DESIGN OF ROBUST LQR-DERIVATIVE CONTROLLERS FOR THE $\mathcal{D}$-STABILIZATION OF LINEAR SYSTEMS

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Abstract— In this paper, sufficient conditions for the $\mathcal{D}$-stabilization of linear time-invariant systems are proposed. The controllers are obtained through the resolution of the linear quadratic regulator (LQR) via linear matrix inequalities (LMIs), and the algebraic Riccati equation's (ARE) formulation is based on the state derivative feedback. In the end, practical implementations are performed to illustrate the efficiency of the proposed technique. During the implementations an uncertainty is considered in the model of the system and, even in the presence of uncertainties, the method proposed is effective.

Keywords— $\mathcal{D}$-stability, Linear Quadratic Regulator (LQR), Linear Matrix Inequalities (LMIs), State Derivative Feedback, Polytopic Uncertainties, Robust Pole Placement.

Resumo— Neste trabalho são propostas condições suficientes para a $\mathcal{D}$-estabilização de sistemas lineares invariantes no tempo. Os controladores são obtidos por meio da resolução do problema do regulador linear quadrático (LQR) via desigualdades matriciais lineares (LMIs), sendo que a formulação da equação de Riccati (ARE) é baseada na realimentação derivativa. Ao final, são feitas implementações práticas para ilustrar a eficiência da técnica proposta. Durante a implementação, é considerada uma incerteza no modelo do sistema e, mesmo na presença de incertezas, o método proposto é eficaz.

Palavras-chave— $\mathcal{D}$-estabilidade, Regulador Linear Quadrático (LQR), Desigualdades Matriciais Lineares (LMIs), Realimentação Derivativa, Incertezas Politópicas, Alocação Robusta de Polos.

1 Introduction

In control systems, stability is a minimal requirement, although, in the greater of practical situations, a good controller should also delivery fast and well-damped time responses, which is made by placing the closed-loop poles in a suitable region of the complex plane (Chilali et al., 1999). According to Leite et al. (2004), it is possible, for the designer, include performance index like damping ratio, bounds in the undamped natural frequency and the damped natural frequency, to improve the response of the system in closed-loop by means of placing the closed-loop poles in a circular region of the complex left half plane with radius r and center in $(-c, 0)$.

Many papers deal with the pole placement in a region of the complex left half plane, for example: Chilali and Gahinet (1996) showed that some regions in the complex left half plane can be described as linear matrix inequalities (LMIs) regions, besides the design of the controllers via $\mathcal{D}$-stability; Faria et al. (2009) presented extended results of the $\mathcal{D}$-stability for uncertain, or not, linear systems using the state derivative feedback; Soliman et al. (2016) designed a saturated controller for uncertain systems considering the $\mathcal{D}$-stability in the problem; Datta (2017) presented an algorithm to compute a state feedback gain for a linear time-invariant, regular descriptor system, using the $\mathcal{D}$-stability concepts, as well as LMI regions. All these papers take in hand the controllers design with LMIs, which presents some advantages like the facility to include performance index and treat, with simplicity, uncertainties present in the system model (Boyd et al., 1994). Also, when feasible, LMIs can be easily solved by software programs present in the mathematical programming literature, such as MatLab® (Gahinet et al., 1994).

Controllers design involving LMIs are widely addressed in the specialized literature, including to solve problems that deal with the state derivative feedback (Faria et al., 2009; da Silva et al., 2012; Beteto et al., 2016; Llins et al., 2017). The advantage of using state derivative feedback in the controllers design is that in some system the second derivative signals are accessible due to the presence of accelerometers. Accelerometers are increasingly used in solving engineering problems (Sabato et al., 2016; Kasprzyk et al., 2017; Zhu et al., 2018). Through the acceleration signal, it is possible to reconstruct the velocity signal with good accuracy, but the same does not occur with the displacement signal (Abdelaziz and Valasek, 2004). So, the signals used in the feedback are velocity and acceleration.

