AN INFINITE HORIZON MODEL PREDICTIVE CONTROL FOR STABLE, INTEGRATING AND UNSTABLE SYSTEMS

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Abstract— Several works in the literature of model predictive control (MPC) have focused on the development of MPC formulations that are suitable for industrial applications. Usually, these controllers are based on state space models whose structures depend on the system type, which may be stable, integrating, or unstable. Thus, the usage of different internal models often yields distinct MPC formulations, which may be an issue for building a more general industrial package. Therefore, in order to consolidate these approaches, the present study addresses the development of a more general infinite horizon model predictive controller (IHMPC). This strategy is based on a novel formulation of state space model for stable, integrating, and unstable systems that is suitable for the IHMPC implementation. Simulation results demonstrated the successful application of the proposed controller to a deisobutanizer distillation column and to an unstable reactor system.

Keywords— Model predictive control, MPC, Infinite horizon, Integrating systems, Unstable systems.

Resumo— Vários trabalhos na literatura de controle preditivo (MPC) têm focado no desenvolvimento de formulações destinadas a aplicações industriais. Normalmente, esses controladores são baseadas em modelos em espaço de estados cujas estruturas dependem do tipo de sistema, que pode ser estável, integrador ou instável. Assim, o emprego de diferentes modelos internos frequentemente leva a formulações de MPC distintas, o que pode ser um empecilho para a elaboração de pacotes industriais mais gerais. Portanto, com o intuito de consolidar tais abordagens, o presente estudo apresenta o desenvolvimento de uma versão mais geral do MPC de horizonte infinito (IHMPC). Essa estratégia é baseada em uma nova formulação de modelo em espaço de estados para sistemas estáveis, integradores e instáveis adequada à implementação do IHMPC. Resultados de simulações demonstraram a aplicação do controlador proposto a uma coluna deisobutanizadora e a um reator instável.

Palavras-chave— Controle preditivo, MPC, Horizonte infinito, Sistemas integradores, Sistemas instáveis.

1 Introduction

The stability properties of the constrained receding horizon control (also known as moving horizon control) was first addressed in (Rawlings and Muske, 1993), in which the authors proposed an infinite horizon controller that is only suitable as a constrained quadratic regulator, since it is designed to drive the system to the origin. Further extension of this regulator has been proposed in (Muske and Rawlings, 1993) and addresses the setpoint tracking and the output feedback through state estimation. However, this formulation requires the knowledge of plant equilibrium point and assumes that, at each sampling time, the system states lie inside a set such that the optimization control problem is feasible, which may not hold in a practical application.

Concerning the development of infinite horizon model predictive controllers (IHMPC) that are suitable for industrial applications, several works have been developed considering systems with stable (Rodrigues and Odloak, 2003b; Odloak, 2004), integrating (Rodrigues and Odloak, 2003a; Carrapiço and Odloak, 2005) and unstable (González et al., 2011; Martins and Odloak, 2016) poles, and with time delay as well (Santoro and Odloak, 2012; Martins et al., 2013). These studies have led to successful industrial applications (Carrapiço et al., 2009; Porfírio and Odloak, 2011).

In these studies, the authors proposed a step response-based state space model to be used as the internal model of the IHMPC. The construction of this model requires transfer functions that describe each output-input relationship and can be summarized as follows:

1. apply a unitary step to system inputs;
2. obtain the analytical form of the step response by partial fractions expansion;
3. define the states related to the incremental form of the inputs;
4. define the states related to the system poles (stable, integrating, or unstable);
5. build up the state space model.

However, the formulation described above depends on previous knowledge about the poles of each transfer function in order to determine the structure of the final model. In other words, although the overall steps to build the model are the same, its structure will be different whether the system has stable, integrating, or unstable poles. Thus, it may be difficult to automatize the model construction for the general case in a computer program without using different subroutines for each type of system, e.g. stable and integrating, stable and unstable with time delay etc. Furthermore, different model structures often lead to distinct MPC formulations, which may be an issue for building a more general industrial package.

In this view, the purpose of the present study is to propose an alternative method for building up state space models suitable for the IHMPC implementation. This new method requires the system discrete time state space model, which in general can be easily obtained by linearization and discretization of rigorous differential equations models or even by means of system identification techniques when experimental data are available. Also, the proposed method deals with systems that have multiplicity of poles and is easy to automatize in a computer program, which is an advantage for developing an industrial package.

This paper is organized as follows. The next section details the proposed method for building the state space model which will be used as the internal model of the IHMPC. Then, in Section 3, we discuss the development of the IHMPC algorithm based on the proposed model formulation. In Section 4, we present simulation results concerning the application of the IHMPC to a deisobutanizer distillation column and to an unstable reactor system. Finally, the conclusions of this study are given in Section 5.

2 The extended output prediction-oriented model

Consider the state space model in the positional form of the inputs defined below, in which $\tilde{x}(k) \in \mathbb{R}^{n_x}$, $\tilde{u}(k) \in \mathbb{R}^{n_u}$, $\tilde{A} \in \mathbb{R}^{n_x \times n_x}$, $\tilde{B} \in \mathbb{R}^{n_x \times n_u}$, $\tilde{C} \in \mathbb{R}^{n_y \times n_x}$ and n_x, n_u and n_y are the number of states, inputs, and outputs, respectively.

