# CONTROL OF A SINGLE-LINK FLEXIBLE-JOINT ROBOT VIA BACKSTEPPING

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**Abstract**— It is usual to consider, in the derivation of robot dynamics, that the arm is a linkage of connected rigid bodies, neglecting, this way, cross-coupling and nonlinear effects that result from the dynamic interactions with the drive system. However, many industrial robots exhibit significant joint flexibility and cannot be considered a pure system of rigid bodies. Thus, control designs that do not consider this effect may lead to poor performance, especially for precise motion systems, where high performance with high speed is required. This paper proposes a control design based on the backstepping technique, where the gains are obtained by solving an optimization problem, which is able to guarantee tracking of the reference trajectory for a single-link robot manipulator with flexible joints.

**Keywords**— Robot manipulator, Trajectory tracking, Backstepping, Nonlinear control.

**Resumo**— A maioria dos modelos dinâmicos de robôs manipuladores assume que o braço robótico é composto por uma união de elos rígidos, negligenciando assim os efeitos do acoplamento dinâmico e os efeitos não lineares resultantes da interação dinâmica com o sistema de acionamento. Porém, vários robôs industriais possuem juntas significativamente flexíveis, fazendo com que o sistema não possa ser considerado como conexão de corpos rígidos. Desta forma, os projetos de controladores que não consideram esse tipo de efeito podem levar o robô a ter um desempenho limitado, principalmente em sistemas que devem seguir uma trajetória desejada com precisão. Este artigo propõe o projeto de um controlador não linear, baseado na técnica backstepping, em que os ganhos são obtidos através da solução de um problema de otimização, capaz de garantir rastreamento de trajetória para um braço robótico de um único elo de junta flexível.

**Palavras-chave**— Robô, Rastreamento de trajetória, Backstepping, Controle não-linear.

## 1 Introduction

Robot manipulators are extensively used in all major industries, for a large variety of applications (Spong, 1992; Spong et al., 2006; Abdallah et al., 1991). One main difficulty in designing controllers, for precise trajectory tracking of the robot end-effector, is produced by the flexibility of the structure, of which the main source are the harmonic drives (Sweet and Good, 1985; Spong, 1990). This aspect is of paramount significance for high performance applications.

Thus, it is important to consider a dynamic model that includes such effects, like the single-link flexible-joint nonlinear dynamic model. For such class of systems, it is important to use a nonlinear control design strategy (Slotine, 1991; Khalil, 2002), which is robust and can be efficiently used for trajectory tracking purposes.

The backstepping procedure is among the most important nonlinear control design techniques, with numerous applications (Kokotovic, 1992; Krstic et al., 1995; Freeman and Kokotović, 1993). It provides a recursive method for stabilizing the origin of a system in strict-feedback form, and can be used for output stabilization or trajectory tracking. Moreover, it is also possible to use backstepping to relax the matching condition on the additive uncertainties required by most control techniques. Application of backstepping in robotics can be found in (Sahab and Modabbernia, 2011; Bridges et al., 1995; Fierro and Lewis, 1995; Lozano and Brogliato, 1992; Ramirez et al., 2003) and references therein.

This paper proposes a nonlinear control law based on the backstepping approach, which is able to solve, in some sense, the trajectory tracking control problem for the single-link flexible-joint robot manipulator. An optimization problem, inspired by the celebrated LQR control design (Kwakernaak and Sivan, 1972), is proposed in order to optimally select the gains used within the backstepping control law. The proposed control problem consists in developing a control law for the robot manipulator that guarantees both uniform ultimate boundedness of the tracking error under bounded disturbance, and asymptotic tracking for the system without uncertainties and disturbances.

## 2 Single-link flexible-joint robot

Figure 1 shows the undamped single-link flexible-joint robot.

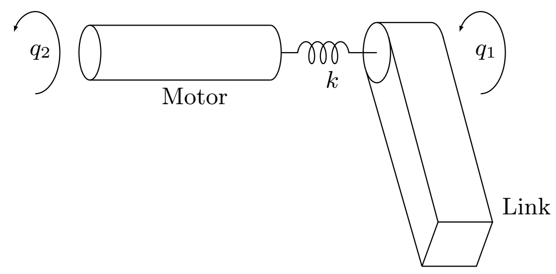


Fig. 1: Single-link flexible-joint robot.