In addition, a robust LQR-state derivative controller with pole placement constraints is proposed in this paper. The design of a controller using linear quadratic regulator (LQR) is very useful once this method seeks the optimal controller that minimizes a given cost function, which is parametrized by the matrices $\mathcal{Q}$ and $\mathcal{R}$ (weighting matrices) that weight the state vector and the signal control vector (Ahmed et al., 2010). Another characteristic of the LQR method is that his model is based on the state-space, beyond it can obtain the optimal control signal by solving the algebraic Riccati equation (ARE) (Ahmed et al., 2010; Kumar and Jerome, 2013). Considering the matrix nature of the ARE, it is possible to solve the ARE by LMIs (Boyd et al., 1994). Ge et al. (2002) and Caun et al. (2015) presented the resolution of the ARE via LMIs. Both of these papers use the state feedback to formulate the ARE. Now, in this paper, the formulation of the ARE is made by the state derivative feedback as seen in (Abdelaziz and Valášek, 2005; Abdelaziz, 2010; Beteto et al., 2016; Beteto et al., 2018).

At the end, a practical implementation is presented to show the efficiency of the proposed technique. During the implementation, a mass uncertain is considered to show the effectiveness of the technique to mitigate the vibrations even when the system is subject to a variation of the mass.

2 State Derivative Feedback for Uncertain and Time-invariant Systems

Consider a controllable, linear, time-invariant and uncertain system described as a convex combination of the polytope vertices:

$$\dot{x}(t) = \sum_{i=1}^{s} \alpha_i (A_i x(t) + B_i u(t)) = A(\alpha)x(t) + B(\alpha)u(t), \quad (1)$$

where s represents the polytope vertices. The parameters $\alpha_i, i = 1, 2, ..., s$ are constant and unknown real numbers belonging to unitary simplex $\mathfrak{A}$ given by

$$\mathfrak{A} = \left\{ \sum_{i=1}^{s} \alpha_i = 1, \quad \alpha_i \geq 0, \quad i = 1, ..., s, \right\}. \quad (2)$$

Then, replacing the control law

$$u(t) = -K\dot{x}(t) \quad (3)$$

in (1) and supposing that $A(\alpha)$ is nonsingular ($det(A(\alpha)) \neq 0, \forall i$) (Abdelaziz and Valasek, 2004), the robust system in closed-loop is given by

$$\dot{x}(t) = (I + B(\alpha)K)^{-1} A(\alpha)x. \quad (4)$$

Considering the state derivative feedback is being used and do not have full access to x_0, an initial conditions polytope will be used:

$$x_0(\beta) = \sum_{k=1}^{p} \beta_k x_{0k}, \quad (5)$$

similar to the unitary simplex (2), being p the vertices of the initial conditions polytope.

3 Pole Placement Region

First, a region for pole placement in the complex left half plane is defined.

Definition 1 *Given a region $\mathcal{D}$ of the complex left half plane, then a matrix $A \in \mathbb{R}^{n\times n}$ is said to be $\mathcal{D}$-stable if all of your eigenvalues are in the $\mathcal{D}$ region (Chilali and Gahinet, 1996).*

As aforementioned, the pole placement in the $\mathcal{D}$ region guarantees performance index. In this paper, a circular region as $\mathcal{D}$ region will be considered (Figure 1). As specified by Leite et al. (2004), the circular region constraints for the closed-loop poles assurance that the system dynamics are bounded by exponentials with decay in the interval $-c \pm r$ and frequencies lower or equal than r, i.e., lower values of r achieve lower transient oscillations.

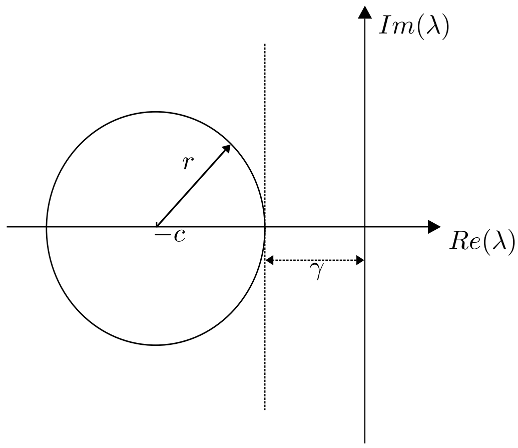

Figure 1: Pole placement region.

