boldsymbol{x}} = \begin{bmatrix} \frac{\mathrm{d}f_{\mathrm{e}}}{\mathrm{d}\hat{\boldsymbol{x}}} & \frac{\mathrm{d}f_{\mathrm{e}}}{\mathrm{d}\boldsymbol{u}} \end{bmatrix}$. The nomenclature $f_{\mathrm{e},k}$ indicates that the economic performance function f_{e} is evaluated with $\boldsymbol{x}_{\mathrm{s}}$ at time step k.

In order to derive a stabilizing model predictive controller with terminal equality constraints, the Jordan decomposition of (3) provides:

$$\boldsymbol{z}(j+1) = \begin{bmatrix} \boldsymbol{J}_{\mathrm{s}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{\mathrm{ns}} \end{bmatrix} \cdot \boldsymbol{z}(j) + \boldsymbol{W} \cdot \boldsymbol{B} \cdot \Delta\boldsymbol{u}(j), \tag{6}$$

$$\begin{bmatrix} \boldsymbol{J}_{\mathrm{s}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{\mathrm{ns}} \end{bmatrix} = \boldsymbol{W} \cdot \boldsymbol{A} \cdot \boldsymbol{V}, \tag{7}$$

where $\boldsymbol{J}_{\mathrm{s}}$ is the Jordan block associated with stable states, $\boldsymbol{J}_{\mathrm{ns}}$ is the Jordan block associated with non-stable states, and $\boldsymbol{W}$ is the generalized eigenvector for which $\boldsymbol{z} = \boldsymbol{W} \cdot \boldsymbol{x}$. The non-stable states are related to integrating or unstable modes (with or without multiplicities). Finally, from $\boldsymbol{W}$ the eigenvectors related to stable states, $\boldsymbol{W}_{\mathrm{s}}$, and non-stable states, $\boldsymbol{W}_{\mathrm{ns}}$, can be obtained.

Thus, a proposed stabilizing economic model predictive controller with optimizing targets, which applies artificial equilibrium states, $\boldsymbol{x}_{\mathrm{s}}$, to provide degrees of freedom for the optimization, can be taken as:

Problem P0

$$\min_{\Delta\boldsymbol{u}_k, \boldsymbol{x}_{\mathrm{s}}} \Phi_k = \sum_{j=0}^{N-1} \left\{ \|\boldsymbol{x}(j) - \boldsymbol{x}_{\mathrm{s}}\|_{\boldsymbol{Q}}^2 + \|\Delta\boldsymbol{u}(j)\|_{\boldsymbol{R}}^2 \right\} + \\ + \|\boldsymbol{x}(N) - \boldsymbol{x}_{\mathrm{s}}\|_{\widetilde{\boldsymbol{Q}}}^2 + \|f_{\mathrm{e},k} - e_{\mathrm{sp}}\|_{P}^2, \tag{8}$$

subject to (3), (4), (5) and:

$$\boldsymbol{x}(j=0) = \boldsymbol{x}(k), \tag{9}$$

$$\boldsymbol{x}(j) \in \mathcal{Z}_{\mathrm{s}},\ j=0,\ldots,N+k_2, \tag{10}$$

$$\Delta\boldsymbol{u}(j) \in \Delta\mathcal{U},\ j=0,\ldots,N-1, \tag{11}$$

$$\boldsymbol{x}_{\mathrm{s}} \in \mathcal{X}_{\mathrm{ss}}, \tag{12}$$

$$\boldsymbol{W}_{\mathrm{ns}} \cdot (\boldsymbol{x}(N) - \boldsymbol{x}_{\mathrm{s}}) = \boldsymbol{0}, \tag{13}$$

where $\Delta\mathcal{U}$ and $\mathcal{Z}_{\mathrm{s}}$ are compact-convex sets related to bound constraints on input increments and states:

$$\Delta\mathcal{U} = \{\Delta\boldsymbol{u} \in \mathbb{R}^{nu} \mid \Delta\boldsymbol{u}_{\min} \leq \Delta\boldsymbol{u} \leq \Delta\boldsymbol{u}_{\max}\}, \tag{14}$$