The equation of motion for this system, taken from (Ghorbel et al., 1989), is given by

$$I\ddot{q}_1 + k(q_1 - q_2) + Mgl\sin(q_1) = 0$$
$$J_m\ddot{q}_2 + k(q_2 - q_1) = u,$$

in which $k$ is the joint stiffness, $I$ is the link inertia, $M$ is the link mass, $l$ is the distance from the shaft to the link center of mass, $J_m$ is the motor inertia, and $u$ is the torque used as control input. The position and angular velocity of the link are respectively denoted by $q_1$ and $\dot{q}_1$. The position and angular velocity of the motor are respectively denoted by $q_2$ and $\dot{q}_2$.

By defining the state vector $x$ as

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} q_1 & \dot{q}_1 & q_2 & \dot{q}_2 \end{bmatrix}^T,$$

the model is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-k}{I}x_1 + \frac{Mgl}{I}\sin(x_1) + \frac{k}{I}x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{k}{J_m}x_1 - \frac{k}{J_m}x_3 + \frac{1}{J_m}u,
\end{aligned}$$

which can be equivalently written as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1(x_1) + \nu_3 x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2(x_1, x_3) + \nu_5 u,
\end{aligned} \tag{1}$$

where

$$f_1(x_1) = \nu_1 x_1 + \nu_2 \sin(x_1), \quad \nu_3 = k/I$$
$$f_2(x_1, x_3) = \nu_4(x_1 - x_3), \quad \nu_5 = 1/J_m,$$

with

$$\nu_1 = -k/I, \quad \nu_2 = Mgl/I, \quad \nu_4 = k/J_m$$

## 3 Controller design

The goal of the controller is to asymptotically drive to zero the position error $e_1$, defined by

$$e_1 = x_1 - x_d, \tag{2}$$

where $x_d$ is the reference position, i.e., the desired link position, for a system without disturbance and precisely known.

The approach proposed in this paper to solve the above trajectory tracking control problem is based on the backstepping procedure. For a detailed explanation of backstepping see (Krstic et al., 1995; Khalil, 2002).

To apply the backstepping procedure, consider the dynamics of the position error given by

$$\dot{e}_1 = \dot{x}_1 - \dot{x}_d = x_2 - \dot{x}_d.$$

Now, define the first Lyapunov function candidate as

$$V_1(e_1) = \frac{1}{2}e_1^2.$$

Then, its derivative is given by

$$\dot{V}_1 = e_1\dot{e}_1 = e_1(x_2 - \phi_1 + \phi_1 - \dot{x}_d).$$

For the next step, let's define $e_2$ by

$$e_2 = x_2 - \phi_1, \tag{3}$$

with $\phi_1$ a virtual input for $x_2$. Then, we have

$$\dot{V}_1 = e_1(e_2 + \phi_1 - \dot{x}_d).$$

Now, choosing the virtual input $\phi_1$ as

$$\phi_1 = \dot{x}_d - \lambda_1 e_1,$$

we obtain

$$\dot{V}_1 = e_1(e_2 - \lambda_1 e_1) = -\lambda_1 e_1^2 + e_1 e_2.$$

Notice that $\dot{e}_2$ is given by

$$\dot{e}_2 = \dot{x}_2 - \dot{\phi}_1 = f_1(x_1) + \nu_3 x_3 - \dot{\phi}_1.$$

To proceed, let us define the second Lyapunov function candidate $V_2$ as

$$V_2(e_1, e_2) = V_1(e_1) + \frac{1}{2}e_2^2.$$

Then, we obtain, after some manipulations,

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + e_2\dot{e}_2 \\
&= -\lambda_1 e_1^2 + e_1 e_2 + e_2(f_1(x_1) + \\
&\qquad \nu_3 x_3 - \dot{\phi}_1) \\
&= -\lambda_1 e_1^2 + e_1 e_2 + e_2(f_1(x_1) + \\
&\qquad \nu_3(x_3 - \phi_2) + \nu_3\phi_2 - \dot{\phi}_1) \\
&= -\lambda_1 e_1^2 + e_1 e_2 + e_2(f_1(x_1) + \nu_3 e_3 \\
&\qquad + \nu_3\phi_2 - \dot{\phi}_1),
\end{aligned}$$

with $e_3$ defined as

$$e_3 = x_3 - \phi_2. \tag{4}$$

In this case, $\phi_2$ is the control law for the virtual input $x_3$. Choosing $\phi_2$ as

