CONTROL OF MULTIPLE INPUT AND MULTIPLE OUTPUT INFINITE DIMENSIONAL SYSTEMS BASED ON MODAL REDUCTION AND FLATNESS THEORY

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Abstract— This paper presents a control design for infinite dimensional systems with multiple inputs and multiple outputs. By using finite differences method, differential flatness theory and model simplifications, we propose to reduce the original problem to the control of a group of independent low order single input systems. The control strategy is illustrated with a heat diffusion problem through numerical simulations.

Keywords— Infinite dimensional systems, MIMO systems, Differential flatness

Resumo— Este trabalho apresenta uma estratégia de controle para sistemas de ordem infinita com múltiplas entradas e múltiplas saídas. A estratégia consiste em utilizar diferenças finitas, planicidade diferencial e simplificações de modelo para reduzir o problema original ao controle de um grupo de sistemas de baixa ordem com uma entrada. A estratégia de controle é ilustrada com um problema de difusão de calor através de simulações numéricas.

Palavras-chave— Sistemas de ordem infinita, Sistemas MIMO, Planicidade Diferencial

1 Introduction

In this work we present a new approach to control of infinite dimensional MIMO systems, which is primarily based on the approach for SISO case presented on (Monteiro et al., 2015), where a simplified model for the infinite dimensional system is obtained via modal reduction and then differential flatness theory is used to make the trajectory planning and tracking problems easier to deal with. Other works related to the use of differential flatness theory for control of infinite dimensional SISO systems can be found in (Fortaleza et al., 2011) and (Fortaleza, 2013). A MIMO case is addressed by (Fliess et al., 1998).

Our goal is to develop a systematic way to break down the MIMO system into several independent low order SISO systems, which are easier to design controllers for. This is accomplished by using differential flatness theory and model simplification techniques. In an attempt to validate the proposed method, numerical simulations were carried out for the control of a heat diffusion process.

This paper is organized as follows. Section 2 presents the infinite dimensional model of the heat diffusion problem and its discretized version, which is what we control. Section 3 describes the procedure to obtain the simplified SISO models for the MIMO system. Section 4 presents the control design for each of the SISO models and how to use them to compute the control action for the original system. The performance of the proposed controller is evaluated through numerical simulations in Section 5. Lastly, Section 6 contains some concluding remarks and directions for future research.

2 Heat Diffusion Process

2.1 Infinite Dimensional Model

The process to be controlled corresponds to the heat diffusion on a metal plate. Assuming we can control punctual heat sources, which represent the system control inputs, and measure the temperature of certain points in the plate, which correspond to the system outputs, our goal is to drive the plate's temperature from any initial condition to a desired homogeneous condition.

The assumptions we make about the system are:

1. The system is linear;
2. The system's physical properties are homogeneous and isotropic;
3. There is no heat flow at the boundaries.

Then, the system is governed by the heat equation

$$\frac{\partial T}{\partial t}(\chi_1, \chi_2, t) = \eta \nabla^2 T(\chi_1, \chi_2, t) + u(\chi_1, \chi_2, t), \tag{1}$$

with boundary conditions

$$\begin{aligned} \frac{\partial T}{\partial \chi_1}(0, \chi_2, t) &= 0, & (2)\\ \frac{\partial T}{\partial \chi_1}(\chi_{1,max}, \chi_2, t) &= 0, & (3)\\ \frac{\partial T}{\partial \chi_2}(\chi_1, 0, t) &= 0, & (4)\\ \frac{\partial T}{\partial \chi_2}(\chi_1, \chi_{2,max}, t) &= 0, & (5) \end{aligned}$$

where T is the temperature, χ_1 and χ_2 are coordinates, $\chi_{1,max}$ and $\chi_{2,max}$ are the length and the height of the plate, t is time, η is the constant heat diffusion coefficient of the plate's material and u is a source function which represents punctual heat sources on the plate.

2.2 Discretization

We now use the finite differences method to obtain a discretized version of (1). According to (LeVeque, 2007), the second-order derivative $d^2f/d\chi^2$ evaluated at $\chi = \chi^*$ can be approximated by

$$\frac{d^2 f}{d\chi^2}(\chi^*) \approx \frac{f(\chi^* + \Delta\chi) - 2f(\chi^*) + f(\chi^* - \Delta\chi)}{\Delta\chi^2}, \tag{6}$$

with error $O(\Delta\chi^2)$ as $\Delta\chi \to 0$.