$$\mathcal{Z}_{\mathrm{s}} = \left\{\boldsymbol{x} \in \mathbb{R}^{nx} \,\middle|\, \begin{bmatrix}\hat{\boldsymbol{x}}_{\min}\\ \boldsymbol{u}_{\min}\end{bmatrix} \leq \begin{bmatrix}\hat{\boldsymbol{x}}\\ \boldsymbol{u}\end{bmatrix} \leq \begin{bmatrix}\hat{\boldsymbol{x}}_{\max}\\ \boldsymbol{u}_{\max}\end{bmatrix}\right\}, \tag{15}$$

and $\mathcal{X}_{\mathrm{ss}}$ is a set to enforce that $\boldsymbol{x}_{\mathrm{s}}$ is an equilibrium point of the system:

$$\mathcal{X}_{\mathrm{ss}} = \{\boldsymbol{x}_{\mathrm{s}} \in \mathcal{Z}_{\mathrm{s}} \mid (\boldsymbol{I} - \boldsymbol{A}) \cdot \boldsymbol{x}_{\mathrm{s}} = \boldsymbol{0}\}. \tag{16}$$

The terminal cost weight, $\widetilde{\boldsymbol{Q}}$, is given by:

$$\widetilde{\boldsymbol{Q}} = \boldsymbol{W}_{\mathrm{s}}^{\top} \cdot \overline{\boldsymbol{Q}} \cdot \boldsymbol{W}_{\mathrm{s}}, \tag{17}$$

where $\overline{\boldsymbol{Q}}$ is the solution of the Lyapunov equation:

$$\overline{\boldsymbol{Q}} = \boldsymbol{V}_{\mathrm{s}}^{\top} \cdot \boldsymbol{Q} \cdot \boldsymbol{V}_{\mathrm{s}} + \boldsymbol{J}_{s}^{\top} \cdot \overline{\boldsymbol{Q}} \cdot \boldsymbol{J}_{s}. \tag{18}$$

Constraint (9) states that $\boldsymbol{x}(0)$ is the initial condition of Problem P0, taken as the observed states at time step k.

Remark 1. In (10), k_2 is specified in order to the feasibility of the bound constraints on states in $\mathcal{Z}_{\mathrm{s}}$ up to time $N+k_2$ ensures feasibility of these constraints on the infinite horizon. Its value can be obtained from the steps described by Rawlings and Muske (1993).

Remark 2. The domain of attraction of Problem P0 is taken as the set of initial conditions $\boldsymbol{x}(0)$ for which the constraints (3), (4), and (9) to (13) are satisfied. This can be summarized in a polyhedron that fulfills the following expressions:

$$\boldsymbol{G}_{\mathrm{ineq}} \cdot \begin{bmatrix}\boldsymbol{x}(0)\\ \boldsymbol{x}_{\mathrm{s}}\\ \Delta\boldsymbol{u}_k\end{bmatrix} \leq \boldsymbol{h}_{\mathrm{ineq}}, \tag{19}$$

$$\boldsymbol{G}_{\mathrm{eq}} \cdot \begin{bmatrix}\boldsymbol{x}(0)\\ \boldsymbol{x}_{\mathrm{s}}\\ \Delta\boldsymbol{u}_k\end{bmatrix} = \boldsymbol{h}_{\mathrm{eq}}, \tag{20}$$

in the space of $\left[\boldsymbol{x}^{\top}(0)\ \boldsymbol{x}_{\mathrm{s}}^{\top}\ \Delta\boldsymbol{u}_k^{\top}\right]^{\top}$. $\boldsymbol{G}_{\mathrm{ineq}}$, $\boldsymbol{G}_{\mathrm{eq}}$, $\boldsymbol{h}_{\mathrm{ineq}}$ and $\boldsymbol{G}_{\mathrm{eq}}$ must be obtained from the aforementioned constraints of Problem P0.

Remark 3. If the terminal constraint (13) is assumed to be (Ferramosca et al., 2009; Krupa et al., 2019):

$$\boldsymbol{x}(N) - \boldsymbol{x}_{\mathrm{s}} = \boldsymbol{0}, \tag{21}$$

the domain of attraction will be less or equal to the domain of the control law proposed for Problem P0. This is a direct consequence of (21) forcing all the states of the system to be at the steady-state, instead of only non-stable components. Let nominate the optimization problem using (21) replacing (13) as Problem P1.