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u(k) \tag{1}$$

$$y(k) = \tilde{C}\tilde{x}(k) \tag{2}$$

We can perform a system decomposition (Nagar and Singh, 2004) in order to obtain the following form of the state space model:

$$\begin{bmatrix} \tilde{x}_{st}(k+1) \\ \tilde{x}_{in}(k+1) \\ \tilde{x}_{un}(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{st} & 0 & 0 \\ 0 & \tilde{A}_{in} & 0 \\ 0 & 0 & \tilde{A}_{un} \end{bmatrix} \begin{bmatrix} \tilde{x}_{st}(k) \\ \tilde{x}_{in}(k) \\ \tilde{x}_{un}(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_{st} \\ \tilde{B}_{in} \\ \tilde{B}_{un} \end{bmatrix} u(k) \tag{3}$$

$$y(k) = \begin{bmatrix} \tilde{C}_{st} & \tilde{C}_{in} & \tilde{C}_{un} \end{bmatrix} \begin{bmatrix} \tilde{x}_{st}(k) \\ \tilde{x}_{in}(k) \\ \tilde{x}_{un}(k) \end{bmatrix} \tag{4}$$

in which $\tilde{A}_{st} \in \mathbb{R}^{n_{st} \times n_{st}}$, $\tilde{A}_{in} \in \mathbb{R}^{n_{in} \times n_{in}}$, $\tilde{A}_{un} \in \mathbb{R}^{n_{un} \times n_{un}}$ are diagonal matrices (or block diagonal in the case of repeated poles) whose main diagonals are formed by n_{st} stable, n_{in} integrating and n_{un} unstable poles of the system, respectively, and $\tilde{x}_{st}(k) \in \mathbb{R}^{n_{st}}$, $\tilde{x}_{in}(k) \in \mathbb{R}^{n_{in}}$, $\tilde{x}_{un}(k) \in \mathbb{R}^{n_{un}}$, $\tilde{B}_{st} \in \mathbb{R}^{n_{st} \times n_u}$, $\tilde{B}_{in} \in \mathbb{R}^{n_{in} \times n_u}$, $\tilde{B}_{un} \in \mathbb{R}^{n_{un} \times n_u}$, $\tilde{C}_{st} \in \mathbb{R}^{n_y \times n_{st}}$, $\tilde{C}_{in} \in \mathbb{R}^{n_y \times n_{in}}$ and $\tilde{C}_{un} \in \mathbb{R}^{n_y \times n_{un}}$.

Now, defining

$$\begin{aligned} \Delta\tilde{x}_{st}(k) &= \tilde{x}_{st}(k) - \tilde{x}_{st}(k-1), \\ \Delta\tilde{x}_{in}(k) &= \tilde{x}_{in}(k) - \tilde{x}_{in}(k-1), \\ \Delta\tilde{x}_{un}(k) &= \tilde{x}_{un}(k) - \tilde{x}_{un}(k-1) \text{ and} \\ \Delta u(k) &= u(k) - u(k-1) \end{aligned}$$

we can write the expressions for the incremental states in a convenient manner as follows

$$\begin{aligned} \Delta\tilde{x}_{st}(k) &= \tilde{A}_{st}\Delta\tilde{x}_{st}(k-1) \\ &+ \left(I - \tilde{A}_{st}\right)^{-1}\left(I - \tilde{A}_{st}\right)\tilde{B}_{st}\Delta u(k-1) \end{aligned} \tag{5}$$

$$\Delta\tilde{x}_{in}(k) = \tilde{A}_{in}\Delta\tilde{x}_{in}(k-1) + \tilde{B}_{in}\Delta u(k-1) \tag{6}$$

$$\begin{aligned} \Delta\tilde{x}_{un}(k) &= \tilde{A}_{un}\Delta\tilde{x}_{un}(k-1) \\ &+ \left(I - \tilde{A}_{un}\right)^{-1}\left(I - \tilde{A}_{un}\right)\tilde{B}_{un}\Delta u(k-1) \end{aligned} \tag{7}$$

Then, the system output can be computed as follows

$$\begin{aligned} y(k) &= \tilde{C}_{st}\left(\tilde{x}_{st}(k-1) + \Delta\tilde{x}_{st}(k)\right) \\ &+ \tilde{C}_{in}\left(\tilde{x}_{in}(k-1) + \Delta\tilde{x}_{in}(k)\right) \\ &+ \tilde{C}_{un}\left(\tilde{x}_{un}(k-1) + \Delta\tilde{x}_{un}(k)\right) \end{aligned} \tag{8}$$