Discretizing the plate in n_1 elements in direction χ_1 and n_2 elements in direction χ_2, see Figure 1, and imposing the condition of no flow in the boundaries, we have

$$\begin{aligned} x_{0,j} &= x_{1,j},\\ x_{n_1-2,j} &= x_{n_1-1,j},\\ x_{i,0} &= x_{i,1},\\ x_{i,n_2-2} &= x_{i,n_2-1}, \end{aligned}$$

for $i = 0, \dots, n_1 - 1$ and $j = 0, \dots, n_2 - 1$, where $x_{i,j}$ is the temperature at the cell on position (i, j).

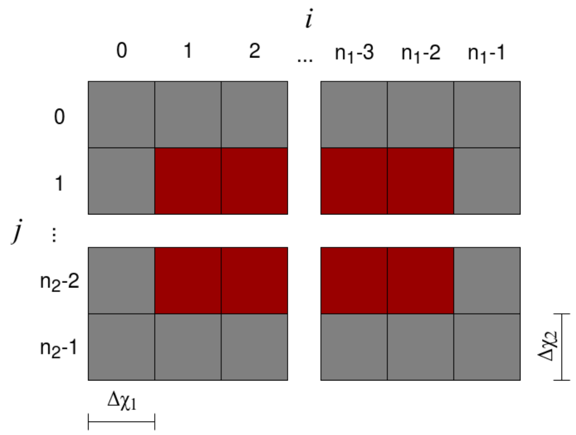

Figure 1: Discretized plate

Defining the state vector to be

$$x = \begin{bmatrix} x_{1,1} & \dots & x_{1,2} & \dots & x_{n_1-2,n_2-2} \end{bmatrix}^T,$$

and assuming $\Delta\chi_1 = \Delta\chi_2$, we have a system of the form

$$\dot{x} = Ax + Bu,$$

with A equals to

$$\frac{\eta}{\Delta\chi_1^2}\begin{bmatrix} -a_{1,1} & 1 & 0_{1\times(n_1-4)} & 1 & & \\ 1 & -a_{2,1} & \ddots & \ddots & \ddots & \\ 0_{(n_1-4)\times 1} & \ddots & \ddots & & & \\ 1 & \ddots & & & & \\ & \ddots & & & & \\ & & & & & \end{bmatrix},$$

with the diagonal elements of A given by

$$a_{i,j} = \begin{cases} 2, & (i,j) = (1,1)\,, (1, n_2-2)\,, \\ & \quad (n_1-2, 1)\,, (n_1-2, n_2-2) \\ 3, & i = 1,\, n_1-2 \text{ and } 1 < j < n_2-1, \text{ or} \\ & j = 1,\, n_2-2 \text{ and } 1 < i < n_1-1 \\ 4, & \text{any other case,} \end{cases}$$

and

$$B = \begin{bmatrix} b_{1,1} & \dots & b_{1,2} & \dots & b_{n_1-2,n_2-2} \end{bmatrix}^T,$$

with $b_{i,j} = 1$ if there is a punctual source at position (i, j) and $b_{i,j} = 0$ otherwise.

3 Model Simplification

3.1 Differential Flatness

According to (Rigatos, 2015), a system with state equation

$$\dot{x} = g(x, u), \tag{7}$$

where g is a smooth vector field, $x \in \Re^n$ is the state vector and $u \in \Re^m$ is the input vector, is differentially flat, or flat, if there is $f \in \Re^m$ such that

$$\begin{aligned} f &= \alpha(x, u, \dot{u}, \dots, u^{(r)}), & (8)\\ x &= \beta(f, \dot{f}, \dots, f^{(q)}), & (9)\\ u &= \gamma(f, \dot{f}, \dots, f^{(q)}), & (10) \end{aligned}$$

where r and q are finite and the functions α, β and γ are smooth. If they exist, the components of f are called the flat outputs of the system (7).