$$\phi_2 = \frac{1}{\nu_3}(-f_1(x_1) + \dot{\phi}_1 - e_1 - \lambda_2 e_2),$$

it immediately follows that

$$\dot{V}_2 = -\lambda_1 e_1^2 - \lambda_2 e_2^2 + \nu_3 e_2 e_3.$$

Notice that $\dot{e}_3$ is given by

$$\dot{e}_3 = \dot{x}_3 - \dot{\phi}_2 = x_4 - \dot{\phi}_2.$$

Now, it remains to design a control law $\phi_3$ for the virtual input $x_4$. For this purpose, let us define the Lyapunov function candidate $V_3$ as

$$V_3(e_1, e_2, e_3) = V_2(e_1, e_2) + \frac{1}{2}e_3^2,$$

for which the time derivative is given by

$$\begin{aligned}\dot{V}_3 &= \dot{V}_2 + e_3\dot{e}_3 = \dot{V}_2 + e_3(x_4 - \dot{\phi}_2)\\ &= -\lambda_1 e_1^2 - \lambda_2 e_2^2 + \nu_3 e_2 e_3 + e_3(e_4 + \phi_3 - \dot{\phi}_2),\end{aligned}$$

with the error $e_4$ given by

$$e_4 = x_4 - \phi_3, \tag{5}$$

where $\phi_3$ is the control law for the virtual input $x_4$. Choosing $\phi_3$ as

$$\phi_3 = \dot{\phi}_2 - \nu_3 e_2 - \lambda_3 e_3$$

gives

$$\dot{V}_3 = -\lambda_1 e_1^2 - \lambda_2 e_2^2 - \lambda_3 e_3^2 + e_3 e_4.$$

Notice that $\dot{e}_4$ is given by

$$\dot{e}_4 = \dot{x}_4 - \dot{\phi}_3 = f_2(x_1, x_3) + \nu_5 u - \dot{\phi}_3.$$

Now, let us define the last Lyapunov function candidate $V_4$ as

$$V_4(e_1, e_2, e_3, e_4) = V_3(e_1, e_2, e_3) + \frac{1}{2}e_4^2.$$

Then, taking its time derivative, we obtain

$$\dot{V}_4 = \dot{V}_3 + e_4(f_2(x_1, x_3) + \nu_5 u - \dot{\phi}_3).$$

The control input $u(t)$ can now be chosen as

$$u(t) = \frac{1}{\nu_5}(-f_2(x_1, x_3) + \dot{\phi}_3 - e_3 - \lambda_4 e_4),$$

With this choice, we finally obtain

$$\dot{V}_4 = -\lambda_1 e_1^2 - \lambda_2 e_2^2 - \lambda_2 e_3^2 - \lambda_4 e_4^2, \tag{6}$$

which is negative definite if $\lambda_i > 0$. Thus, it follows that the origin $e = 0$ is asymptotically stable and, consequently, the tracking error $e(t)$ converges to zero, i.e., $e_1 \to 0$, $e_2 \to 0$, $e_3 \to 0$, and $e_4 \to 0$.

Notice that the backstepping methodology is a systematic design procedure, which allows different choices for the virtual control inputs and Lyapunov functions.

### *3.1 The control law summary*

Summarizing, the final control law $u(t)$ is thus given by

$$u(t) = \frac{1}{\nu_5}(-f_2(x_1, x_3) + \dot{\phi}_3 - e_3 - \lambda_4 e_4), \tag{7}$$

with

$$\begin{aligned} e_1 &= x_1 - x_d, \quad & e_2 &= x_2 - \phi_1,\\ e_3 &= x_3 - \phi_2, \quad & e_4 &= x_4 - \phi_3,\end{aligned}$$

and

$$\begin{aligned}\dot{e}_1 &= \dot{x}_1 - \dot{x}_d = x_2 - \dot{x}_d,\\ \ddot{e}_1 &= \dot{x}_2 - \ddot{x}_d = f_1(x_1) + \nu_3 x_3 - \ddot{x}_d,\\ \dddot{e}_1 &= \dot{f}_1 + \nu_3 x_4 - \dddot{x}_d,\\ \dot{e}_2 &= \dot{x}_2 - \dot{\phi}_1 = f_1(x_1) + \nu_3 x_3 - \dot{\phi}_1.\end{aligned}$$

