Comparison of Control Strategies for Opinion Dynamics Models

Gabriel Gentil * **Amit Bhaya** *

* *Department of Electrical Engineering, Federal University of Rio de Janeiro, RJ (e-mail: gabrielgent@gmail.com; amit@nacad.ufrj.br).*

Abstract: This paper conducts a comparative analysis of control strategies applied to opinion dynamics models, focusing on the computational efficiency of Riccati and One Step Ahead Optimal Control (OSAOC) methodologies. Building upon existing literature, three methodologies are explored for opinion control, two of them utilizing game-theoretic Riccati equations and one using the OSAOC approach, employing Jacobi and Gauss-Seidel procedures for the last one. Findings reveal that OSAOC significantly reduces computational costs compared to methods employing Riccati equations, particularly as network size increases (nodes > 50).

Keywords: Optimal control theory, One Step Ahead Optimal Control, Riccati Equation, Opinion Dynamics Control, Game Theory.

1. INTRODUCTION

For linear systems with quadratic performance indices, the problem of regulation (the LQR problem) has an elegant solution using the Riccati equation approach (Anderson and Moore, 1989). This approach is popular in diverse applications, including the control of opinion dynamics models (Veetaseveera et al., 2021; Jiang et al., 2023; Gentil and Bhaya, 2024). Despite the appeal of the Riccati approach, it has some drawbacks for high-dimensional models. In fact, the computational complexity associated with solving Riccati equations for large-scale systems poses practical limitations, making real-time implementation cumbersome (Mohammadpour and Grigoriadis, 2010).

In One Step Ahead Optimal Control (OSAOC) (Kaszkurewicz and Bhaya, 2022), the idea is to make control decisions based on one step ahead predictions of system behavior, rather than optimizing control actions over an entire future trajectory. To accomplish this, the controller utilizes current information about the system state and employs a discrete-time model that naturally yields the system state at the next time step. The controller then determines the optimal control action that minimizes a specified performance criterion, such as cost or error. By iteratively applying this process at each time step, OSAOC enables the system to adapt and respond in real-time, making it suitable for dynamic and uncertain environments in which accurate long-term predictions are challenging. This approach also provides a computationally efficient solution for implementing optimal control strategies in practical circumstances.

This article presents a comparative analysis between Riccati and OSAOC methodologies. To achieve this, two control approaches utilizing Riccati and one using OSAOC through Jacobi and Gauss-Seidel procedures are used in opinion dynamic models (see (Hassani et al., 2022; Proskurnikov and Tempo, 2017, 2018) for more references on opinion dynamic models). Jiang et al. (2023) investigates the influence of external control on opinion dynamics in a social network. The model considers a game-theoretic framework in which players aim to minimize a function representing the deviation of opinions from a target and control costs. The paper employs a dynamic programming approach to calculate optimal control strategies in the context of opinion dynamics in a social network. Specifically, the Bellman equation (Bellman (1957)) is utilized to determine optimal control and equilibrium in the linear-quadratic games considered in the study. In Veetaseveera et al. (2021), the same problem is examined, but with two markets conceptualized as the players. Riccati equations are used to calculate the control, and two control policies are introduced: Uniform Broadcasting and Targeted Advertising. Finally, Gentil and Bhaya (2024) present an OSAOC approach where the player focuses solely on determining the best immediate move without estimating the opponent's move. In addition comparing the computational time and memory usage of these three methods, we also present preliminary results on the effects of incorporating opponent move estimation within OSAOC.

The main contribution of this article is to demonstrate the computational advantage of using OSAOC as a control method, comparing it with two existing methodologies presented in the literature.

The structure of this paper is as follows. After a general introduction in section 1, in section 2 provides a brief overview of how controls are obtained for each methodology studied. For brevity, we will refer to the methodology presented by Veetaseveera et al. (2021) as Riccati Control and the one presented by Jiang et al. (2023) as Bellman Control. This section also includes preliminary results for OSAOC with the estimation of opponent control. Section

⋆ This research was partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) Finance Code 001, CNPq BPP/PQ-Sr grant no. 313335/2022-2 [AB] and CNPq doctoral fellowship grant no. 163968/2021-7 [GG].

3 presents the benchmarking results of the three control methods and, finally, in section 4 concludes the article.