If $f \in \Re^m$ is a flat output vector of (7), then (7) is equivalent via a endogenous feedback to the m inpendent chains of integrators

$$\begin{aligned} f_1^{(\kappa_1)} &= v_1, & (11)\\ &\vdots \\ f_m^{(\kappa_m)} &= v_m, & (12) \end{aligned}$$

where $\kappa_1, \ldots, \kappa_m$ are finite integers and $v_1, \ldots, v_m$ are new inputs (refer to (Levine, 2009) for a proof). This is the property we use to reduce a MIMO control problem to a group of SISO control problems.

For a linear time-invariant system,

$$\dot{x} = Ax + Bu, \tag{13}$$

with $\dim(x) = n$ and $\dim(u) = m$, (Levine, 2009) proves that flatness is equivalent to controllability, i.e., (13) is flat if, and only if, its controllability matrix,

$$C_k = \left[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{n-1}B \end{array} \right], \tag{14}$$

is full row-rank. Notice that C_k can be written as

$$\left[\begin{array}{ccccc} b_1 \ldots & b_m & \ldots & A^{n-1}b_1 & \ldots & A^{n-1}b_m \end{array} \right],$$

where b_i is the i-th column of B. Then, if C_k is full row-rank, we can construct the reduced controllability matrix, C_{kr}, as

$$\left[\begin{array}{ccccccc} b_1 & \ldots & A^{\kappa_1 - 1}b_1 & \ldots & b_m & \ldots & A^{\kappa_m - 1}b_m \end{array} \right],$$

with controllability indexes, i.e. $\kappa_1, \ldots, \kappa_m$, non-negative integers such that their sum is $n = \dim(x)$.

Define a matrix Φ such that its element on the i-th row and j-th column is given by

$$\begin{cases} 1, & \text{if } k_1 + \ldots + k_i = j \\ 0, & \text{otherwise,} \end{cases}$$

Then, according to (Sira-Ramirez and Agrawal, 2004),

$$f = \Phi C_{kr}^{-1} x, \tag{15}$$

is a valid flat output vector, and (13) is equivalent to m single-input systems of the form

$$\dot{z}_i = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ 0 & \ldots & \ldots & 0 \end{bmatrix} z_i + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v_i \tag{16}$$

for $i = 1, \ldots, m$, where

$$z_i = \left[\begin{array}{ccc} f_i & \ldots & f_i^{(\kappa_i - 1)} \end{array} \right]^T.$$

Moreover, there is an invertible matrix M such that

$$\begin{bmatrix} z_1 \\ f_1^{(\kappa_1)} \\ \vdots \\ z_m \\ f_m^{(\kappa_m)} \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix} \tag{17}$$

Defining M_i as the row of M associated with f_i, $i = 1, \ldots, m$, we have

$$f = \begin{bmatrix} M_1 \\ \vdots \\ M_m \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \Phi C_{kr}^{-1} x,$$

hence,

$$\begin{bmatrix} M_1 \\ \vdots \\ M_m \end{bmatrix} = \left[\begin{array}{cc} \Phi C_{kr}^{-1} & 0_{m \times m} \end{array} \right],$$

where $0_{m \times m}$ is a m by m matrix with all its elements equal to 0. The remaining rows of M can be find by differentiating the flat outputs.

3.1.1 Non-Differentially Flat Systems

The system we used to perform the simulation belongs to a class of non-differentially flat systems due to the fact that its controllability matrix is not full row-rank. Hence, we created the C_{kr} matrix with the available linear independent columns from C_k and replaced the inverses of C_{kr} and M by pseudo-inverses in the formulas whenever needed.

3.2 Order Reduction

Using differential flatness theory, we obtained several potentially high-order SISO systems. From this, the following subsections show the procedure to obtain simplified models for each of the SISO systems.

3.2.1 Canonical form

To make the systems of the form (16) suitable for the simplification techniques of the next subsection it is necessary to represent them in a way such that the eigenvalues of the state matrix of each of them are not all zeros. To do this, we write the expressions for $v_i = f_i^{(\kappa_i)}$ in terms of x and u using matrix M. Then, we use M^{-1} to write the terms in x back in terms of f and its time derivatives.