The term $\phi_1$ and its time derivatives are given by

$$\begin{aligned}\phi_1 &= \dot{x}_d - \lambda_1 e_1,\\ \dot{\phi}_1 &= \ddot{x}_d - \lambda_1 \dot{e}_1,\\ \ddot{\phi}_1 &= \dddot{x}_d - \lambda_1 \ddot{e}_1,\\ \dddot{\phi}_1 &= x_d^{(4)} - \lambda_1 \dddot{e}_1.\end{aligned}$$

The term $\phi_2$ and its derivatives are given by

$$\begin{aligned}\phi_2 &= \frac{1}{\nu_3}(-f_1(x_1) + \dot{\phi}_1 - e_1 - \lambda_2 e_2),\\ \dot{\phi}_2 &= \frac{1}{\nu_3}(-\dot{f}_1 + \ddot{\phi}_1 - \dot{e}_1 - \lambda_2 \dot{e}_2),\\ \ddot{\phi}_2 &= \frac{1}{\nu_3}(-\ddot{f}_1 + \dddot{\phi}_1 - \ddot{e}_1 - \lambda_2 \ddot{e}_2).\end{aligned}$$

The term $\phi_3$ and its derivatives are given by

$$\begin{aligned}\phi_3 &= \dot{\phi}_2 - \nu_3 e_2 - \lambda_3 e_3,\\ \dot{\phi}_3 &= \ddot{\phi}_2 - \nu_3 \dot{e}_2 - \lambda_3 \dot{e}_3.\end{aligned}$$

### *3.2 The closed-loop system*

With the control law above, it is possible to derive the dynamics of the closed-loop error. Let us define the tracking error state vector by

$$e = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix}^T,$$

where the error signal $e_1$, $e_2$, $e_3$, and $e_4$ are respectively given by (2), (3), (4) and (5). Then, after some manipulations, the dynamics of the closed-loop error is given by

$$\dot{e} = \begin{bmatrix} -\lambda_1 & 1 & 0 & 0\\ -1 & -\lambda_2 & \nu_3 & 0\\ 0 & -\nu_3 & -\lambda_3 & 1\\ 0 & 0 & -1 & -\lambda_4 \end{bmatrix} e, \tag{8}$$

which is linear and time invariant.

Clearly, enforcing the gains to be positive $\lambda_i > 0$, as requested from Lyapunov stability condition (6), ensures that the error dynamics is asymptotically stable and, thus, $e_i \to 0$ for any initial conditions. This can be easily shown by choosing a Lyapunov function of the form $V(e) = e^T e/2$. However, this is a conservative choice, since the matrix above can be stable with $\lambda_i$ not necessarily positive. For instance, take $\lambda_1 = 0$, $\lambda_2 = \nu_3 > 0$,

$\lambda_4 = 1$, then system matrix (8) is Hurwitz for any $\lambda_3 > -1$.

It is important to note that the first state from error dynamics (8), which is the tracking error $e_1 = q_1 - x_d$ between link angular position $q_1$ and desired position $x_d$, has an immediate physical meaning. The second state $e_2$ is given by $e_2 = (\dot{q}_1 - \dot{x}_d) + \lambda_1 e_1$, which is clearly the combination (weighted by $\lambda_1$) of velocity tracking error $\dot{q}_1 - \dot{x}_d$ with position tracking error $e_1$ for the link. The physical interpretation for the other two states, $e_3$ and $e_4$, related to the motor's angular position and velocity, is not immediate.

### *3.3 Control law in terms of $e(t)$*

In terms of the trajectory error $e(t)$, the control law $u(t)$ is given by

$$u(t) = \frac{1}{\nu_3 \nu_5}\left(\sum_{i=1}^{5} \gamma_i + \sum_{i=1}^{4} \alpha_i e_i\right),$$