Let us define the following states

$$\begin{aligned} x_{\Delta}(k-1) &= \tilde{C}_{st}\tilde{x}_{st}(k-1) + \tilde{C}_{in}\tilde{x}_{in}(k-1) \\ &+ \tilde{C}_{un}\tilde{x}_{un}(k-1) \end{aligned} \tag{9}$$

$$x_{st}(k-1) = \Delta\tilde{x}_{st}(k-1) \tag{10}$$

$$x_{in}(k-1) = \Delta\tilde{x}_{in}(k-1) \tag{11}$$

$$x_{un}(k-1) = \Delta\tilde{x}_{un}(k-1) \tag{12}$$

Now, we may write the output $y(k)$ as function of the above variables as follows

$$\begin{aligned} y(k) &= x_{\Delta}(k-1) + \tilde{C}_{st}\Big(\tilde{A}_{st}x_{st}(k-1) \\ &- \left(I - \tilde{A}_{st}\right)^{-1}\tilde{A}_{st}\tilde{B}_{st}\Delta u(k-1)\Big) \\ &+ \tilde{C}_{in}\tilde{A}_{in}x_{in}(k-1) + \tilde{C}_{un}\Big(\tilde{A}_{un}x_{un}(k-1) \\ &- \left(I - \tilde{A}_{un}\right)^{-1}\tilde{A}_{un}\tilde{B}_{un}\Delta u(k-1)\Big) \\ &+ \Big(\tilde{C}_{st}\left(I - \tilde{A}_{st}\right)^{-1}\tilde{B}_{st} + \tilde{C}_{in}\tilde{B}_{in} \\ &+ \tilde{C}_{un}\left(I - \tilde{A}_{un}\right)^{-1}\tilde{B}_{un}\Big)\Delta u(k-1) \end{aligned} \tag{13}$$

The states defined in 9, 10, 11 and 12 can be updated recursively according to 14, 15, 16 and 17, respectively.

$$x_\Delta(k) = x_\Delta(k-1) + \tilde{C}_{in}\tilde{A}_{in}x_{in}(k-1) + \left(\tilde{C}_{st}\left(I-\tilde{A}_{st}\right)^{-1}\tilde{B}_{st} + \tilde{C}_{in}\tilde{B}_{in} + \tilde{C}_{un}\left(I-\tilde{A}_{un}\right)^{-1}\tilde{B}_{un}\right)\Delta u(k-1) \tag{14}$$

$$x_{st}(k) = \tilde{A}_{st}x_{st}(k-1) - \left(I-\tilde{A}_{st}\right)^{-1}\tilde{A}_{st}\tilde{B}_{st}\Delta u(k-1) \tag{15}$$

$$x_{in}(k) = \tilde{A}_{in}x_{in}(k-1) + \tilde{B}_{in}\Delta u(k-1) \tag{16}$$

$$x_{un}(k) = \tilde{A}_{st}x_{un}(k-1) - \left(I-\tilde{A}_{un}\right)^{-1}\tilde{A}_{un}\tilde{B}_{un}\Delta u(k-1) \tag{17}$$

Finally, we may write the following state-space model:

$$x(k+1) = Ax(k) + B\Delta u(k) \tag{18}$$
$$y(k) = Cx(k) \tag{19}$$

with

$$A = \begin{bmatrix} I_{n_y} & 0 & C_{in}A_{in} & 0 \\ 0 & A_{st} & 0 & 0 \\ 0 & 0 & A_{in} & 0 \\ 0 & 0 & 0 & A_{un} \end{bmatrix},$$
$$x(k) = \begin{bmatrix} x_\Delta(k) \\ x_{st}(k) \\ x_{in}(k) \\ x_{un}(k) \end{bmatrix}, \quad B = \begin{bmatrix} B_\Delta \\ B_{st} \\ B_{in} \\ B_{un} \end{bmatrix},$$
$$C = \begin{bmatrix} I_{n_y} & C_{st} & 0 & C_{un} \end{bmatrix}$$

in which

$$A_{st} = \tilde{A}_{st},\ A_{in} = \tilde{A}_{in},\ A_{un} = \tilde{A}_{un},$$
$$B_\Delta = \tilde{C}_{st}\left(I-\tilde{A}_{st}\right)^{-1}\tilde{B}_{st} + \tilde{C}_{in}\tilde{B}_{in} + \tilde{C}_{un}\left(I-\tilde{A}_{un}\right)^{-1}\tilde{B}_{un},$$
$$B_{st} = -\left(I-\tilde{A}_{st}\right)^{-1}\tilde{A}_{st}\tilde{B}_{st},\ B_{in} = \tilde{B}_{in},$$
$$B_{un} = -\left(I-\tilde{A}_{un}\right)^{-1}\tilde{A}_{un}\tilde{B}_{un},$$
$$C_{st} = \tilde{C}_{st},\ C_{in} = \tilde{C}_{in},\ C_{un} = \tilde{C}_{un}.$$

Here in this representation, $x_\Delta(k)$ corresponds to the integrating states related to the incremental form of inputs and $x_{st}(k)$, $x_{in}(k)$ and $x_{un}(k)$ stand for the states related to stable, integrating and unstable modes of the original system.

3 The IHMPC formulation

The infinite horizon model predictive controller is based on the following quadratic cost function:

$$V(k) = \sum_{j=0}^{\infty}\|y(k+j|k) - y_{sp,k}\|_Q^2 + \sum_{j=0}^{\infty}\|\Delta u(k+j|k)\|_R^2 \tag{20}$$

in which $Q \in \mathbb{R}^{n_y\times n_y}$ and $R \in \mathbb{R}^{n_u\times n_u}$ are positive definite weighting matrices, $y(k+j|k)$ is the output prediction at time step $k+j$ computed at time step k, $y_{sp,k} \in \mathbb{R}^{n_y}$ is the output setpoint at time step k, $\Delta u(k+j|k)$ is the vector of input moves with $\Delta u(k+j|k) = 0$ for $j \geq m$, in which m is the control horizon.