For a matrix $A \in \mathbb{R}^{n\times n}$ have all of your eigenvalues inside of the circle it is necessary that the inequality (6) be respected (Haddad and Bernstein, 1992).

$$A^T P + PA + 2\gamma P + \frac{1}{r}(A + \gamma I)^T P(A + \gamma I) < 0. \quad (6)$$

Observation 1 *Inequality (6) was verified in (Haddad and Bernstein, 1992) using a Common Quadratic Lyapunov Function (CQLF), $V(x(t)) = x(t)^T Px(t) > 0$, $P = P^T \in \mathbb{R}^{n\times n}$, with $\dot{V}(x(t)) = \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) < 0$, $\forall x(t) \neq 0$.*

If (6) is feasible, it is possible to assure that all eigenvalues (λ) of A belong to the disk of radius r, decay rate greater or equal to γ and center in $(-c, 0)$ ($c = \gamma + r$). According to da Silva et al. (2012), the circular region guarantees an overshoot and decay rate limitation for the closed-loop system.

Note that the inequality (6) deal with a matrix A precisely know. For a matrix A uncertain, $A \in \mathfrak{A}$, a sufficient condition for the pole placement in the region of Figure 1 is achieved by using the result presented in Leite et al. (2004). Then, the inequality (7) is equivalent to the inequality (6).

$$\begin{bmatrix} A_i^T P + PA_i + 2\gamma P & (A_i + \gamma I)^T P \\ P(A_i + \gamma I) & -rP \end{bmatrix} < 0. \quad (7)$$

The next section presents sufficient conditions to the $\mathcal{D}$-stabilization of the uncertain system (1) using the state derivative feedback.

4 Robust LQR-State Derivative Controller

For the demonstration of LQR-state derivative controller in terms of LMI, the Propriety 1 is used. The ARE formulation can be found in (Abdelaziz and

Valášek, 2005; Abdelaziz, 2010; Beteto et al., 2016; Beteto et al., 2018).

Propriety 1 *A matrix M is invertible if $M+M^T<0$ for any nonsymmetric matrix M ($M \neq M^T$) (Slotine et al., 1991).*

Theorem 1 (Robust LQR-State Derivative Controller via LMI, $\mathcal{D}$-stability) *Let A_i nonsingular ($det(A_i)\neq 0$) and given $\mathcal{Q}\in\mathbb{R}^{n\times n}$, $\mathcal{R}\in\mathbb{R}^{m\times m}$, $x_{0k}\in\mathbb{R}^n$, $r>0$ and $\gamma>0$, the system (1) is $\mathcal{D}$-stable and has optimized performance if there exist symmetric matrix $X>0\in\mathbb{R}^{n\times n}$ and a matrix $Y\in\mathbb{R}^{m\times n}$ satisfying*

$$\min_{X=X^T>0,\,Y} \mu$$

Subject to

$$\begin{bmatrix}\mu & x_{0k}^T\\ x_{0k} & X\end{bmatrix}\geq 0,\quad k=1,2,...,p, \tag{8}$$

$$\Psi_{ii}=\begin{bmatrix}\Lambda_{ii} & * & * & * & *\\ \Gamma_i^T & -rX & * & * & *\\ \Omega_i^T & 0 & -X/(2\gamma) & * & *\\ XA_i^T & 0 & 0 & -\mathcal{Q}^{-1} & *\\ YA_i^T & 0 & 0 & 0 & -\mathcal{R}^{-1}\end{bmatrix}<0,\quad i=1,2,...,s, \tag{9}$$