with $\alpha_i$ and $\gamma_i$ given by

$$\begin{aligned}
\alpha_1 &= 1 + \nu_3^2 - \nu_3\nu_4 - 3\lambda_1^2 + \lambda_1^4 + \nu_4(\lambda_1^2 - 1) \\
&\quad - 2\lambda_1\lambda_2 - \nu_1(\nu_4 + \lambda_1^2 - 1) - \lambda_2^2 \\
\alpha_2 &= -(\lambda_1 + \lambda_2)(\nu_4 - \lambda_1^2 + \lambda_2^2) + \nu_1(\lambda_1 + \lambda_2) \\
&\quad + \nu_3^2(\lambda_1 + 2\lambda_2 + \lambda_3) - 2(\lambda_1^2 - 1)(\lambda_1 + \lambda_2) \\
\alpha_3 &= \nu_3(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 + (\lambda_1 + \lambda_2)\lambda_3 + \lambda_3^2) \\
&\quad + \nu_3(\nu_4 - \nu_1 - \nu_3^2 - 2) \\
\alpha_4 &= -\nu_3(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \\
\gamma_1 &= -\nu_3\nu_4 x_d + \nu_4 \ddot{x}_d + x_d^{(4)} - \nu_1\nu_4 x_d - \nu_1 \ddot{x}_d \\
\gamma_2 &= -\nu_2 \ddot{x}_d \cos(x_d + e_1) - \nu_2\nu_4 \sin(x_d + e_1) \\
&\quad + \nu_2 \dot{x}_d^2 \sin(x_d + e_1) \\
&\quad + (e_1\lambda_1 + e_2)^2 \nu_2 \sin(x_d + e_1) \\
\gamma_3 &= (-\nu_2(\lambda_1^2 - 1)\cos(x_d + e_1) \\
&\quad - 2\nu_2(\dot{x}_d + e_2)\lambda_1 \sin(x_d + e_1))e_1 \\
\gamma_4 &= (\nu_2(\lambda_1 + \lambda_2)\cos(x_d + e_1) \\
&\quad + 2\nu_2(\dot{x}_d - e_1\lambda_1)\sin(x_d + e_1))e_2 \\
\gamma_5 &= -\nu_3\nu_2 \cos(x_d + e_1) e_3
\end{aligned}$$

In steady state, after the transient vanished, i.e., after the error dynamics converged to zero ($e(t) \to 0$), the control law $u(t)$ reaches steady state response $u_{ss}(t)$, which is given by

$$\begin{aligned}
u_{ss}(t) = \frac{1}{\nu_3\nu_5}&(x_d^{(4)} - \nu_1\nu_4 x_d - \nu_3\nu_4 x_d \\
&+ \nu_4 \ddot{x}_d - \nu_1 \ddot{x}_d - \nu_2 \ddot{x}_d \cos(x_d) \\
&\quad - \nu_2\nu_4 \sin(x_d) + \nu_2 \dot{x}_d^2 \sin(x_d).
\end{aligned}$$

This expression will be central in the next section to define a performance metric used to optimize the choice of the controller parameters $\lambda_1, \ldots, \lambda_4$.

## 4 Optimizing the control gains

This section describes the approach used to design the gains $\lambda_i$ in the error dynamics given by (8), so that robot link $q_1$ is guaranteed to track desired trajectory $x_d$ with a prescribed amount of control energy.

Inspired by the celebrated LQR control problem, we define as our performance metric the cost function $J$ given by

$$J = \int_0^{T_{max}} e^T Q e + (u - u_{ss})^T R (u - u_{ss})\, \mathrm{d}t, \quad (9)$$

for some positive semidefinite matrix $Q = Q^T$, and positive definite matrix $R = R^T$. The time-horizon for the cost function is given by $T_{max}$.

Some important issues deserve our attention. Even though (9) is quadratic with respect the to control input $u(t)$ and the error $e(t)$, subject to the linear error dynamics (8), the optimization problem, considering the controller gains $\lambda_i$ as decision variables, is not, in general, convex. Therefore, methods based on the solution of algebraic Riccati equations may not be suitable for this problem. Consequently, we rely on local optimization methods. For all the numerical examples in the sections to come, optimization was performed with the BFGS quasi-Newton method (Nocedal and Wright, 2006), where the gradient was estimated using finite differences. This method is implemented, for instance, in *fminunc* function in MATLAB or *scipy.optimize.fmin_ bfgs* in Python. Finally, the term $(u - u_{ss})$ is used due to the fact tat the control law $u_{ss}(t)$ is necessary to maintain the system at zero error after steady state.

## 5 Numerical simulations

This section presents the numerical simulations for the single-link flexible-joint robot given by (1) using the backstepping control law given by (7), with the gains $\lambda_i$ computed in the previous section.