The terms that multiply f_i and its time derivatives are now allocated at the state matrix, and the others are grouped with u to form a new input w_i, this will result in systems of the form

$$\dot{z}_i = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ * & * & \ldots & * \end{bmatrix} z_i + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} w_i, \tag{18}$$

where each $*$ represents an scalar which are all possibly distinct. There are matrices $\Theta_u \in \Re^{m \times m}$ and $\Theta_x \in \Re^{m \times n}$ ($m = \dim(u)$, $n = \dim(x)$), such that

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \Theta_x x + \Theta_u u. \tag{19}$$

3.2.2 Modal reduction

Consider a SISO system with state vector $x \in \Re^n$, input $u \in \Re$ and output $y \in \Re$ of the form

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx. \end{cases} \tag{20}$$

We can perform a similarity transformation to represent (20) in its modal form, i.e., there is V such that (20) can be written as

$$\begin{cases} \dot{x}_M = A_M x_M + B_M u, \\ y = C_M x, \end{cases} \tag{21}$$

where $x_M = V^{-1}x$, $B_M = V^{-1}B$, $C_M = CV$, and $A_M = V^{-1}AV$, with A_M of the form

$$\begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \begin{matrix} \mathrm{Re}(\lambda_k) & \mathrm{Im}(\lambda_k) \\ -\mathrm{Im}(\lambda_k) & \mathrm{Re}(\lambda_k) \end{matrix} & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix},$$

where λ_i is the i-th eigenvalue of A. Notice that y can now be interpreted as a sum of the outputs of several parallel subsystems that receive the same input.

For the system (21) we compute the static gain associated with each subsystem and, for some desired $n_R < n$, we create the reduced model

$$\begin{cases} \dot{x}_R = A_R x_R + B_R u, \\ y_R = C_R x_R + D_R u, \end{cases} \tag{22}$$

with $x_R \in \Re^{n_R}$ and D_R the sum of static gains of the removed subsystems. We choose which subsystems to remove in such a way that the absolute value of D_R is minimal. Notice we can only remove subsystems associated with stable eigenvalues.

3.2.3 Input Delay

(Fortaleza, 2009) proposes to use the model

$$\begin{cases} \dot{x}_D = A_R x_D + B_D u(t-\varepsilon), \\ y_D = C_R x_R + D_D u(t-\varepsilon), \end{cases} \tag{23}$$

where

$$\begin{aligned} B_D &= A_R \left(e^{\varepsilon A_R} - I\right) A_R^{-1} B_R + B_R, \\ D_D &= C_R \left(e^{\varepsilon A_R} - I\right) A_R^{-1} B_R + D_R, \end{aligned}$$

instead of (22) to represent the original SISO system (20), which we do in this paper. The input delay ε is choosen as the time at which the unit-step response of (22) is closer to the initial condition of (20) in order to reduce the direct transfer. For any time $t > \varepsilon$, the unit-step reponses of (22) and (23) are equal (Fortaleza, 2009).

4 Control Design

4.1 Control SISO Delayed Systems

We use the same control approach as (Monteiro et al., 2015) for each of the SISO systems obtained after the simplifications exposed on the previous section. The strategy consists on using a Smith predictor for each of the SISO systems in order to design controllers as if they had no input delay, refer to (Bahill, 1983).

Figure 2 shows the general scheme for the Smith predictor. Notice we used the delayed model output, instead of inserting a delayed input into the model, this is possible due to the time-invariance of the system. For this paper we used state feedback control with an integral action, in order to get zero steady-state error, as can be seen in Figure 3.

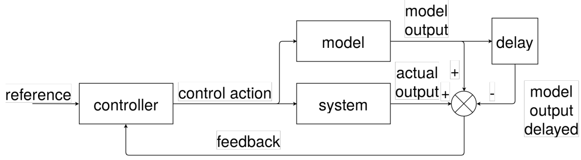

Figure 2: Smith predictor scheme

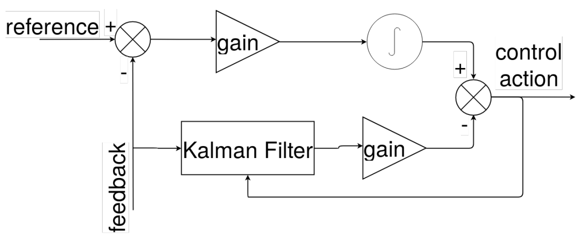

Figure 3: Controller block of Figure 2

4.2 Flat output observer

Since it is most often not possible to measure the state-vector x, only the output y, we developed a way to estimate the flat outputs of the system without directly estimate the state vector. This is done because an observer of the order of the original system would be too sensitive to measuring noises, whereas a lower order observer is expected to be more robust.