$$\Psi_{ij}+\Psi_{ji}=\begin{bmatrix}\Lambda_{ij}+\Lambda_{ji} & * & * & * & *\\ \Gamma_{ij}^T+\Gamma_{ji}^T & -2rX & * & * & *\\ 2X+Y^TB_i^T+Y^TB_j^T & 0 & -X/\gamma & * & *\\ XA_i^T+XA_j^T & 0 & 0 & -2\mathcal{Q}^{-1} & *\\ YA_i^T+YA_j^T & 0 & 0 & 0 & -2\mathcal{R}^{-1}\end{bmatrix}\leq 0,\quad \begin{matrix} i=1,2,...,s-1,\\ j=i+1,...,s,\end{matrix} \tag{10}$$

where $\Lambda_{ii}=A_iX+XA_i^T+B_iYA_i^T+A_iY^TB_i^T$, $\Gamma_{ii}^T=XA_i^T+\gamma(X+Y^TB_i^T)$, $\Omega_i^T=X+Y^TB_i^T$, $\Lambda_{ij}+\Lambda_{ji}=A_iX+XA_i^T+A_jX+XA_j^T+B_jYA_i^T+A_iY^TB_j^T+B_iYA_j^T+A_jY^TB_i^T$, $\Gamma_{ij}^T+\Gamma_{ji}^T=XA_i^T+XA_j^T+\gamma(2X+Y^TB_i^T+Y^TB_i^T)$. The state derivative feedback gain can be given by

$$K=YX^{-1}. \tag{11}$$

Proof: Multiplying by β_k, $k=1,2,...,p$, summing all terms and applying the Schur complement (more details about the Schur complement can be seen in (Boyd et al., 1994)) in LMI (8), it has been

$$x_0(\beta)^TX^{-1}x_0(\beta)\leq\mu. \tag{12}$$

In many practical situations, the objective (8) can be modified by (12), where μ is the specified upper bound and $X^{-1}=P$. More details can be seen in (Ge et al., 2002; Beteto et al., 2018).

Now, multiplying (9) by α_i^2 and summing all terms in i, with $i=1,2,...,s$, multiplying (10) by $\alpha_i\alpha_j$ and summing all terms in i, with $i=1,2,...,s-1$ and j, with $j=i+1,...,s$, and summing both

$$\sum_{i=1}^{s}\alpha_i^2(\Psi_{ii})+\sum_{i=1}^{s-1}\alpha_i\sum_{j=i+1}^{s}\alpha_j(\Psi_{ij}+\Psi_{ji})<0. \tag{13}$$

Generically $\sum_{i=1}^s\alpha_i\sum_{j=1}^s\alpha_j(M_iN_j)=\sum_{i=1}^s\alpha_i^2(M_iN_i)+\sum_{i=1}^{s-1}\alpha_i\sum_{j=i+1}^s\alpha_j(M_iN_j+M_jN_i)$, from (13), (14) follows.

$$\sum_{i=1}^{s}\alpha_i\sum_{j=1}^{s}\alpha_j\begin{bmatrix}\Lambda_{ij} & * & * & * & *\\ \Gamma_{ij}^T & -rX & * & * & *\\ \Omega_j^T & 0 & -X/(2\gamma) & * & *\\ XA_i^T & 0 & 0 & -\mathcal{Q}^{-1} & *\\ YA_i^T & 0 & 0 & 0 & -\mathcal{R}^{-1}\end{bmatrix}$$

$$\Leftrightarrow\begin{bmatrix}\Lambda(\alpha) & * & * & * & *\\ \Gamma(\alpha)^T & -rX & * & * & *\\ \Omega(\alpha)^T & 0 & -X/(2\gamma) & * & *\\ XA(\alpha)^T & 0 & 0 & -\mathcal{Q}^{-1} & *\\ YA(\alpha)^T & 0 & 0 & 0 & -\mathcal{R}^{-1}\end{bmatrix}<0, \tag{14}$$

where, taking the sum in (14), the following equivalences are obtained: $\Lambda_{ij}=A_iX+XA_i^T+B_jYA_i^T+A_iY^TB_j^T\Leftrightarrow\Lambda(\alpha)=A(\alpha)X+X(\alpha)^T+B(\alpha)YA(\alpha)^T+A(\alpha)Y^TB(\alpha)^T$, $\Gamma_{ij}^T=XA_i^T+\gamma(X+Y^TB_j^T)\Leftrightarrow\Gamma(\alpha)^T=XA(\alpha)^T+\gamma(X+Y^TB(\alpha)^T)$ and $\Omega_i^T=X+Y^TB_i^T\Leftrightarrow\Omega(\alpha)^T=X+Y^TB(\alpha)^T$.