The numerical data used for this simulation, taken from (Ghorbel et al., 1989), are: $k = 31.0$ [N.m/rad]; $Mgl = 0.8$ [N.m]; $J_m = 0.004$ [Kg.m$^2$]; $I = 0.031$ [Kg.m$^2$]. The reference trajectory $x_d(t)$, which the robot arms have to follow, is given by

$$x_d(t) = 1 + \sin(2t + \pi/2)\cos(3t) \quad \text{[rad]}.$$

It is assumed the robot is under rest with initial conditions $x_0 = 0$. For the cost function (9), the terminal time is taken to be $T_{max} = 7$ seconds, matrix $R$ is given by $R = 10$, and matrix $Q$ is given by $Q = [q_{ij}]$, with $q_{11} = 1$, $q_{22} = q_{33} = q_{44} = 0.01$, and $q_{ij} = 0$ otherwise.

Using the optimization procedure proposed in Section 4, the following control gains are obtained for $R = 10$:

$$\begin{aligned}&[\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4] = \\ &\qquad [3.28,\ -703.66,\ 1.4164\times 10^3,\ 25.20]\,.\end{aligned}$$

Figure 2 shows the position [rad] of the link and the motor ($x_1$ and $x_3$, respectively) and the velocity [rad/s] of the link and the motor ($x_2$ and $x_4$, respectively). One can observe that the amplitude of the link position and the rotor position are quite close to each other. The same observation holds for the velocity profile.

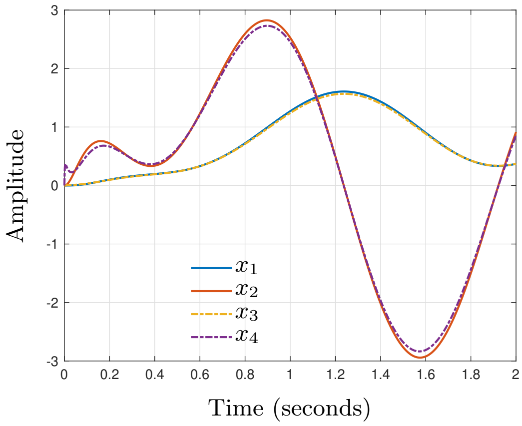


Fig. 2: Link and motor positions and velocities.

Figure 3 shows the position [rad] and velocity [rad/s] of the link ($x_1$ and $x_2$) and the position [rad] and velocity [rad/s] of the desired reference trajectory ($x_d$ and $\dot{x}_d$).

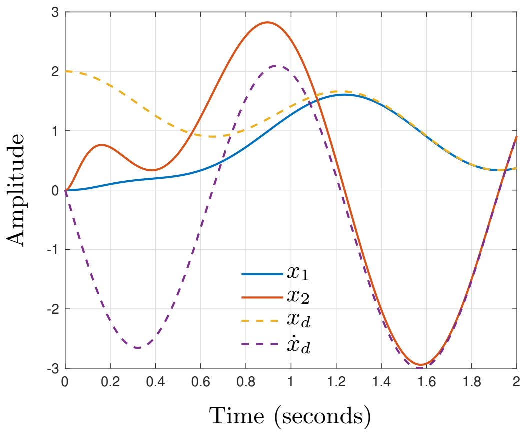


Fig. 3: Link and reference trajectory.

Despite of the large difference between the link initial condition at zero and the desired reference initial condition at $x_d(0) = 2$, the position and velocity tracking error of the link, given respectively by $e_1(t) = x_1 - x_d$ and $e_2(t) = x_2 - \dot{x}_d$, converged asymptotically to zero, as was expected. This fact is immediately seen in Figure 4, in which the time response (the time to reach a tube of radius 0.01 around the origin) of tracking errors $e_1(t)$ and $e_2(t)$ is found to be approximately 1.7 seconds.

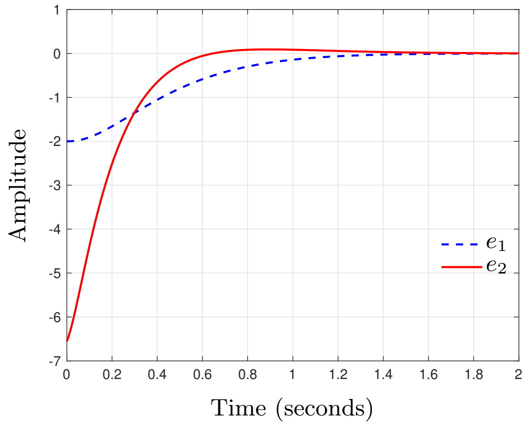


Fig. 4: Tracking error $e_1(t)$ and $e_2(t)$ for $R = 10$.