2. DE GROOT MODEL OF OPINION DYNAMICS WITH CONTROL

Let $\mathbf{A}$ be given $n \times n$ row stochastic matrix. Assuming the influence of p players on the opinion x_i of the ith agent ($i = 1, \ldots, n$) is encapsulated in an $n \times p$ matrix denoted as $\mathbf{B}$, where each column is identified as $\mathbf{b}_i$ ($i = 1, \ldots, p$), and the vector of agent opinions is represented as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Each player's influence or control action is designated as u_i, with the vector of control actions denoted as $\mathbf{u} \in \mathbb{R}^p$. Hence, the generalized DeGroot opinion dynamics game, involving n agents influenced by p players, is expressed as follows:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{1}$$

This model has been studied in Veetaseveera et al. (2021); Jiang et al. (2023); Gentil and Bhaya (2024) and, as mentioned earlier, this article will undertake a comparative analysis of these three methodologies for computing controls. Among these, two methodologies will be derived via Riccati equations, while the remaining two will employ the One Step Ahead Optimal Control approach. Notably, for the One Step Ahead Optimal Control methodology, we will utilize two distinct opinion update procedures: Jacobi and Gauss-Seidel, while for Riccati-based controls only the Jacobi procedure will be analyzed.

To ensure readability, the control computations involved in each methodology are presented briefly below, omitting all proofs that can be found in the original articles.

2.1 One Step Ahead Optimal Control (OSAOC)

The one-step-ahead index for the ith player at instant k, represented as $J_i(k)$, is defined following the standard procedure for quadratic indices, where γ_i denotes the control weight and $\mathbf{g}_i$ is the goal for the ith player:

$$J_i(k) = (\mathbf{x}(k+1) - \mathbf{g}_i)^{\mathrm{T}}(\mathbf{x}(k+1) - \mathbf{g}_i) + \gamma_i u_i^2. \tag{2}$$

In the Jacobi procedure, each player optimizes its index J_i based on the state vector $\mathbf{x}(k)$, known to all players. Following the computation of optimal controls u_i by all players, the resultant control vector $\mathbf{u} = (u_1, \ldots, u_p)$ is substituted into Equation 1 right-hand side for state computation. This computed state is subsequently utilized in the succeeding iteration of the Jacobi procedure.

By taking the partial derivative of Equation 2 with respect to control and setting it to zero, we obtain the control

$$u_i^{os}(k) = \frac{\mathbf{b}_i^{\mathrm{T}}(\mathbf{g}_i - \mathbf{A}\mathbf{x}(k))}{\mathbf{b}_i^{\mathrm{T}}\mathbf{b}_i + \gamma_i} \tag{3}$$

and state the main OSAOC result as follows:

Theorem 1. (Gentil and Bhaya (2024)). If $\mathbf{A}_{cl}^J$, defined below in (5), has a spectral radius, $\rho(\mathbf{A}_{cl}^J)$, strictly less than one, and the players influence disjoint sets of nodes, (i.e., $\mathbf{b}_i^T\mathbf{b}_j = 0, i \neq j$), then the dGc dynamics (1) using OSAOC (2), under the Jacobi game playing procedure, is asymptotically stable and opinions converge to the equilibrium point $\mathbf{x}^*$ defined as follows:

$$\mathbf{x}^* = \left(\mathbf{I} - \mathbf{A}_{cl}^J\right)^{-1} \left(\sum_{i=1}^{p} \tilde{\mathbf{P}}_{\boldsymbol{b}_i} \mathbf{g}_i\right), \tag{4}$$

where:

$$\tilde{\mathbf{P}}_{\mathbf{b}_i} = \frac{\mathbf{b}_i\mathbf{b}_i^{\mathrm{T}}}{\mathbf{b}_i^{\mathrm{T}}\mathbf{b}_i + \gamma_i}\ ,\ \mathbf{A}_{cl}^J = \left(\mathbf{I} - \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\right)\mathbf{A}. \tag{5}$$

For the Gauss-Seidel procedure, if players are numbered from 1 to p, an update sequence is defined as a permutation π of the integers 1 to p, denoted as $i \mapsto \pi(i)$. In the Gauss-Seidel update process at iteration k, according to this sequence, the player labeled $\pi(1)$ is the first to update the state $\mathbf{x}(k)$ using the one-step-ahead optimal control defined in (3).

Theorem 2. (Gentil and Bhaya (2024)). If $\mathbf{A}_{cl}^{GS}$, defined below in (7), has spectral radius, $\rho(\mathbf{A}_{cl}^J)$, strictly less than one, and the players influence disjoint sets of nodes, (i.e., $\mathbf{b}_i^T\mathbf{b}_j = 0, i \neq j$), then the dGc dynamics (1) using OSAOC (2), under the Gauss-Seidel game playing procedure, is asymptotically stable and opinions converge to the equilibrium point $\mathbf{x}^*$ defined as follows:

$$\mathbf{x}^* = \left(\mathbf{I} - \mathbf{A}_{cl}^{GS}\right)^{-1} \mathbf{d}^{GS}, \tag{6}$$

where:

$$\mathbf{A}_{cl}^{GS} = \left(\prod_{i=1}^{p}[(\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(i)}})\mathbf{A}]\right) \tag{7}$$

$$\mathbf{d}^{GS} = \sum_{m=1}^{p-1}\left(\prod_{i=1}^{m}[(\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p-i+1)}})\mathbf{A}]\right)\tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p-m)}}\mathbf{g}_{\pi(p-m)} + \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p)}}\mathbf{g}_{\pi(p)} \tag{8}$$