Remark 4. Aiming practical implementations, the domain of attraction of Problem P0 can be enlarged by including slack variables, $\boldsymbol{\delta}$, to soften bound constraints as:

Problem P2

$$\min_{\Delta\boldsymbol{u}_k,\boldsymbol{x}_{\mathrm{s}},\boldsymbol{\delta}} \Phi_k = \sum_{j=0}^{N-1}\left\{\|\boldsymbol{x}(j) - \boldsymbol{x}_{\mathrm{s}}\|_{\boldsymbol{Q}}^2 + \|\Delta\boldsymbol{u}(j)\|_{\boldsymbol{R}}^2\right\} + \\ + \|\boldsymbol{x}(N) - \boldsymbol{x}_{\mathrm{s}}\|_{\widetilde{\boldsymbol{Q}}}^2 + \|f_{\mathrm{e},k} - e_{\mathrm{sp}}\|_{P}^2 + \|\boldsymbol{\delta}\|_{\boldsymbol{S}}^2 \tag{22}$$

subject to (3), (4), (5), (9), (10), (11), (12), (13), in addition to $\mathcal{Z}_{\mathrm{s}}$ being softened by including such slack variables solely on the bound constraints of the original states of the system:

$$\mathcal{Z}_{\mathrm{s}} = \left\{\boldsymbol{x} \in \mathbb{R}^{nx} \,\middle|\, \begin{bmatrix}\hat{\boldsymbol{x}}_{\min}\\ \boldsymbol{u}_{\min}\end{bmatrix} \leq \begin{bmatrix}\hat{\boldsymbol{x}} + \boldsymbol{\delta}\\ \boldsymbol{u}\end{bmatrix} \leq \begin{bmatrix}\hat{\boldsymbol{x}}_{\max}\\ \boldsymbol{u}_{\max}\end{bmatrix}\right\}. \tag{23}$$

Furthermore, as $\boldsymbol{\delta}$ is penalized in the cost function with a high weight matrix $\boldsymbol{S}$, it is used only if an unmeasured disturbance could turn Problem P2 unfeasible. Therefore, Problem P2 provides a larger or equal domain of attraction to the domain obtained for Problem P0. It is important to highlight that the slack can be bounded to limit this enlargement.

2.1 Stability

Problem P0 is nominally stable, provided that at time step k, the initial condition $\boldsymbol{x}(0)$ belongs to the domain of attraction of the optimization problem. Theorem 1 addresses the stabilizing properties.

Theorem 1. Consider an undisturbed process with stable and non-stable poles. If the solution to Problem P0 is feasible at time step k and satisfies Remark 1, then it will remain feasible at successive time steps. Thus, the successive solutions drive the closed-loop system asymptotically to a steady-state where the cost function Φ_k reaches its lowest achievable value.

Proof. The proof of this theorem builds on the concepts proposed by Rawlings and Muske (1993), i.e it is demonstrated that this formulation is recursively feasible and the cost function is a Lyapunov-like function for the closed-loop system. In this sense, consider that $\boldsymbol{x}(0)$ belongs to the domain of attraction of Problem P0 and $\left[\Delta\boldsymbol{u}_k^{*,\top}\ \boldsymbol{x}_{\mathrm{s}}^{*,\top}\right]^{\top}$ is a feasible solution for the problem at time step k. The optimal cost function at this time step is:

$$\Phi_k^* = \sum_{j=0}^{N-1}\left\{\|\boldsymbol{x}(j) - \boldsymbol{x}_{\mathrm{s}}^*\|_{\boldsymbol{Q}}^2 + \|\Delta\boldsymbol{u}^*(j)\|_{\boldsymbol{R}}^2\right\} + \\ + \|\boldsymbol{x}(N) - \boldsymbol{x}_{\mathrm{s}}^*\|_{\widetilde{\boldsymbol{Q}}}^2 + \left\|\left.\frac{\mathrm{d}f_{\mathrm{e}}}{\mathrm{d}\boldsymbol{x}}\right|_{\mathrm{ss}} \cdot \boldsymbol{x}_{\mathrm{s}}^* + f_{\mathrm{e}}|_{\mathrm{ss}} - e_{\mathrm{sp}}\right\|_{P}^2. \tag{24}$$