Figure 5 and Figure 6 show, respectively, tracking errors $e_3(t)$ and $e_4(t)$, for $R = 10$. Both errors converged to zero at a fast rate. The time response are, approximately, 1.6 seconds for $e_3$ and 0.5 seconds for $e_4$.

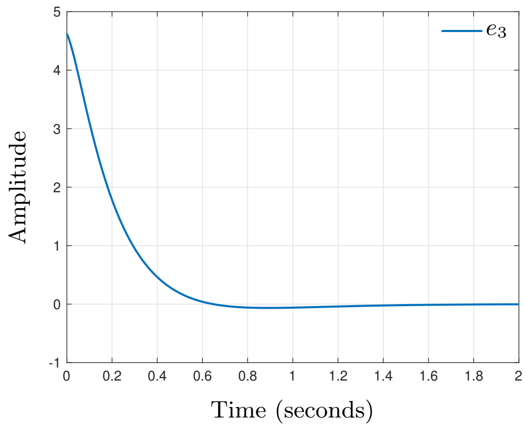


Fig. 5: Tracking error $e_3(t)$ for $R = 10$.

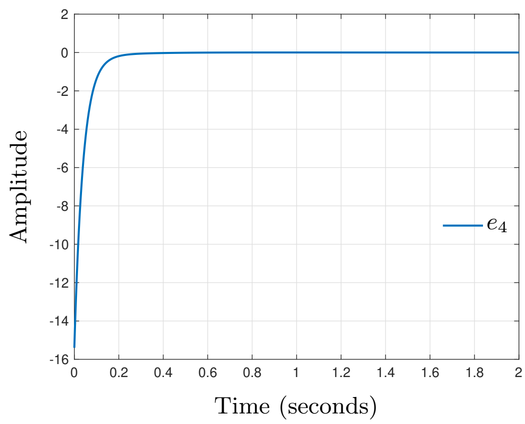


Fig. 6: Tracking error $e_4(t)$ for $R = 10$.

The control law $u(t)$ and the steady state component $u_{ss}(t)$ are shown in Figure 7. For the weighting $R = 10$, one has that the control law at the initial time is $u(0) = 1.24$ [N.m] and the $L^2$-norm of the difference between the control law $u(t)$ and its steady state component $u_{ss}(t)$ is given by $\|u - u_{ss}\|_{L^2} = \int_0^{T_{\max}} (u(\tau) - u_{ss}(\tau))^2 \, \mathrm{d}\tau = 0.42$.

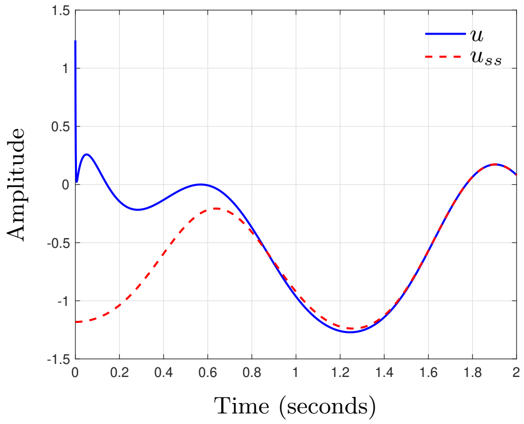


Fig. 7: Control law $u(t)$ for $R = 10$.

Other values of $R$ would provide a trade-off between control effort and convergence rates. For example, for $R = 0.1$, the controller gains obtained are

$$\begin{aligned} &\left[\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4\right] = \\ &\qquad \left[4.16,\ -979.04,\ 1.0205 \times 10^3.\ 146.99\right]. \end{aligned}$$

Figure 8 shows the control law $u(t)$ and its steady state component $u_{ss}(t)$. For this new value of $R$, the control effort is less penalized and one obtains $\|u - u_{ss}\|_{L^2} = 1.39$, which is about three times higher than the previous case. The control law at the initial time is $u(0) = 5.11$ [N.m].

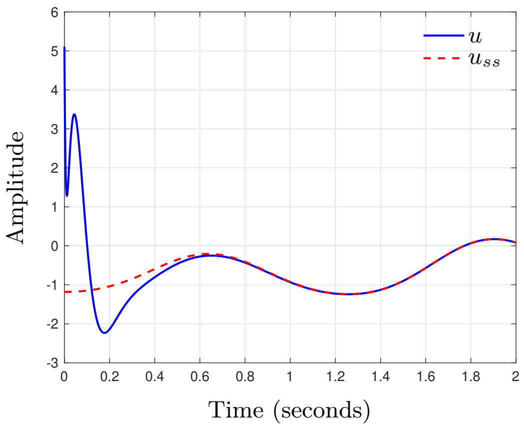


Fig. 8: Control law $u(t)$ for $R = 0.1$.