Applying the Schur complement recursively, replacing $Y=KX$ and organizing

$$\begin{aligned}&(I+B(\alpha)K)XA(\alpha)^T+A(\alpha)X(I+B(\alpha)K)^T+\\&(A(\alpha)X+\gamma(I+B(\alpha)K)X)(1/r)X^{-1}(A(\alpha)X+\\&\gamma(I+B(\alpha)K)X)^T+((I+B(\alpha)K)X)(2\gamma)X^{-1}\\&((I+B(\alpha)K)X)^T+A(\alpha)X(K^T\mathcal{R}K+\mathcal{Q})XA(\alpha)^T<0.\end{aligned} \tag{15}$$

Now, applying Propriety 1 in (15) it is concluded that matrices $(I+B(\alpha)K)$, X and $A(\alpha)$ are invertible. Then, premultiplying by $A(\alpha)^{-1}$ and posmultiplying by $A(\alpha)^{-T}$:

$$\begin{aligned}&A(\alpha)^{-1}(I+B(\alpha)K)X+X(I+B(\alpha)K)^TA(\alpha)^{-T}+\\&(X+\gamma A(\alpha)^{-1}(I+B(\alpha)K)X)(1/r)X^{-1}(X+\gamma A(\alpha)^{-1}\\&(I+B(\alpha)K)X)^T+(A(\alpha)^{-1}(I+B(\alpha)K)X)(2\gamma)X^{-1}\\&(A(\alpha)^{-1}(I+B(\alpha)K)X)^T+X(K^T\mathcal{R}K+\mathcal{Q})X<0.\end{aligned} \tag{16}$$

Using the dual form $[(A(\alpha)^{-1}+A(\alpha)^{-1}B(\alpha)K)X]\rightarrow[X(A(\alpha)^{-1}+A(\alpha)^{-1}B(\alpha)K)^T]$, and considering $X=P^{-1}$:

$$\begin{aligned}&P^{-1}(I+B(\alpha)K)^TA(\alpha)^{-T}+A(\alpha)^{-1}(I+B(\alpha)K)P^{-1}+\\&(P^{-1}+\gamma A(\alpha)^{-1}(I+B(\alpha)K)P^{-1})^T(1/r)P(P^{-1}+\gamma\\&A(\alpha)^{-1}(I+B(\alpha)K)P^{-1})+(A(\alpha)^{-1}(I+B(\alpha)K)P^{-1})^T\\&(2\gamma)P(A(\alpha)^{-1}(I+B(\alpha)K)P^{-1})+\\&P^{-1}(K^T\mathcal{R}K+\mathcal{Q})P^{-1}<0.\end{aligned} \tag{17}$$

Premultiplying by $A(\alpha)^T(I+B(\alpha)K)^{-T}P$, posmultiplying by $P(I+B(\alpha)K)^{-1}A(\alpha)$ and replacing $A_{cl}(\alpha)=(I+B(\alpha)K)^{-1}A(\alpha)$

$$\begin{aligned}&A_{cl}(\alpha)^TP+PA_{cl}(\alpha)+(A_{cl}(\alpha)+\gamma I)^T(1/r)P(A_{cl}(\alpha)+\\&\gamma I)+2\gamma P+A_{cl}(\alpha)^T(K^T\mathcal{R}K+\mathcal{Q})A_{cl}(\alpha)<0.\end{aligned} \tag{18}$$