The first term of 20 can be written as follows

$$\sum_{j=0}^{\infty}\|y(k+j|k) - y_{sp,k}\|_Q^2 = \sum_{j=0}^{m}\|y(k+j|k) - y_{sp,k}\|_Q^2 + \sum_{j=m+1}^{\infty}\|y(k+j|k) - y_{sp,k}\|_Q^2 \tag{21}$$

Since there are no control actions for $j \geq m$, it is easy to show that we can compute $y(k+j|k)$ for $j \geq m+1$ according to

$$y(k+j|k) = CA^{j-m}x(k+m|k) \tag{22}$$

Then, we have that

$$\sum_{j=m+1}^{\infty}\|y(k+j|k) - y_{sp,k}\|_Q^2 = \sum_{j=m+1}^{\infty}\|CA^{j-m}x(k+m|k) - y_{sp,k}\|_Q^2 \tag{23}$$

However, if we expand the right-hand side of 22 according to 24, it is easy to see that only $C_{st}A_{st}^{j-m}x_{st}(k+m|k)$ goes to zero as $j \to \infty$ since $\lim_{j\to\infty} A_{st}^j = 0$.

$$CA^{j-m}x(k+m|k) = x_\Delta(k+m|k) + C_{st}A_{st}^{j-m}x_{st}(k+m|k) + C_{in}\left(\sum_{i=1}^{j}A_{in}^{i-m} - \sum_{i=0}^{m-1}A_{in}^{-i}\right)x_{in}(k+m|k) + C_{un}A_{un}^{j-m}x_{un}(k+m|k) \tag{24}$$

Therefore, in order to prevent the cost function to be unbounded, the following terminal equality constraints must be included in the control optimization problem:

$$x_\Delta(k+m|k) - y_{sp,k} = 0 \tag{25}$$

$$x_{in}(k+m|k) = 0 \tag{26}$$

$$x_{un}(k+m|k) = 0 \tag{27}$$

It is easy to show that $x(k+m|k)$ can be computed according to

$$x(k+m|k) = A^m x(k) + W\Delta u_k \tag{28}$$

with

$$W = \begin{bmatrix} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{bmatrix}$$
$$\Delta u_k = \begin{bmatrix} \Delta u(k|k)^T & \cdots & \Delta u(k+m-1|k)^T \end{bmatrix}^T$$

Then, the constraints given in 25, 26 and 27 can be written in a compact form as follows:

$$N\left(A^m x(k) + W\Delta u_k\right) - \gamma_k = 0 \tag{29}$$

with

$$N = \begin{bmatrix} I_{n_y} & 0 & 0 & 0 \\ 0 & 0 & I_{n_{in}} & 0 \\ 0 & 0 & 0 & I_{n_{un}} \end{bmatrix} \text{ and } \gamma_k = \begin{bmatrix} y_{sp,k} \\ 0_{n_{in}} \\ 0_{n_{un}} \end{bmatrix}.$$

Now, imposing the constraint given in 29, we can simplify 23 according to

$$\begin{aligned} &\sum_{j=m+1}^{\infty} \|y(k+j|k) - y_{sp,k}\|_Q^2 \\ &= \sum_{j=m+1}^{\infty} \|C_{st}A_{st}^{j-m}x_{st}(k+m|k)\|_Q^2 \\ &= \|x_{st}(k+m|k)\|_{\bar{Q}}^2 \end{aligned} \tag{30}$$

in which $\bar{Q} \in \mathbb{R}^{n_{st}\times n_{st}}$ is computed as solution of the Lyapunov equation given in

$$\bar{Q} - A_{st}^T\bar{Q}A_{st} = A_{st}^T C_{st}^T Q C_{st} A_{st} \tag{31}$$

Therefore, the controller can be defined as solution of the following problem:

Problem P1

$$\begin{aligned} \min_{\Delta u_k} V_1(k) = &\sum_{j=0}^{m} \|y(k+j|k) - y_{sp,k}\|_Q^2 \\ &+ \sum_{j=0}^{m-1} \|\Delta u(k+j|k)\|_R^2 \\ &+ \|x_{st}(k+m|k)\|_{\bar{Q}}^2 \end{aligned} \tag{32}$$

subject to 18, 19 and 29 and

$$\Delta u(k+j|k) \in \mathcal{U}, \qquad j = 0, \ldots, m-1 \tag{33}$$
$$\Delta u(k+j|k) = 0, \qquad j \geq m \tag{34}$$

in which

$$\mathcal{U} = \left\{ \begin{aligned} \Delta u_{min} \leq \Delta u(k+j|k) \leq \Delta u_{max} \\ u_{min} \leq u(k-1) + \sum_{i=0}^{j} \Delta u(k+i|k) \leq u_{max} \end{aligned} \right\}$$

However, we have no guarantee that the terminal equality constraint 29 holds for any control horizon m, which may lead the optimization problem to be infeasible and then compromise its practical application due to lack of reliability. Hence, appropriate slack variables must be included in order to soften the hard terminal constraints. Therefore, constraints 25, 26 and 27 are rewritten as follows:

$$x_{\Delta}(k+m|k) - y_{sp,k} + \delta_{\Delta,k} = 0 \tag{35}$$

$$x_{in}(k+m|k) + \delta_{in,k} = 0 \tag{36}$$

$$x_{un}(k+m|k) + \delta_{un,k} = 0 \tag{37}$$

which again can be written in a compact form as given in 38.

$$N\left(A^m x(k) + W\Delta u_k\right) - \gamma_k + \delta_k = 0 \tag{38}$$

with $\delta_k = \begin{bmatrix} \delta_{\Delta,k}^T & \delta_{in,k}^T & \delta_{un,k}^T \end{bmatrix}^T$, $\delta_{\Delta,k} \in \mathbb{R}^{n_y}$, $\delta_{in,k} \in \mathbb{R}^{n_{in}}$ and $\delta_{un,k} \in \mathbb{R}^{n_{un}}$.