Moving at time step $k+1$, it is shown that the solution inherited from k, $\left[\Delta\widetilde{\boldsymbol{u}}_{k+1}^{\top}, \boldsymbol{x}_{\mathrm{s}}^{*,\top}\right]^{\top}$, where $\Delta\widetilde{\boldsymbol{u}}_{k+1}^{\top} = \left[\Delta\boldsymbol{u}^{\top}(1), \ldots, \Delta\boldsymbol{u}^{\top}(N-1), \boldsymbol{0}^{\top}\right]^{\top}$ remains feasible. It is straightforward to demonstrate that constraints (10) to (12) are satisfied. Then, taking (13) at time step k:

$$\boldsymbol{W}_{\mathrm{ns}} \cdot (\boldsymbol{x}(N) - \boldsymbol{x}_{\mathrm{s}}^*) = \boldsymbol{0}, \\ \boldsymbol{W}_{\mathrm{ns}} \cdot \left(\boldsymbol{A}^N \cdot \boldsymbol{x}(0) + \boldsymbol{\Theta}_{\boldsymbol{N}} \cdot \Delta\boldsymbol{u}_k^* - \boldsymbol{x}_{\mathrm{s}}^*\right) = \boldsymbol{0}, \\ \boldsymbol{z}_{\mathrm{ns}}(N) = \boldsymbol{W}_{\mathrm{ns}} \cdot \boldsymbol{x}_{\mathrm{s}}^*, \tag{25}$$

where $\boldsymbol{\Theta}_{\boldsymbol{N}}$ is $\left[\boldsymbol{A}^{N-1} \cdot \boldsymbol{B}, \cdots, \boldsymbol{B}\right]$. Moving at the next time step, $k+1$, (13) gives:

$$\boldsymbol{W}_{\text{ns}} \cdot \left(\boldsymbol{A}^{N} \cdot \boldsymbol{x}(1) + \boldsymbol{\Theta}_{\boldsymbol{N}} \cdot \Delta\widetilde{\boldsymbol{u}}_{k+1} - \boldsymbol{x}_{\text{s}}^{*} \right) = \boldsymbol{0},$$
$$\boldsymbol{W}_{\text{ns}} \cdot \left(\boldsymbol{A} \cdot \left(\boldsymbol{A}^{\boldsymbol{N}} \cdot \boldsymbol{x}(0) + \boldsymbol{\Theta}_{\boldsymbol{N}} \cdot \Delta\boldsymbol{u}_{k}^{*} \right) - \boldsymbol{x}_{\text{s}}^{*} \right) = \boldsymbol{0},$$
$$\boldsymbol{J}_{\text{ns}} \cdot \boldsymbol{z}_{\text{ns}}(N) = \boldsymbol{W}_{\text{ns}} \cdot \boldsymbol{x}_{\text{s}}^{*}, \tag{26}$$

as $\boldsymbol{z}_{\text{ns}}(N)$ is at an equilibrium point, see (25), then $\boldsymbol{z}_{\text{ns}}(N) = \boldsymbol{J}_{\text{ns}} \cdot \boldsymbol{z}_{\text{ns}}(N)$, resulting in (26) to be equivalent to (25). Therefore, the inherited solution satisfies (13), and Problem P0 is recursively feasible.

Taking the difference of Φ_k^* and the cost function applying the inherited solution, $\widetilde{\Phi}_{k+1}$:

$$\Phi_k^* - \widetilde{\Phi}_{k+1} = \|\boldsymbol{x}(0) - \boldsymbol{x}_{\text{s}}^*\|_{\boldsymbol{Q}}^2 + \|\Delta\boldsymbol{u}(0)\|_{\boldsymbol{R}}^2 + \\ + \|f_{\text{e},k} - e_{\text{sp}}\|_P^2 - \|f_{\text{e},k+1} - e_{\text{sp}}\|_P^2, \tag{27}$$

since $f_{\text{e},k+1}$ is equal to $f_{\text{e},k}$, due to (5), it gives:

$$\Phi_k^* - \widetilde{\Phi}_{k+1} = \|\boldsymbol{x}(0) - \boldsymbol{x}_{\text{s}}^*\|_{\boldsymbol{Q}}^2 + \|\Delta\boldsymbol{u}(0)\|_{\boldsymbol{R}}^2. \tag{28}$$