2.2 Riccati Control

Veetaseveera et al. (2021) proposed a payoff function of player m with the form:

$$J_m(\mathbf{K}_m, \mathbf{K}_{-m}) = \sum_{t=0}^{\infty} \delta^t(||\mathbf{x}(k) - \mathrm{d}_m\mathbf{1}_n||^2 + u_m(k)^{\mathrm{T}}\mathbf{R}_m u_m(k)) \tag{9}$$

where $\delta \in (0, 1)$ is the discount factor to ensure convergence, d_m is the target for player m, $\mathbf{R}_m$ is the control weight for the same player, and $\boldsymbol{K}_m$ is the static feedback strategy for player m. Note that (9) is the infinite horizon version of (2).

Let $\beta_m(\mathbf{K}_{-m})$ denote the best response function defined as:

$$\beta_m(\mathbf{K}_{-m}) = \mathrm{argmin}_{\mathbf{K_m}} J(\mathbf{K}_m, \mathbf{K}_{-m}). \tag{10}$$

Typically, the best response function is set-valued since the arg min is not always unique. However, the subsequent proposition defines a unique best response in this case and outlines the method for determining its value. The reader is referred to Veetaseveera et al. (2021) for further details.

Proposition 1. (Veetaseveera et al. (2021)). The best response function is unique and can be evaluated as

$$\beta_m(\mathbf{K}_{-m}) = \delta(\tilde{\mathbf{B}}\mathbf{P}_m\tilde{\mathbf{B}}_m + \mathbf{R}_m)^{-1}\tilde{\mathbf{B}}_m^{\mathrm{T}}\mathbf{P}_m\tilde{\mathbf{D}}_m, \quad (11)$$

where:

$$\tilde{\mathbf{D}}_m = \begin{bmatrix} \mathbf{D} - \mathbf{B}_{-m}\mathbf{G}_{-m} & \mathbf{B}_{-m}(\mathbf{G}_{-m}\mathbf{1}_n(d_{-m} - d_m) - \mathbf{M}_{-m}) \\ \mathbf{0}_n^{\mathrm{T}} & 1 \end{bmatrix} \quad (12)$$

$$\tilde{\mathbf{B}}_m = \begin{bmatrix} \mathbf{B}_m \\ \mathbf{0}_n^{\mathrm{T}} \end{bmatrix} \quad (13)$$

and $\mathbf{P}_m$ is the solution to the Algebraic Riccati equation

$$\mathbf{P}_m = \tilde{\mathbf{D}}_m^{\mathrm{T}}(\delta\mathbf{P}_m - \delta^2\mathbf{P}_m\tilde{\mathbf{B}}_m(\delta\tilde{\mathbf{B}}_m^{\mathrm{T}}\mathbf{P}_m\tilde{\mathbf{B}}_m + \mathbf{R}_m)^{-1}\tilde{\mathbf{B}}_m^{\mathrm{T}}\mathbf{P}_m)\tilde{\mathbf{D}}_m + \mathbf{Q}_m \quad (14)$$

where $\mathbf{Q}_m = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ \mathbf{0}_n^{\mathrm{T}} & 1 \end{bmatrix}$.

Using the above proposition, the optimal control is defined:

$$u_m^* = -\delta(\delta\tilde{\mathbf{B}}_m^{\mathrm{T}}\mathbf{P}_m\tilde{\mathbf{B}}_m + \mathbf{R}_m)^{-1}\tilde{\mathbf{B}}_m^{\mathrm{T}}\mathbf{P}_m\tilde{\mathbf{D}}_m \begin{bmatrix} \mathbf{x}(k) - \mathbf{1}_n d_m \\ 1 \end{bmatrix} \quad (15)$$

Algorithm 1 presents an iterative asynchronous best response process aimed at discovering a Nash equilibrium for the game G. Although the convergence of the iterative best response dynamics is not ensured for all game classes, but the numerical experiments presented in Veetaseveera et al. (2021) point towards its consistency.