Left multiplying Eq. (17) by M^{-1} we obtain

$$M^{-1} \begin{bmatrix} z_1 \\ f_1^{(\kappa_1)} \\ \vdots \\ z_m \\ f_m^{(\kappa_m)} \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix}. \tag{24}$$

Assuming a slow system in order to neglect the time derivatives of flat outputs, we have

$$M_R \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} \approx \begin{bmatrix} x \\ u \end{bmatrix}, \qquad (25)$$

where $M_R \in \Re^{(n+m)\times m}$ contains the columns of M^{-1} that multiply $f_1, \ldots, f_m$ in Eq. (24). We can partition M_R as

$$M_R = \begin{bmatrix} M_x \\ M_u \end{bmatrix},$$

where $M_x \in \Re^{n\times m}$ contains the rows associated with x and $M_u \in \Re^{m\times m}$ contains the rows associated with u. This way, the output y can be approximated as

$$y = Cx \approx CM_x f.$$

The i-th SISO system obtained after performing all the simplifications previously explained is

$$\begin{cases} \dot{z}_{Di} = A_{Ri} z_{Di} + B_{Di} w_i(t-\varepsilon_i) \\ f_{Di} = C_{Ri} z_{Di} + D_{Di} w_i(t-\varepsilon_i) \end{cases},$$

where f_{Di} is an approximation for f_i, $i = 1, \ldots, m$. Thus,

$$\begin{aligned} f &\approx \begin{bmatrix} C_{R1} z_{D1} + D_{D1} w_1(t-\varepsilon_1) \\ \vdots \\ C_{Rm} z_{Dm} + D_{Dm} w_m(t-\varepsilon_m) \end{bmatrix}, \\ y &\approx CM_x \begin{bmatrix} C_{R1} z_{D1} + D_{D1} w_1(t-\varepsilon_1) \\ \vdots \\ C_{Rm} z_{Dm} + D_{Dm} w_m(t-\varepsilon_m) \end{bmatrix}. \end{aligned}$$

Now, we can build the asymptotic state observer

$$\begin{cases} \begin{bmatrix} \dot{\hat{z}}_{D1} \\ \vdots \\ \dot{\hat{z}}_{Dm} \end{bmatrix} = L(y-\hat{y}) + A_R \begin{bmatrix} \hat{z}_{D1} \\ \vdots \\ \hat{z}_{Dm} \end{bmatrix} \\ \qquad + B_D \begin{bmatrix} w_1(t-\varepsilon_1) \\ \vdots \\ w_m(t-\varepsilon_m) \end{bmatrix}, \\ \hat{y} = C_{obs} \begin{bmatrix} \hat{z}_{D1} \\ \vdots \\ \hat{z}_{Dm} \end{bmatrix} + D_{obs} \begin{bmatrix} w_1(t-\varepsilon_1) \\ \vdots \\ w_m(t-\varepsilon_m) \end{bmatrix}, \end{cases} \qquad (26)$$

where L is to be chosen, $C_{obs} = CM_x C_R$, $D_{obs} = CM_x D_D$ and

$$\begin{aligned} A_R &= \operatorname{diag}(A_{R1}, \ldots, A_{Rm}), \\ B_D &= \operatorname{diag}(B_{D1}, \ldots, B_{Dm}), \\ C_R &= \operatorname{diag}(C_{R1}, \ldots, C_{Rm}), \\ D_D &= \operatorname{diag}(D_{D1}, \ldots, D_{Dm}). \end{aligned}$$