The matrix $\boldsymbol{Q}$ is assumed positive semi-definite, while $\boldsymbol{R}$ is assumed positive-definite, consequently, Φ_k^* must be greater or equal to $\widetilde{\Phi}_{k+1}$. Additionally, the inherited solution is only a feasible solution at time step $k+1$, as a result the optimal cost function must comply with $\Phi_{k+1}^* \leq \widetilde{\Phi}_{k+1} \leq \Phi_k^*$. This demonstrates that the cost function is non-increasing along its time evolution, i.e a Lyapunov-like function, resulting in a recursively feasible control law. □

Remark 5. The stability proof of Problem P1 can be straightforwardly derived from Theorem 1, and regarding Problem P2, this proof can be obtained by using $\left[\Delta\widetilde{\boldsymbol{u}}_{k+1}^{\top}, \boldsymbol{x}_{\text{s}}^{*,\top}, \boldsymbol{\delta}^{*,\top}\right]^{\top}$ as the inherited solution at time step $k+1$.

Remark 6. If $\boldsymbol{x}(0)$ belongs to the domain of attraction, Theorem 1 states that Φ_k is a Lyapunov-like function, consequently the polyhedron of the domain of attraction, defined in Remark 2, is an invariant set for the closed-loop system. The bigger this set, the more flexible is the controller to accommodate system trajectories.

Remark 7. Taking into account that the cost function of Problem P2 is a Lyapunov-like function, it is non-increasing for any positive scalar P and at steady-states its minimum value is given by:

$$\Phi_\infty = \left\| \left.\frac{\mathrm{d}f_{\text{e}}}{\mathrm{d}\boldsymbol{x}}\right|_{\text{ss}} \cdot \boldsymbol{x}_{\text{s}} + f_{\text{e}}|_{\text{ss}} - e_{\text{sp}} \right\|_P^2, \tag{29}$$

if e_{sp} is unreachable, i.e. Problem P2 finds an equilibrium point $\boldsymbol{x}_{\text{s}}$ for which the economic performance is as close as possible of e_{sp}, in order to minimize Φ_∞.

On the one hand, if the economic target, e_{sp}, corresponds to a reachable steady-state, Problem P2 is able to zero Φ_∞, assuming that there are enough degrees of freedom in the input variables, i.e.:

$$f_{\text{e}}|_{\text{ss}} + \left.\frac{\mathrm{d}f_{\text{e}}}{\mathrm{d}\boldsymbol{x}}\right|_{\text{ss}} \cdot \boldsymbol{x}_{\text{s}} = e_{\text{sp}}. \tag{30}$$

3. CASE STUDY

The case study focuses on a Continuous Stirred Tank Reactor (CSTR) processing A→B, which can show unstable behavior. The dimensionless model of the system is borrowed from Nagrath et al. (2002):

$$\frac{dy_1}{d\tau} = u_1 \cdot (1 - y_1) - 0.072 \cdot y_1 \cdot \kappa, \tag{31}$$
$$\frac{dy_2}{d\tau} = u_1 \cdot (-y_2) - 0.3 \cdot (y_2 - y_3) + 0.0576 \cdot y_1 \cdot \kappa, \tag{32}$$
$$\frac{dy_3}{d\tau} = \frac{u_2 \cdot (-1 - y_3)}{0.1} + \frac{\delta \cdot (y_2 - y_3)}{0.05}, \tag{33}$$
$$\kappa = \exp\left(\frac{y_2}{1 + y_2/20}\right), \tag{34}$$