Algorithm 1 Sequencial Gain computation

Require: $\delta = 0.999; \epsilon = 10^{-6}$
Ensure: $\mathbf{K}_1^0, \mathbf{K}_2^0, k = 0;$
  while $||\mathbf{K}_1^{k+1} - \mathbf{K}_1^k|| > \epsilon$ or $||\mathbf{K}_2^{k+1} - \mathbf{K}_2^k|| > \epsilon$ **do**
    $\mathbf{K}_1^{k+1} \leftarrow \beta_1(\mathbf{K}_2^k);$
    $\mathbf{K}_2^{k+1} \leftarrow \beta_2(\mathbf{K}_1^{k+1})$
    $k \leftarrow k + 1$
  end while

2.3 Bellman Control

The payoff function is similar to Equation 9, therefore it will be omitted here, but details are in Jiang et al. (2023). In this section, the Bellman equation is used to achieve the controller and can be written as:

$$V_m(k, \mathbf{x}) = \min_{u_m(k)} [\mathbf{x}(k)^{\mathrm{T}}\mathbf{x}(k) + \mathbf{q}_m\mathbf{x}(k) + n\mathrm{d}_m^2 + \gamma_m(u_m(k))^2 + \delta\mathrm{V_m(k+1, x(k+1))}], \quad (16)$$

with $\mathbf{q}_m = (-2\mathrm{d}_m, \ldots, -2\mathrm{d}_m) \in \mathbb{R}^n$. Let $V_m(k, \mathbf{x})$ have the form:

$$V_m(k, \mathbf{x}) = \mathbf{x}(k)^{\mathrm{T}}\mathbf{K}_m\mathbf{x}(k) + \mathbf{k}_m\mathbf{x}(k) + k_{0m}, \quad (17)$$

where $\mathbf{K}_m \in \mathbb{R}^{n\times n}$ is a symmetric matrix and $\mathbf{k}_m \in \mathbb{R}^n$ is a row vector. Performing the necessary calculations leads to the optimal control:

$$u_m^*(k) = \alpha_m(\boldsymbol{\eta}\mathbf{x}(k) + \lambda_m + \boldsymbol{\beta}_m U^{-m}(k)), \quad (18)$$

where:

$$\alpha_m = -(\gamma_m + \delta\mathbf{B}_{i'}^{\mathrm{T}}\mathbf{K}_m\mathbf{B}_{i'})^{-1}\delta,$$
$$\boldsymbol{\eta}_m = \mathbf{B}_{i'}^{\mathrm{T}}\mathbf{K}_m\mathbf{A},$$
$$\lambda_m = \frac{1}{2}\mathbf{k}_m\mathbf{B}_{i'},$$
$$\boldsymbol{\beta}_m = \mathbf{B}_{i'}^{\mathrm{T}}\mathbf{K}_m.$$

Since Equation 18 is linear, we have:

$$u_m^*(k) = \mathbf{c}_m\mathbf{x}(k) + c_{0m}, \quad (19)$$

where $c_m = (c_{1m}, c_{2m}, \ldots, c_{nm}) \in \mathbb{R}^n$ and $c_{0m} \in \mathbb{R}^1$.

The matrix $\mathbf{K}_m$ and vector $\mathbf{k}_m$ are found using Riccati equation:

$$\mathbf{K}_m = \mathbf{I} + \gamma\mathbf{c}_m^{\mathrm{T}}\mathbf{c}_m + \delta\left(\mathbf{A} + \mathbf{B}\mathbf{c}_k\right)^{\mathrm{T}}\mathbf{K}_m\left(\mathbf{A} + \mathbf{B}\mathbf{c}_k\right) \quad (20)$$

$$\mathbf{k}_m = \mathbf{q}_m + 2\gamma_m\mathbf{c}_m\mathbf{c}_{0m} + 2\delta(\mathbf{B}\mathbf{c}_{0k})^{\mathrm{T}}\mathbf{K}_m(\mathbf{A} + \mathbf{B}\mathbf{c}_k) + \delta\mathbf{k}_m(\mathbf{A} + \mathbf{B}\mathbf{c}_k) \quad (21)$$

With $\mathbf{c}_k = (c_1, c_2, \ldots, c_p) \in \mathbb{R}^{p\times n}$ and $\mathbf{c}_{0k} = (c_{01}, c_{02}, \ldots, c_{0p}) \in \mathbb{R}^p$ The process to compute the controllers is similar to that presented in Algorithm 1. The reader is referred to Jiang et al. (2023) for further details.

2.4 Comparison of models

Both Jiang et al. (2023) and Veetaseveera et al. (2021) rely on Riccati equations, resulting in computationally intensive gain computations and yielding similar outcomes. Conversely, OSAOC employs only matrix-vector products, thereby offering a distinct computational advantage. Another key distinction lies in the estimation of the opponent's control action. In Riccati-based models, the best response strategy is normally applied, whereas in OSAOC-based models under the orthogonality assumption, players do not need to use the opponent's action when calculating their control.

Initial findings are presented for a scenario featuring two players using OSAOC strategy, but without the orthogonality assumption, accompanied by the following assumptions regarding dynamics and objectives:

- Dynamics:
  $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{b}_1u_1(k) + \mathbf{b}_2u_2(k)$; initial state x_0 (22)
- Player 1 wishes to drive $\mathbf{x}(k)$ to g_1, as $k \to \infty$.
- Player 2 wishes to drive $\mathbf{x}(k)$ to g_2, as $k \to \infty$.