Supposing the LMI (9) feasible, the portion $A_{cl}(\alpha)^T(K^T\mathcal{R}K+\mathcal{Q})A_{cl}(\alpha)$ is positive. Then

$$\begin{aligned}&A_{cl}(\alpha)^TP+PA_{cl}(\alpha)+(A_{cl}(\alpha)+\gamma I)^T(1/r)P(A_{cl}(\alpha)+\\&\gamma I)+2\gamma P<-A_{cl}(\alpha)^T(K^T\mathcal{R}K+\mathcal{Q})A_{cl}(\alpha),\end{aligned} \tag{19}$$

$$\Downarrow$$

$$\begin{aligned}&A_{cl}(\alpha)^TP+PA_{cl}(\alpha)+(A_{cl}(\alpha)+\gamma I)^T(1/r)P(A_{cl}(\alpha)+\\&\gamma I)+2\gamma P<0.\end{aligned} \tag{20}$$

Applying the Schur complement in (20), (21) follows.

$$\begin{bmatrix} A_{cl}(\alpha)^T P + P A_{cl}(\alpha) + 2\gamma P & (A_{cl}(\alpha)+\gamma I)^T \\ (A_{cl}(\alpha)+\gamma I) & -rP \end{bmatrix} < 0, \tag{21}$$

which is equivalent to (7).

$\square$

Thus, sufficient conditions to assure the $\mathcal{D}$-stability of the uncertain system (1) can be obtained using the proposed theorem.

5 Illustrative Example

Consider the active suspension of a car seat, presented in (Reithmeier and Leitmann, 2003; Assunção et al., 2007). The dynamic equation of this system can be described in the state-space form using the state vector $x(t) = [x_1\ x_2\ \dot{x}_1\ \dot{x}_2]^T$ as:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -30 & 3.3333 & -1.6667 & 0.3333 \\ 41.6667 & -41.6667 & 4.1667 & -4.1667 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.0007 & -0.0007 \\ 0 & 0.0083 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t). \tag{22}$$

To solve the LMIs was used MatLab® software, together with "LMILab" solver and the YALMIP interface (Lofberg, 2004). The weighting matrices and the parameters γ and r chosen were: $\mathcal{Q} = diag(1, 1, 10, 1000)$, $\mathcal{R} = diag(1, 10)$, $r = 50$ and $\gamma = 1$. The range of the states are: $-0.5 \leq x_1, x_2 \leq 0.5\ (m)$ and $-2 \leq x_3, x_4 \leq 2\ (m/s)$. Therefore, the initial conditions polytope has sixteen vertices ($p = 16$). For the simulation, initial conditions are $x_0 = [0.1\ 0.3\ 0\ 0]^T$. The following controllers were designed, K_1 - (da Silva et al., 2012) and K_2 - Theorem 1:

$$K_1 = 10^4 \times \begin{bmatrix} 1.0963 & 0.5172 & 0.1973 & 0.2043 \\ -0.0420 & 0.0789 & -0.0124 & 0.0144 \end{bmatrix}, \tag{23}$$

$$K_2 = 10^3 \times \begin{bmatrix} 2.5989 & 0.3103 & -0.0765 & -0.0294 \\ 0.3419 & 0.0526 & 0.0292 & 0.0016 \end{bmatrix}. \tag{24}$$

Note in Figure 2 that the proposed theorem is recommended for practical implementations, once that have settling time similar to the result achieved by the theorem presented in (da Silva et al., 2012), but have lower controls signals. Stands out that the norm of controller K_2 ($||K_2|| = 2.6413 \times 10^3$) is smaller than the norm of controller K_1 ($||K_1|| = 1.2450 \times 10^4$). Besides, in LQR technique, when the signal control is prioritized the states suffer a penalty and vice-versa.

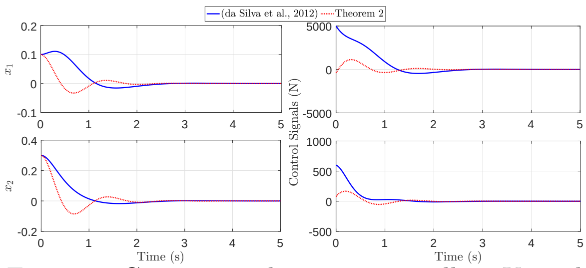

Figure 2: Comparison between controllers K_1 and K_2.