where τ is the dimensionless time, y_1 is the reactant A concentration, y_2 is the reactor temperature, y_3 is the cooling jacket temperature, u_1 is the feed flow rate of reactor and u_2 is the feed flow rate of the cooling fluid in the jacket. This system is linearized to obtain a continuous state-space in the following unstable steady-state: 0.6364 ($y_{1_{\text{ss}}}$), 1.9146 ($y_{2_{\text{ss}}}$), -0.4823 ($y_{3_{\text{ss}}}$), 0.7232 ($u_{1_{\text{ss}}}$) and 2.7779 ($u_{2_{\text{ss}}}$). Furthermore, such a system is discretized with sampling time 0.05. Thus, the discrete state space model obtained is:

$$\hat{\boldsymbol{x}}(j+1) = \begin{bmatrix} 9.44 \cdot 10^{-1} & -1.08 \cdot 10^{-2} & -4.99 \cdot 10^{-5} \\ 1.64 \cdot 10^{-1} & 1.04 \cdot 10^{0} & 7.41 \cdot 10^{-3} \\ 1.51 \cdot 10^{-2} & 1.48 \cdot 10^{-1} & 1.85 \cdot 10^{-1} \end{bmatrix} \cdot \hat{\boldsymbol{x}}(j) + \\ + \begin{bmatrix} 1.82 \cdot 10^{-2} & 4.85 \cdot 10^{-6} \\ -9.60 \cdot 10^{-2} & -1.20 \cdot 10^{-3} \\ -8.81 \cdot 10^{-3} & -1.25 \cdot 10^{-1} \end{bmatrix} \cdot \boldsymbol{u}(j) \tag{35}$$

$$\boldsymbol{y}(j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \hat{\boldsymbol{x}}(j). \tag{36}$$

Table 1 represents the desired operational zones of the system states, the constraints imposed on the manipulated variables, as well as constraints of slacks.

Table 1. Desired operational zones and constraints imposed on the inputs.

$\hat{\boldsymbol{x}}_{\max}$	$\begin{bmatrix}0.2 & 2.0 & 1.0\end{bmatrix}^\top$
$\hat{\boldsymbol{x}}_{\min}$	$\begin{bmatrix}-0.2 & -1.0 & -1.0\end{bmatrix}^\top$
$\boldsymbol{u}_{\max}$	$\begin{bmatrix}4.0 & 6.0\end{bmatrix}^\top$
$\boldsymbol{u}_{\min}$	$\begin{bmatrix}-0.5 & -1.\end{bmatrix}^\top$
$\Delta\boldsymbol{u}_{\max}$	$\begin{bmatrix}1.00 & 2.00\end{bmatrix}^\top$
$\boldsymbol{\delta}_{\max}$	$40\% \cdot \hat{\boldsymbol{x}}_{\max}$
$\boldsymbol{\delta}_{\min}$	$40\% \cdot \hat{\boldsymbol{x}}_{\min}$

In this case study, the aim of the control strategy is to regulate the reaction rate:

$$f_{\text{e}} = y_1 \cdot \exp\left(\frac{y_2}{1 + y_2/20}\right), \tag{37}$$

and the following scenarios are explored for the economic target: (i) an unreachable high reaction rate, (ii) an unreachable low reaction rate, and (iii) an achievable reaction rate.

The controller parameters are: $\boldsymbol{Q} = \text{diag}([1.\ 1.\ 0.5\ 0.1\ 0.1])$, $\boldsymbol{R} = \text{diag}([5\ 5])$, $\boldsymbol{S} = \text{diag}\left(\begin{bmatrix}10^2 & 10^2 & 10^2\end{bmatrix}\right)$ and $P = 5$. It must be emphasized that the slack weights are orders of magnitude higher than the other tuning weights, complying with Remark 4. Additionally, in this case study, it turns to be sufficient to define k_2 as zero, complying with Remark 1.

Figure 1 depicts the domains of attraction for Problems P0, P1, and P2 applying a control horizon of 2, as well as Problem P0 with the control horizon 5. On the one hand, the domain of attraction of Problem P1 is the smallest among all the Problems, due to imposing that all states $\boldsymbol{x}(N)$ must be at the artificial steady-state, as declared by Remark 3. On the other hand, Problem P2 allows some flexibility to accommodate unmeasured disturbances that may drive the system outside the bound constraints, which would turn Problem P0 unfeasible. Furthermore, even though increasing the control horizon of Problem P0 enlarges its domain of attraction, as depicted with $N = 5$, it must comply with the bound constraints, while the slack variables soften these constraints and allow a larger domain of attraction that can better accommodate disturbances.