Player 1, at time k, knows $\mathbf{x}(k)$ and estimates player 2's action $\hat{u}_2(k)$ under the Jacobi or simultaneous play assumption. And player 2, at time k, also knows the value of $\mathbf{x}(k)$ and uses estimate $\hat{u}_1(k)$. Thus, the indices are:

$$J_1 = (\mathbf{x}(k+1) - \mathbf{g}_1)^{\mathrm{T}}(\mathbf{x}(k+1) - \mathbf{g}_1) + \gamma_1u_1(k)^2 \quad (23)$$

$$J_2 = (\mathbf{x}(k+1) - \mathbf{g}_2)^{\mathrm{T}}(\mathbf{x}(k+1) - \mathbf{g}_2) + \gamma_2u_2(k)^2 \quad (24)$$

By setting the partial derivative with respect to the first player's control to zero, we obtain:

$$\begin{aligned}\frac{\partial J_1}{\partial u_1} &= \frac{\partial}{\partial u_1}((\mathbf{A}\mathbf{x}(k)+\mathbf{b}_1u_1(k)+\mathbf{b}_2\hat{u}_2(k)-\mathbf{g}_1)^{\mathrm{T}} \\ &\quad (\mathbf{A}\mathbf{x}(k)+\mathbf{b}_1u_1(k)+\mathbf{b}_2\hat{u}_2(k)-\mathbf{g}_1)+\gamma_1u_1(k)^2) \\ &= 2\mathbf{b}_1^{\mathrm{T}}\left(\mathbf{A}\mathbf{x}(k)+\mathbf{b}_1u_1(k)+\mathbf{b}_2\hat{u}_2k-\mathbf{g}_1\right)+2\gamma_1u_1(k),\end{aligned}$$

Solving for the optimal u_1 gives:

$$u_1^{os}(k) = \frac{\mathbf{b}_1^{\mathrm{T}}\left[\mathbf{g}_1-(\mathbf{A}\mathbf{x}(k)+\mathbf{b}_2\hat{u}_2(k))\right]}{\mathbf{b}_1^{\mathrm{T}}\mathbf{b}_1+\gamma_1} \tag{25}$$

Similarly, the control for the second player is:

$$u_2^{os}(k) = \frac{\mathbf{b}_2^{\mathrm{T}}\left[\mathbf{g}_2-(\mathbf{A}\mathbf{x}(k)+\mathbf{b}_1\hat{u}_1(k))\right]}{\mathbf{b}_2^{\mathrm{T}}\mathbf{b}_2+\gamma_2} \tag{26}$$

In fact, for the ith Player, the control is:

$$u_i^{os}(k) = \frac{\mathbf{b}_i^{\mathrm{T}}\left[\mathbf{g}_i-\left(\mathbf{A}\mathbf{x}(k)+\sum_{j=1,i\neq j}^{p}\mathbf{b}_j\hat{u}_j(k)\right)\right]}{\mathbf{b}_i^{\mathrm{T}}\mathbf{b}_i+\gamma_i} \tag{27}$$

Plugging Equation 27 into dynamics Equation 22:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k)+\sum_{i=1}^{p}\frac{\mathbf{b}_i\mathbf{b}_i^{\mathrm{T}}}{\mathbf{b}_i^{\mathrm{T}}\mathbf{b}_i+\gamma_i} \\ &\quad \left(\mathbf{g}_i-\left(\mathbf{A}\mathbf{x}(k)+\sum_{j=1,i\neq j}^{p}\mathbf{b}_j\hat{u}_j(k)\right)\right) \\ &= \mathbf{A}\mathbf{x}(k)-\sum_{i=1}^{p}\left(\tilde{\mathbf{P}}_{\mathbf{b}_i}\mathbf{A}\mathbf{x}(k)\right) \\ &\quad +\sum_{i=1}^{p}\tilde{\mathbf{P}}_{\mathbf{b}_i}\left(\mathbf{g}_i-\sum_{j=1,i\neq j}^{p}\mathbf{b}_j\hat{u}_j(k)\right) \\ \mathbf{x}(k+1) &= \left(\mathbf{I}-\sum_{i=1}^{p}\tilde{\mathbf{P}}_{\mathbf{b}_i}\right)\mathbf{A}\mathbf{x}(k)+ \\ &\quad \sum_{i=1}^{p}\tilde{\mathbf{P}}_{\mathbf{b}_i}\left(\mathbf{g}_i-\sum_{j=1,i\neq j}^{p}\mathbf{b}_j\hat{u}_j(k)\right)\end{aligned} \tag{28}$$

Using above strategy, consider a simple "2 agents 2 player" example with $\mathbf{A} = \frac{1}{2}\mathbf{1}\mathbf{1}^T$). First, consider a Jacobi procedure where both players influence agent 1. Player 1's goal is $g_1 = 0.9$ and Player 2's goal is $g_2 = 0.3$. Figure 1 presents the opinions for this case. In this scenario, each player's estimation of the opponent's strategy is simply the previous action. In other words, we consider the simple estimate $\hat{u}_2(k) = u_2(k-1)$ and it results in instability.