Also, the slack variables must be included in the first term of the cost function so it remains bounded and the terminal penalty can be employed. Then, we have that

$$\begin{aligned} &\sum_{j=m+1}^{\infty} \left\|y(k+j|k) - y_{sp,k} + CA^{j-m}N^T\delta_k\right\|_Q^2 \\ &= \sum_{j=m+1}^{\infty} \left\|CA^{j-m}\left(x(k+m|k) + N^T\delta_k\right) - y_{sp,k}\right\|_Q^2 \\ &= \sum_{j=m+1}^{\infty} \left\|C_{st}A_{st}^{j-m}x_{st}(k+m|k)\right\|_Q^2 \\ &= \|x_{st}(k+m|k)\|_{\bar{Q}}^2 \end{aligned} \tag{39}$$

Finally, the control optimization problem can be redefined as stated below.

Problem P2

$$\min_{\Delta u_k,\ \delta_k} V_2(k), \tag{40}$$

$$\begin{aligned} V_2(k) = &\sum_{j=0}^{m} \left\|y(k+j|k) - y_{sp,k} + CA^{j-m}N^T\delta_k\right\|_Q^2 \\ &+ \sum_{j=0}^{m-1} \|\Delta u(k+j|k)\|_R^2 \\ &+ \|x_{st}(k+m|k)\|_{\bar{Q}}^2 + \|\delta_k\|_S^2 \end{aligned}$$

subject to 18, 19, 33, 34 and 38.

In the above problem, the penalty matrix of slack variables is such that $S = \begin{bmatrix} S_{\Delta} & & \\ & S_{in} & \\ & & S_{un} \end{bmatrix}$ in which $S_{\Delta} \in \mathbb{R}^{n_y}$ is a positive semi-definite weighting matrix related to $\delta_{\Delta,k}$, while $S_{in} \in \mathbb{R}^{n_{in}}$ and $S_{un} \in \mathbb{R}^{n_{un}}$ are positive definite weighting matrices related to $\delta_{in,k}$ and $\delta_{un,k}$, respectively.

In the case when the slack variables related to integrating and unstable modes are zeroed, a stabilizing control law can be obtained by simplifying Problem P2 as the following optimization problem:

Problem P3

$$\min_{\Delta u_k,\ \delta_{\Delta,k}} V_3(k), \tag{41}$$

$$\begin{aligned} V_3(k) = &\sum_{j=0}^{m} \|y(k+j|k) - y_{sp,k} + \delta_{\Delta,k}\|_Q^2 \\ &+ \sum_{j=0}^{m-1} \|\Delta u(k+j|k)\|_R^2 \\ &+ \|x_{st}(k+m|k)\|_{\bar{Q}}^2 + \|\delta_{\Delta,k}\|_{S_\Delta}^2 \end{aligned}$$

subject to 18, 19, 33, 34 and

$$N\left(A^m x(k) + W\Delta u_k\right) - \gamma_k + \delta_k = 0 \tag{42}$$

with $\delta_k = \begin{bmatrix}\delta_{\Delta,k}^T & 0_{n_{in}}^T & 0_{n_{un}}^T\end{bmatrix}^T$.

The following lemma concerns about the recursive feasibility of Problem 3, while its convergence is assured by the theorem stated next.

Lemma 1 *The feasibility of Problem P3 at time step k implies the optimization problem will remain feasible for any subsequent time step $k+j>k$.*

Proof: The main idea to this proof has been provided in (Muske and Rawlings, 1993) and a similar procedure will be followed here.

Let $(\Delta u_k^*,\ \delta_{\Delta,k}^*)$ denote the solution to Problem P3 at a given time step k. Then, only the first control move is inject into the plant according to the receding horizon principle. At time step $k+1$, let us consider the non-optimal candidate solution given as follows:

$$\begin{aligned} &\Delta\tilde{u}_{k+1} = \\ &\begin{bmatrix}\Delta u^*(k+1|k)^T & \cdots & \Delta u^*(k+m-1|k)^T & 0^T\end{bmatrix} \\ &\tilde{\delta}_{\Delta,k+1} = \delta_{\Delta,k}^* \end{aligned}$$

It is clear that $(\Delta\tilde{u}_k,\ \tilde{\delta}_{\Delta,k})$ satisfies the inequality constraints given in 33 and 34 since the last control movement is null. We can also show the equality constraint 42 is satisfied. To do so, note that $x(k+m+1|k+1) = Ax(k+m|k)$, then we have

$$\begin{aligned} Nx(k+m+1|k+1) - \gamma_k + {\delta_k^*}^T = \\ NAx(k+m|k) - \gamma_k + {\delta_k^*}^T \end{aligned}$$

with ${\delta_k^*}^T = \begin{bmatrix}{\delta_{\Delta,k}^*}^T & 0_{n_{in}}^T & 0_{n_{un}}^T\end{bmatrix}^T$.