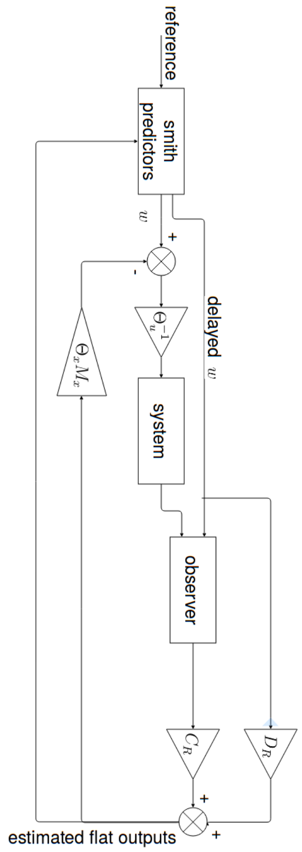

Figure 4: Complete Control System

Finally, the estimation of the flat outputs is given by

$$\hat{f} = C_R \begin{bmatrix} \hat{z}_{D1} \\ \vdots \\ \hat{z}_{Dm} \end{bmatrix} + D_D \begin{bmatrix} w_1(t-\varepsilon_1) \\ \vdots \\ w_m(t-\varepsilon_m) \end{bmatrix}. \qquad (27)$$

4.3 Compute control of the complete system

Solving (19) for u, we have

$$u = \Theta_u^{-1}(w - \Theta_x x). \qquad (28)$$

If x cannot be measured we use the approximation

$$x \approx M_x \hat{f}.$$

Figure 4 shows an schematic for the complete control system used.

5 Simulations

For the simulation, we used a 32 by 32 grid, resulting in a system with $30 \times 30 = 900$ states, the heat diffusion coefficient is $\eta = 0.1$. For the sake of simplicity we chose $\Delta\chi_1 = \Delta\chi_2 = 1$. The system has 15 inputs and 15 outputs on matched locations, i.e., $B = C^T$. The initial condition is $x(0) = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}^T$ and the desired final value is $x(\infty) = \begin{bmatrix} 100 & 100 & \ldots & 100 \end{bmatrix}^T$.