There are also instances where utilizing this estimate does not destabilize, but fails to produce discernible advantages. Consider an alternative scenario employing the same opinion dynamic network and controls, but this time employing the Gauss-Seidel update method, which results in the following closed-loop dynamics:

$$\mathbf{x}(k+1) = \mathbf{A}_{cl}^{GS}\mathbf{x}(k)+\mathbf{d}^{GS}, \tag{29}$$

where:

$$\mathbf{A}_{cl}^{GS} = \left(\prod_{i=1}^{p}[(\mathbf{I}-\tilde{\mathbf{P}}_{\mathbf{b}_{\pi(i)}})\mathbf{A}]\right) \tag{30}$$

$$\begin{aligned}\mathbf{d}^{GS} &= \sum_{m=1}^{p-1}\left(\prod_{i=1}^{m}[(\mathbf{I}-\tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p-i+1)}})\mathbf{A}]\right)\tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p-m)}} \\ &\quad \left(\mathbf{g}_{\pi(p-m)}-\sum_{r=1,r\neq p-m}^{p}\mathbf{b}_r\hat{u}_r(k)\right) \\ &\quad +\tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p)}}\left(\mathbf{g}_{\pi(p)}-\sum_{r=1,r\neq p}^{p}\mathbf{b}_r\hat{u}_r(k)\right)\end{aligned} \tag{31}$$

Utilizing the same matrix $\mathbf{A}$ as in the previous example, Player 1 now influences agent 1, while Player 2 influences agent 2, maintaining the same objectives as in the previous scenario. Figure 2 illustrates the evolution of opinions under Gauss-Seidel updates, with the second player steering the node influenced by it to the desired target value.

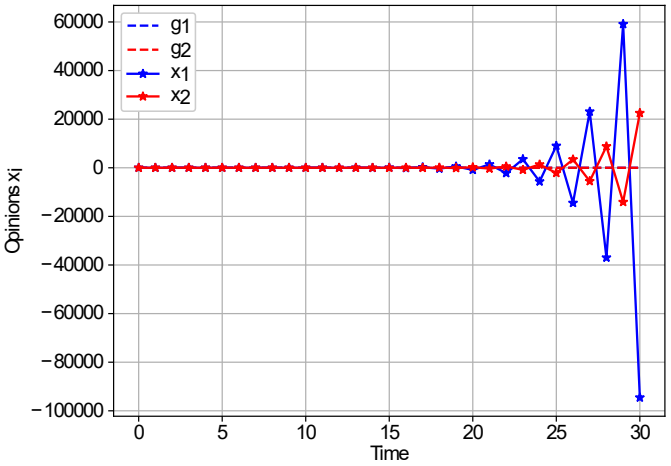

Figure 1. DeGroot model with players estimating the opponent action using Jacobi procedure.

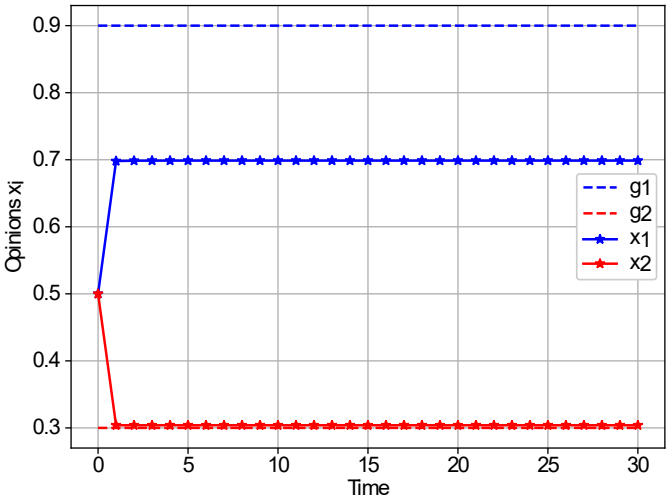

Figure 2. DeGroot model with players estimating the opponent action using Gauss-Seidel procedure.

3. BENCHMARK COMPARISON

For a comparative analysis, we adopt Uniform Broadcasting (UB) as proposed by Veetaseveera et al. (2021). In UB, each player exercises influence over all nodes through a single control parameter u, i.e., disseminating a unified advertisement (control) to impact all agents. We assume that the objectives of the players are characterized by opposing values, i.e. Player 1 pursues an objective denoted by $g_1 = 2$, while Player 2 aims for $g_2 = -2$. For all methods, the control discount was set at $\gamma = 0.01$. Specifically, for the Riccati and Bellman approaches, δ was set at 0.999 and ϵ at 10^{-6}, values chosen in Veetaseveera et al. (2021). For all simulations, Erdös-Renyi random graphs are constructed (Erdös and Rényi, 1959). Figure 3 shows the evolution of opinions among 10 agents, and it becomes evident that methodologies using Riccati equations exhibit similar patterns. However, the larger differences lie in the computational demands for generating results, as discussed below.