Thus, since $x_{in}(k+m|k)$ and $x_{un}(k+m|k)$ are zero, it yields that

$$x_\Delta(k+m|k) - y_{sp,k} + \delta_{\Delta,k}^* = 0$$

Therefore, by induction, it is easy to see that the problem will remain feasible at every time step $k+j>k$. □

Theorem 1 *For an undisturbed stable, integrating and unstable system with (A,B) stabilizable and $m \geq (n_{in}+n_{un})$ the steady state corresponding to the system reference $y_{sp,k}$ is an asymptotically stable solution to Problem P3, provided $y_{sp,k}$ is reachable and Q and R are positive definite weighting matrices.*

Proof: Consider that $V_3^*(k)$ denotes the optimal cost function value at time step k and $\tilde{V}_3(k+1)$ corresponds to the cost value for $(\Delta\tilde{u}_k,\ \tilde{\delta}_{\Delta,k})$, as defined above. Thus, we may write the following relationship:

$$\begin{aligned} V_3^*(k) - \tilde{V}_3(k+1) = \|y(k|k) - y_{sp,k} + \delta_{\Delta,k}\|_Q^2 \\ + \|\Delta u(k|k)\|_R^2 \end{aligned} \tag{43}$$

Since the right-hand side of 43 is positive, it is easy to see that $\tilde{V}_3(k+1) \leq V_3^*(k)$, which also implies $V_3^*(k+1) \leq V_3^*(k)$. Consequently, the sequence that comprises optimal cost values at subsequent time steps is non-increasing and bounded below by zero. Therefore, $V_3^*(k)$ converges to zero, which implies that Δu_k also converges to zero and that the system output converges to $y_{sp,k}$. □

4 Application examples

Deisobutanizer distillation column

As our first example, we consider a deisobutanizer distillation column, which has been described in (Alvarez et al., 2009). This distillation column is part of the alkylation unit in the oil refinery of PETROBRAS/Cubatão.

A simplified version of the experimental model is given as follows:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{bmatrix} = \begin{bmatrix} \frac{2.3}{s} & \frac{-0.7\times10^{-3}}{s} & \frac{0.2}{s} \\ \frac{4.7}{9.3s+1} & \frac{1.4\times10^{-3}}{6.8s+1} & \frac{0.4}{11.6s+1} \\ \frac{1.9}{10.1s+1} & \frac{61\times10^{-3}}{6.6s+1} & \frac{0.2}{12.3s+1} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \end{bmatrix}$$

The output variables of this process are y_1 (%) the liquid level of the top drum, y_2 (°C) the temperature of tray 68 and y_3 (%) the percentage of flooding in the column, while the manipulated inputs are u_1 (ton/h) the steam flow rate to the reboiler, u_2 (m^3/d) the reflux flow rate and u_3 (°C) the feed temperature, which is supposed to be manipulated (Alvarez et al., 2009).

For simulation purpose, we considered the system has the same input constraints as given in (Alvarez et al., 2009), which are

$$\begin{aligned} u_{max} &= \begin{bmatrix}5.5 & 2800 & 90\end{bmatrix}^T \\ u_{min} &= \begin{bmatrix}4 & 2400 & 85\end{bmatrix}^T \\ \Delta u_{max} &= \begin{bmatrix}0.2 & 25 & 0.5\end{bmatrix}^T \\ \Delta u_{min} &= -\Delta u_{max}. \end{aligned}$$

The tunning parameters of the controller are

$$\begin{aligned} m &= 3 \\ Q &= diag\left(\begin{bmatrix}1 & 1 & 1\end{bmatrix}\right) \\ R &= diag\left(\begin{bmatrix}1 & 1 & 1\end{bmatrix}\right) \times 10^{-1} \\ S_\Delta &= diag\left(\begin{bmatrix}1 & 1 & 1\end{bmatrix}\right) \times 10^{3} \\ S_{in} &= 1 \times 10^5 \end{aligned}$$

The simulation started from a point in which $y_{sp} = \begin{bmatrix}47 & 52.5 & 91\end{bmatrix}^T$, $y = y_{sp}$ and $u = \begin{bmatrix}4.7 & 2650 & 88.5\end{bmatrix}^T$. Then, the output set-point was changed to $y_{sp} = \begin{bmatrix}49 & 52.5 & 90.5\end{bmatrix}^T$ at time step $t = 10$ min.

The process response is shown in Figures 1 and 2 for the controlled and manipulated variables, respectively. Since the first output is pure integrating with respect to all the three inputs, it rapidly converged to its reference as an effect of the high penalty selected for the slack variable related to the integrating modes. Also, from Figure 3, we observe the monotonically decreasing behavior of the controller cost function along the simulation.