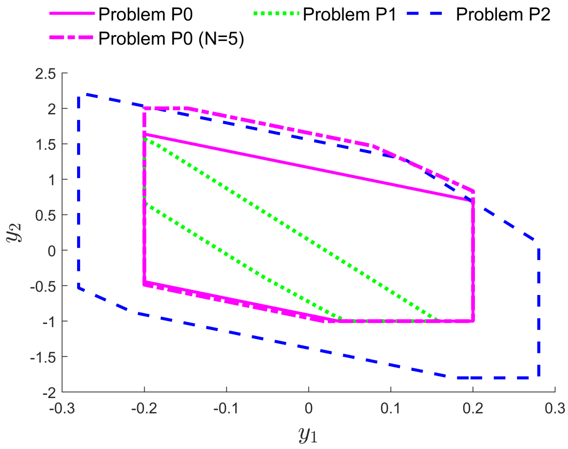

Figure 1. Domains of attraction for Problems P0 ($N = 2$, $N = 5$), P1 ($N = 2$) and P2 ($N = 2$).

Figures 2 to 5 depict the behaviour of the closed-loop system with Problem P2, applying a control horizon 2, initialized at $\boldsymbol{x}(0) = [0.23\ -1.3\ -1.2\ 0\ 0]$. It must be emphasized that this initial condition is only feasible for Problem P2. In order to demonstrate the impacts of disturbance, an unmeasured disturbance is simulated at 85 τ.

The strategy is able to track the economic target, e_{sp}, as illustrated by Figure 2. Until 100 τ such a target is unreachable for the closed-loop system, but the strategy remains feasible while driving the process to a steady-state condition close to it. This result exemplifies Remark 7. Furthermore, after 100 τ, e_{sp} becomes reachable and the closed-loop system is driven towards it without offset, which exemplifies Remark 7.

Figure 3 presents the closed-loop behavior of the outputs, which performance is directly related to the desired economic target: the higher the target, the lower the reactant concentration y_1, and the higher the reactor temperature y_2. Such a performance relationship is also present on the inputs in Figure 4. Moreover, it is explicit that the domain of attraction for Problem P0, Figure 1, cannot accommodate the conditions imposed by the initial states and the unmeasured disturbance at 85 τ, even for a control horizon $N = 5$.

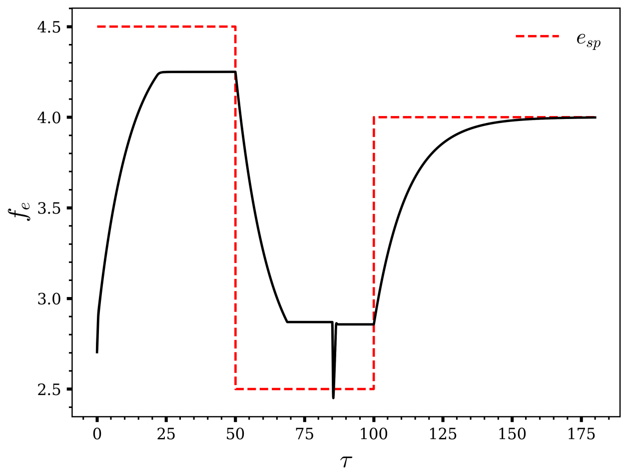

Figure 2. The behaviour of economic function f_e.

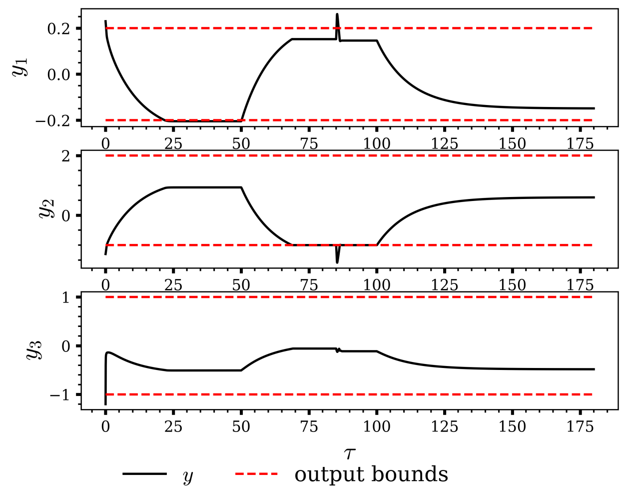

Figure 3. Outputs of the system with Problem P2 and $N = 2$.