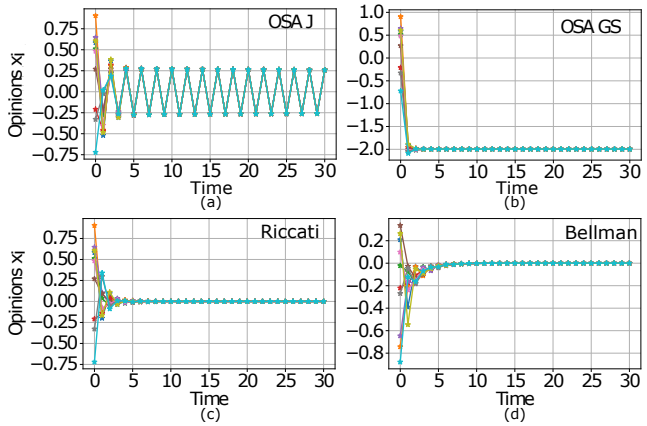

Figure 3. Evolution of opinions for a 10-node graph using the controls: (a) One Step Ahead with Jacobi Procedure (OSA J), (b) One Step Ahead with Gauss-Seidel Procedure (OSA GS), (c) Riccati Control and (d) Bellman Control.

For the Riccati and Bellman control strategies, the agents reach consensus at 0, for OSA GS, the second player achieves its objectives, since under the GS procedure, the second player has the advantage over the first player due to the fixed order, as mentioned in Gentil and Bhaya (2024), and for OSA J, the opinions oscillate around zero.

The *BenchmarkTools* package in Julia is utilized for the subsequent comparisons, using the same process under similar conditions to ensure that the hardware and operating system are the same for every test.

Figure 4 shows the comparison time for the methods used. It is evident that as the network size increases (nodes > 50), the computational time required for Riccati Control is one or more orders of magnitude greater than that for OSAOC. Moreover, roughly speaking, the computational time for Riccati Control grows exponentially, while OSAOC's time increases linearly, as illustrated in Figure 5. The Bellman Control methodology is not represented in the graphs due to its substantial convergence time; for instance, it takes over 250 seconds to converge for a network with 10 nodes. Notably, using the same discount factor δ and error ϵ settings for convergence (Algorithm 1), Riccati Control converges faster than Bellman Control and produces similar results (Figure 3).

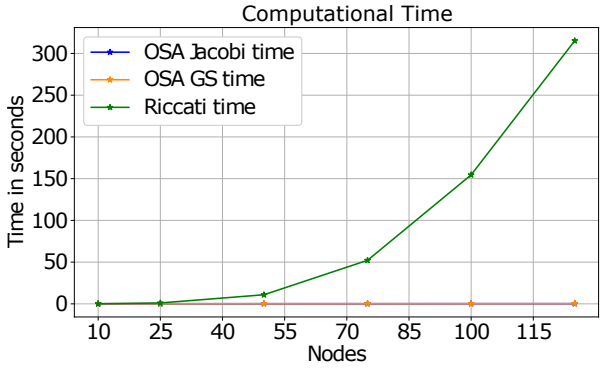

Figure 4. Comparison of computational times for OSA under Jacobi and GS control, and Riccati strategies.

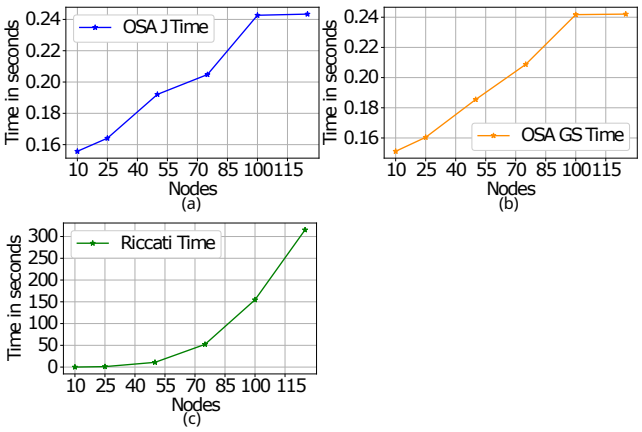

Figure 5. Computational time for each strategy for $10, 25, 50, 100$ and 125 nodes graph for (a) OSA J, (b) OSA GS, and (c) Riccati.