Unstable reactor system

In the sequence, we provide the application of the proposed IHMPC to an unstable reactor system, in which a liquid phase, exothermic, irreversible, first order chemical reaction occurs, with $A \rightarrow B$. The system model based on dimensionless variables are detailed in (Nagrath et al., 2002) and is replicated here as follows:

$$\begin{aligned} \frac{dx_1}{d\tau} &= q\left(x_{1f} - x_1\right) - \phi x_1 \kappa, \quad \kappa = \exp\left(\frac{x_2}{1 + x_2/\gamma}\right) \\ \frac{dx_2}{d\tau} &= q\left(x_{2f} - x_2\right) - \delta\left(x_2 - x_3\right) - \beta\phi x_1 \kappa \\ \frac{dx_3}{d\tau} &= \frac{q_c\left(x_{3f} - x_3\right)}{\delta_1} + \frac{\delta\left(x_2 - x_3\right)}{\delta_1 \delta_2} \end{aligned}$$

in which τ corresponds to the time, x_1 is the concentration of reactant A, x_2 is the reactor temperature, x_3 is the jacket temperature, q is the reactor feed flow rate and q_c represents the jacket flow rate. Also, the dimensionless parameters are given as follows: $\beta = 8.0$, $\gamma = 20$, $\delta = 0.3$, $\delta_1 = 0.1$, $\delta_2 = 0.5$, $\phi = 0.072$, $x_{1f} = 1.0$, $x_{2f} = 0.0$ and $x_{3f} = -1.0$. The authors in (Martins and Odloak, 2016) performed the linearization of the CSTR model around an open-loop unstable operating point, which resulted in the following transfer matrix:

$$G(s) = \begin{bmatrix} \frac{0.45(s+22.44)(s+0.58)}{(s+22.59)(s+0.89)(s-0.62)} & \frac{1.04}{(s+22.59)(s+0.89)(s-0.62)} \\ \frac{-2.75(s+22.53)(s+0.76)}{(s+22.59)(s+0.89)(s-0.62)} & \frac{-3(s+1.81)}{(s+22.59)(s+0.89)(s-0.62)} \\ \frac{-16.51(s+0.76)}{(s+22.59)(s+0.89)(s-0.62)} & \frac{-10(s+3.57)(s-0.12)}{(s+22.59)(s+0.89)(s-0.62)} \end{bmatrix}$$

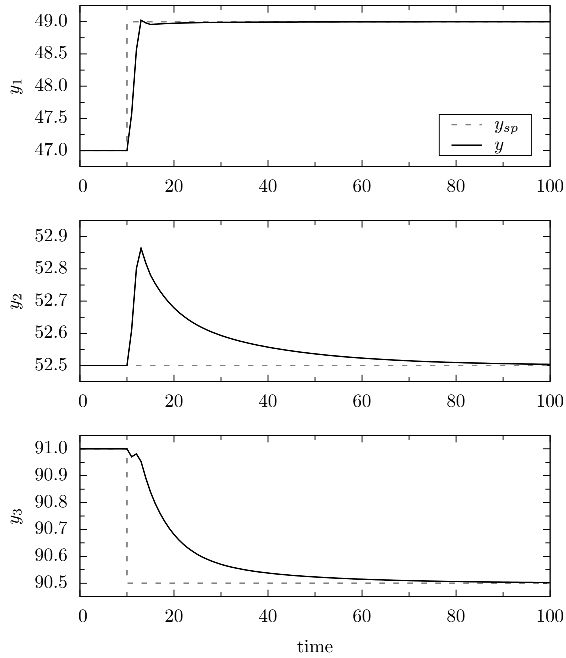

Figure 1: Controlled variables of the distillation column.

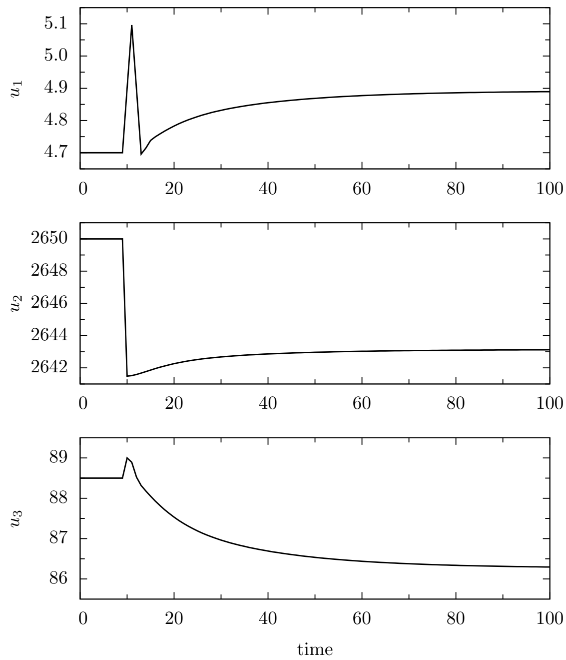

Figure 2: Manipulated variables of the distillation column.

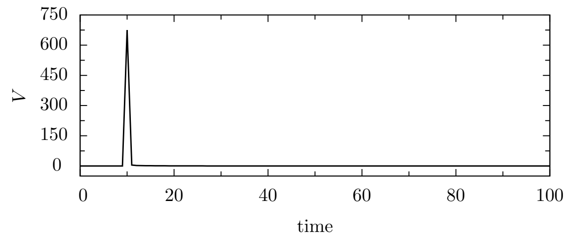

Figure 3: Cost function of the IHMPC applied to the deisobutanizer distillation column.