Therefore, at the cost of larger control signals, the tracking error can be driven to the origin at a faster rate, as illustrated in Figure 9, which shows the tracking error $e_1(t)$ and $e_2(t)$ for $R = 0.1$. For this case, the time response is approximately 0.9 seconds, which is almost half the time response of the tracking error in the previous case for $R = 10$.

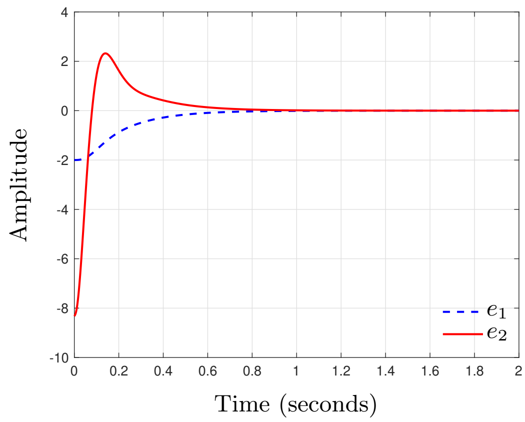


Fig. 9: Tracking error $e_1(t)$ and $e_2(t)$ for $R = 0.1$.

To verify that the proposed control law guarantees uniform ultimate boundedness of the tracking error $e(t)$ under additive bounded disturbance, it is assumed that the equation of motion of the robot manipulator is subject to an independent Gaussian noise torque disturbance.

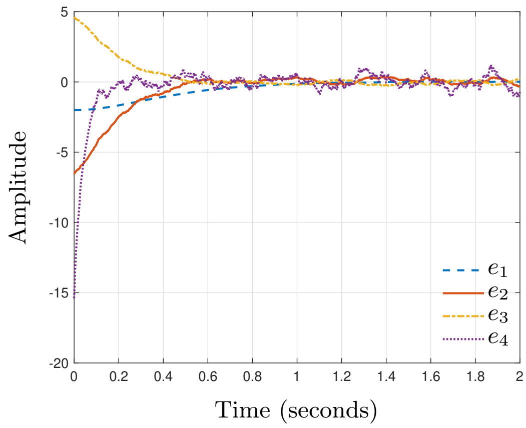


Fig. 10: Tracking error $e(t)$, under disturbance torques, for $R = 10$.

Figure 10 shows the tracking error $e(t)$, under this noise disturbance, for the gains computed using the weighting $R = 10$. One observes that $e(t)$ is ultimately bounded for the bounded disturbance applied to the equation of motion. Figure 11 shows the control law $u(t)$ for this case.

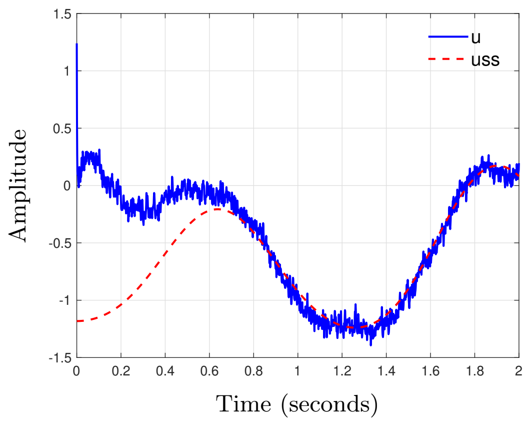


Fig. 11: Control law $u(t)$ under disturbance.

## 6 Conclusions

A backstepping design was used to guarantee reference trajectory to a single-link robot manipulator with a flexible joint. As shown trough numerical simulation, the tracking error converged to zero within a reasonable time.

A challenging part of this procedure is the selection of the control gains, the free parameters available in the design, that need to be chosen in such a way that the closed-loop system is stable and can also be used for performance improvement. Imposing performance in a nonlinear control design is not so straightforward as in the linear case.

It is also important to emphasize that the proposed backstepping application can be extended to multiple-link robot manipulators, and the linear design for the backstepping gains can also be computed using many classical optimization and control techniques.

## Acknowledgments

The authors are partially supported by the Brazilian agencies FAPESP, CAPES and CNPq.

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