The memory usage, shown in Figure 6, follows the same pattern as the time comparison. For Riccati Control, memory usage grows exponentially, while for OSAOC, it grows linearly, as depicted in Figure 5. Similarly, the memory required for Bellman Control calculations is much higher than that required for the other methodologies, thus it is not shown in Figure 6.

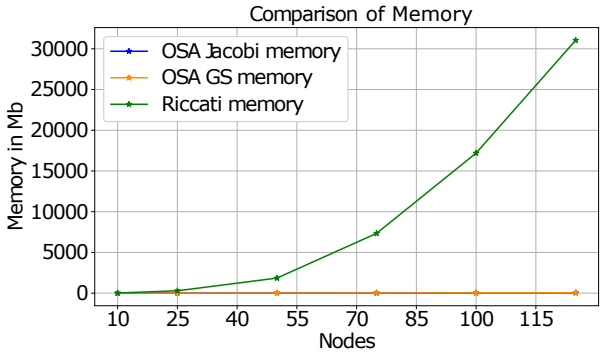

Figure 6. Comparison memory required for OSA under Jacobi and GS, and Riccati control strategies.

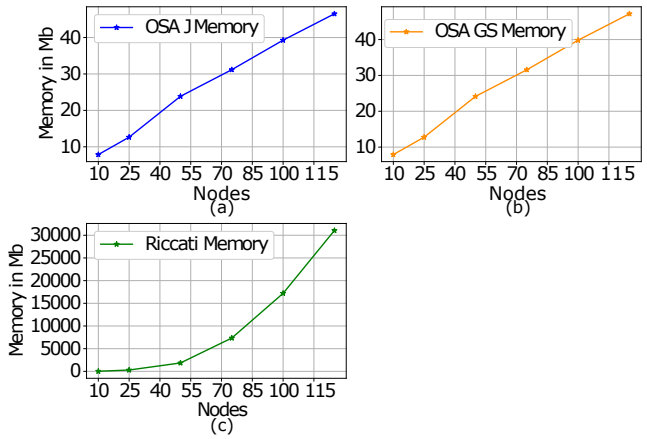

Figure 7. Memory required for each strategy for $10, 25, 50, 100$ and 125 nodes graph for (a) OSA J, (b) OSA GS, and (c) Riccati.

4. CONCLUSION

In summary, the comparative analysis presented in this paper shows the efficacy of the One Step Ahead Optimal Control (OSAOC) methodology in handling large-scale models, as well as its considerable computational advantages over Riccati-based equations in this context. Preliminary findings, shown in subsection 2.4, suggest that incorporating simple opponent strategy estimation within the OSAOC framework does not yield significant improvements and may even introduce instability. Future endeavors will focus on further validation of these findings through additional tests and theoretical estimates. Moreover, we propose expanding comparative studies to encompass traditional control concerns, considering factors like system noise and control action delay.

REFERENCES

Anderson, B.D.O. and Moore, J.B. (1989). *Optimal Control: Linear Quadratic Methods*. Prentice-Hall, Engelwood Cliffs, N.J.

Bellman, R. (1957). *Dynamic Programming*. Princeton University Press, Princeton, NJ.

Erdös, P. and Rényi, A. (1959). On random graphs i. *Publicationes Mathematicae Debrecen*, 6, 290–297.

Gentil, G. and Bhaya, A. (2024). Opinion dynamic games under one step ahead optimal control. *IEEE Transactions on Computational Social Systems*, 11(3), 4202–4213. doi:10.1109/TCSS.2024.3364611. Correction published in Vol. 11, n. 4, p. 5554.

Hassani, H., Razavi-Far, R., Saif, M., Chiclana, F., Krejcar, O., and Herrera-Viedma, E. (2022). Classical dynamic consensus and opinion dynamics models: A survey of recent trends and methodologies. *Information Fusion*, 88, 22–40.

Jiang, H., Mazalov, V.V., Gao, H., and Wang, C. (2023). Opinion dynamics control in a social network with a communication structure. *Dynamic Games and Applications*, 13, 412–434.

Kaszkurewicz, E. and Bhaya, A. (2022). *Business Dynamics Models: Optimization-Based One Step Ahead Optimal Control*. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Mohammadpour, J. and Grigoriadis, K.M. (2010). *Efficient Modeling and Control of Large-Scale Systems*. Springer New York, 1 edition.

Proskurnikov, A.V. and Tempo, R. (2017). A tutorial on modeling and analysis of dynamic social networks. part I. *Annual Reviews in Control*, 43, 65–79.

Proskurnikov, A.V. and Tempo, R. (2018). A tutorial on modeling and analysis of dynamic social networks. part II. *Annual Reviews in Control*, 45, 166–190.

Veetaseveera, J., Varma, V.S., and Morarescu, I.C. (2021). A dynamic game formulation for control of opinion dynamics over social networks. In *Network games, control and optimization*, volume 1354, 252–260.