In this example, we considered the following input constraints:

$$u_{max} = \begin{bmatrix}5 & 10\end{bmatrix}^T$$
$$u_{min} = \begin{bmatrix}0 & 0\end{bmatrix}^T$$
$$\Delta u_{max} = \begin{bmatrix}1 & 2\end{bmatrix}^T$$
$$\Delta u_{min} = -\Delta u_{max}.$$

Note that in this case we have only two inputs to manipulate, which implies that there are no degrees of freedom to control all the three outputs. Therefore, we considered here that y_3 is not a controlled variable, which can be easily handled by zeroing the weights related to this output. The tunning parameters of the controller are given as follows:

$$m = 4$$
$$Q = diag\left(\begin{bmatrix}1 & 1 & 0\end{bmatrix}\right)$$
$$R = diag\left(\begin{bmatrix}1 & 1\end{bmatrix}\right)$$
$$S_\Delta = diag\left(\begin{bmatrix}1 & 1 & 0\end{bmatrix}\right) \times 10^4$$
$$S_{un} = diag\left(\begin{bmatrix}1 & 1\end{bmatrix}\right) \times 10^4$$

In the application of the IHMPC, we considered the system initial condition is $y_{sp} = \begin{bmatrix}0.8933 & 0.5193 & -0.5950\end{bmatrix}^T$, $y = y_{sp}$ and $u = \begin{bmatrix}1.0 & 1.65\end{bmatrix}^T$. At time step $t = 2$, the desired system reference was changed to $y_{sp} = \begin{bmatrix}0.6 & 2.3 & -0.5950\end{bmatrix}^T$.

Figure 4 depicts the response of the system outputs, while the manipulated variables are shown in Figure 5. As expected, only the first two outputs converged to their setpoints since y_3 was not considered as a controlled variable. According to Figure 5, the manipulated variables temporarily reached their lower limits after the setpoint change, which was properly handled by the optimization problem. In addition, the value of cost function as shown in Figure 6 was also non-increasing, which demonstrates the controller stability.

5 Conclusion

In this study, an alternative formulation of a state space model suitable for IHMPC implementation was provided. In addition, we presented the development of the IHMPC for stable, integrating and unstable systems. The simulation examples showed the proposed controller was successfully applied to both an integrating and an unstable system.

Finally, the proposed IHMPC can be easily modified in order to accommodate output zones and input targets, which is very common in process industries. In addition, future works may comprehend the extension of the model presented here considering time delayed systems as well as a robust version of the proposed IHMPC for dealing with system uncertainties.

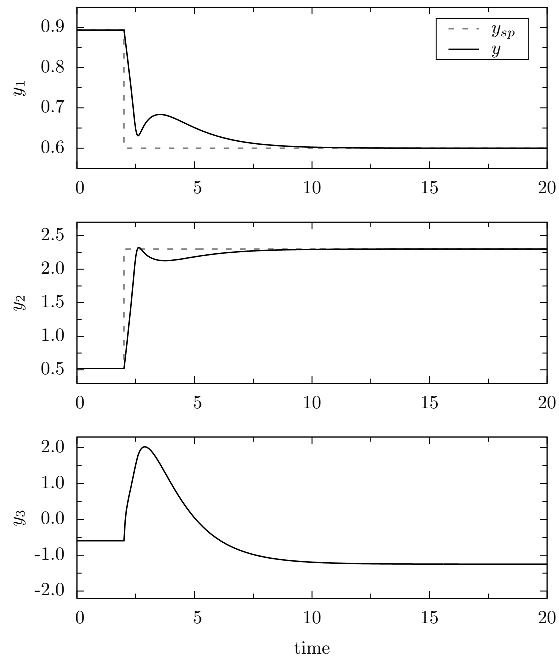

Figure 4: Output response of the unstable reactor.

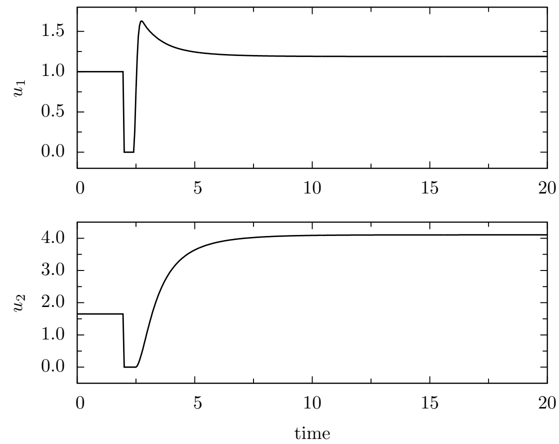

Figure 5: Manipulated variables of the unstable reactor.

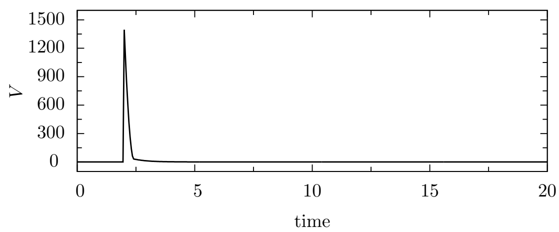

Figure 6: Cost function of the IHMPC applied to the unstable reactor.

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