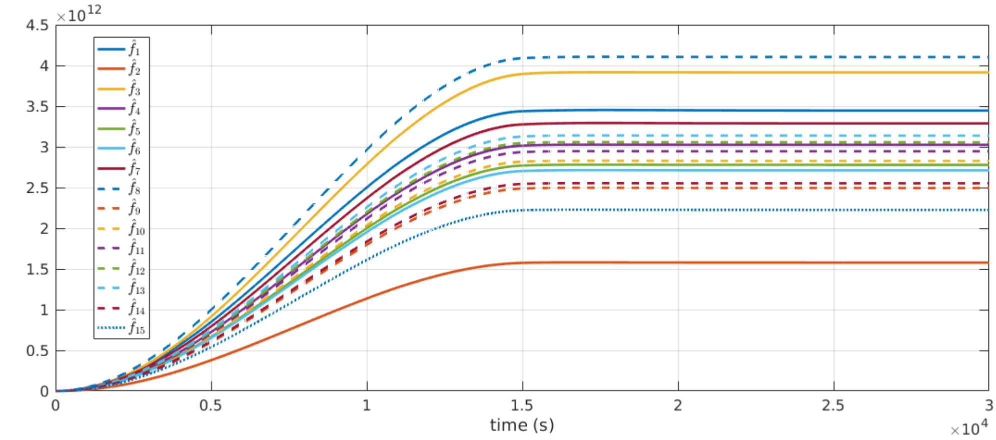

Figure 5: Estimated Flat Outputs

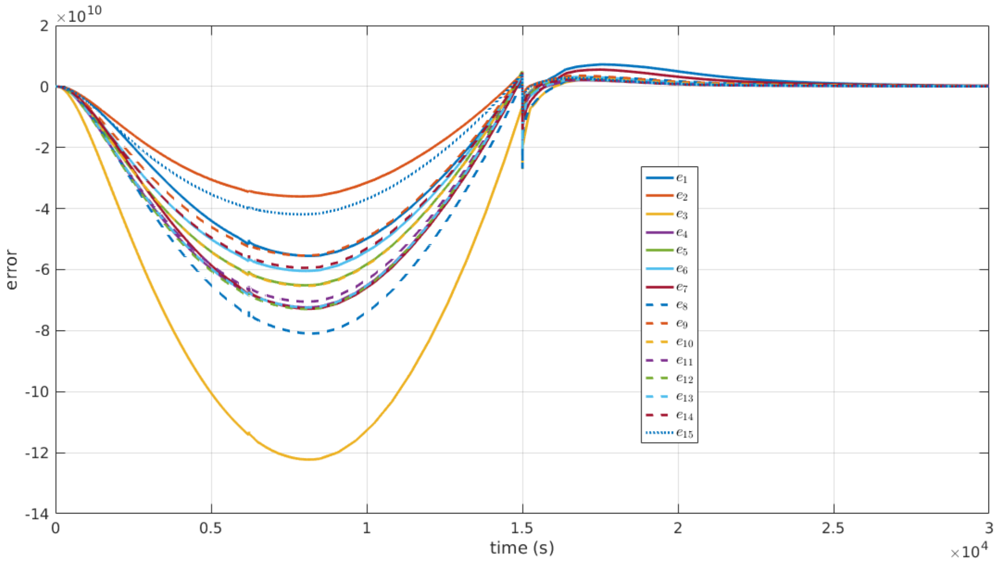

Figure 6: Tracking Error

The reference trajectories for each flat output is a cubic polynomial for $0 \leq t \leq t_f = 15000$ s and a constant value for $t > t_f$. The cubic polynomials were chosen so that the reference trajectories have first time derivative equals to zero at times $t = 0$ and $t = t_f$, and the initial and final values of $\hat{f}$ are computed by inverting CM_x (notice $\hat{y} = CM_x\hat{f}$). Furthermore, saturation is considered in the simulations for the elements of the control action, u, so that $|u_i| \leq 1$.

The rank of the controllability matrix of the system is 105. We used all the 15 controllability indexes equal to 7 and the reduced order systems are of order 2. All the controllers were designed so that the closed loop poles are at -0.01 rad/s, -0.055 rad/s and -0.1 rad/s (notice the closed loop systems are of order 3 due to the integrator in the controller). The gain L is such that the eigenvalues of $A_R^T - C_{obs}^T L^T$ are evenly spaced values between -1 rad/s and -2 rad/s.

Figure 5 shows the estimate of the flat outputs and Figure 6 shows the tracking error for estimated flat outputs. Figure 7 shows the system actual output, y. Asymptotic convergence is obtained, as we intended.

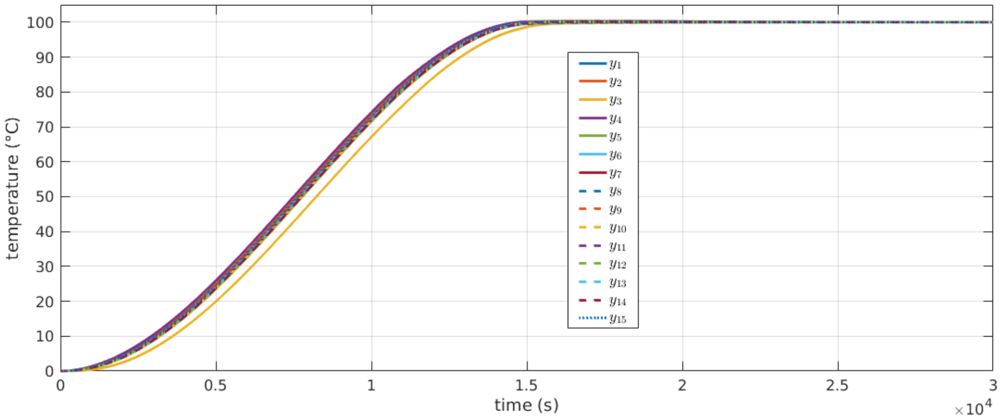

Figure 7: Actual Output of the System

6 Conclusion

This paper presented a new approach for control of infinite dimensional MIMO systems based on differential flatness theory and modal reduction. The main aspects of this work are finding low order SISO systems that well represent the original MIMO system in order to make the design of controllers easier and creating and state observer that is ultimately used to estimate the flat outputs of the system without need for estimate the state vector itself.

Our future perspective is to derive sufficient conditions that guarantee performance, or at least stability, of the original system for a controller based on our method.

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