One particular aspect of Problem P2 is the bound constraints imposed on the original states working as desired operational zones, similar to the zone control approach, due to the inclusion of slack variables. In this context, the economic target aims to drive the process to a desired operational condition inside these bounds.

Finally, Figure 5 depicts the cost function and its discrete difference only for the time steps related to the response to the initial condition and economic target change: 0 τ, 50 τ, 100 τ. The cost function is non-increasing, behaving as a Lyapunov-like function, exemplifying the results of Theorem 1.

4. CONCLUSION

This work presented a one-layer gradient-based economic MPC, which stabilizing properties are derived from terminal equality constraints that force non-stable states to be at any admissible artificial steady-state. The Jordan decomposition is applied to segregate states in stable and non-stable modes, and such a task can be performed without additional parametrization.

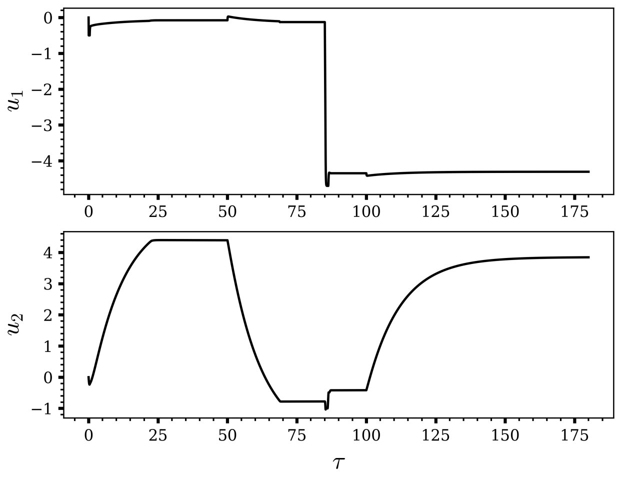

Figure 4. Inputs of the system with Problem P2 and $N = 2$.

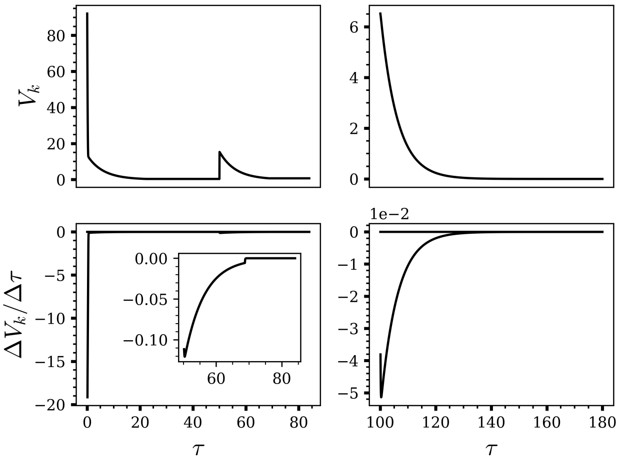

Figure 5. Cost function of Problem P2.

In this design, bound constraints on the original states can be softened in order to circumvent unfeasible scenarios caused by process disturbances. This enlarges the domain of attraction of the resulting optimization problem and can be useful to overcome practical implementation issues. It is also demonstrated that forcing both stable and non-stable modes to be at a steady-state at the end of the control horizon can reduce the domain of attraction of the controller.

Finally, the proposed approach is able to drive the closed-loop system towards an economic target. If such a target is unreachable, the controller retains its stabilizing properties and drives the system towards a region as close as possible of this target.

Future developments can assess the global feasibility of the formulation, i.e ensure solution for the optimization problem in the mismatch case at any time step, and a robustly stabilizing formulation.

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