In the following section, a practical application of the proposed methodology is shown to verify the efficiency to mitigate vibrations.

6 Practical Implementation

Consider a model of active suspension of a $\frac{1}{4}$ of the vehicle, produced by Quanser®, whose schematic is present in Figure 3. The system consists of two masses, M_s represents $\frac{1}{4}$ the mass of the vehicle body, and M_{us} represents the tire assembly of the vehicle. The mass M_s is supported by the spring k_s and the damper b_s. The mass M_{us} is supported by the spring k_{us} and the damper b_{us}. To reduce the oscillations caused by runway irregularities the active suspension system is used, which is composed of the masses M_s and M_{us}, and a motor (actuator) connected between them and controlled by force F_c.

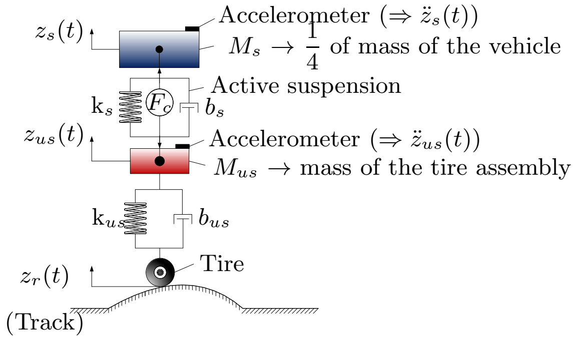

Figure 3: Model of active suspension of a $\frac{1}{4}$ of the vehicle.

The dynamic of the system is given by (Quanser, 2009):

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\rho k_s & -\rho b_s & 0 & \rho b_s \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{M_{us}} & \frac{b_s}{M_{us}} & -\frac{k_{us}}{M_{us}} & -\frac{(b_s+b_{us})}{M_{us}} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & \rho \\ -1 & 0 \\ \frac{b_{us}}{M_{us}} & -\frac{1}{M_{us}} \end{bmatrix} u(t). \tag{25}$$

with $x(t) = \begin{bmatrix} z_s(t) - z_{us}(t) \\ \dot{z}_s(t) \\ z_{us}(t) - z_r(t) \\ \dot{z}_{us}(t) \end{bmatrix}$, $u(t) = \begin{bmatrix} \dot{z}_r(t) \\ F_c \end{bmatrix}$ and $\rho = \frac{1}{M_s}$. The value of constants can be seen in Table 1 (Quanser, 2009).

Table 1: Active suspension parameters.

Parameters	Symbol	Value
Mass of $\frac{1}{4}$ of the total body of vehicle (kg)	M_s	2.45
Mass of the tire assembly (kg)	M_{us}	1
Spring stiffness constant (N/m)	k_s	900
Spring stiffness constant (N/m)	k_{us}	2500
Damping coefficient (Ns/m)	b_s	7.5
Damping coefficient (Ns/m)	b_{us}	5

Note in the state space representation of the system that there are two control inputs, one referring to the track surface velocity ($\dot{z}_r$) and one referring to the force (F_c) applied to the active suspension actuator. For this work, only the control input (F_c) will be taken into account.

The mass can be changed due to two equal loads (each one weights 0.4975 kg), which make up the mass M_s. In this way, the mass M_s can belong to the range $1.455 \leq M_s \leq 2.45\ (kg)$. Considering the parameter $\rho = 1/M_s$, the range can be modified to $1/M_{smax} \leq$

$1/M_s \leq 1/M_{smin} \Rightarrow \rho_{min} \leq \rho \leq \rho_{max} \Rightarrow 0.4082 \leq \rho \leq 0.6873$.

Thus, we have two polytope vertices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.35 & -3.0612 & 0 & 3.0612 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad (26)$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.56 & -5.1546 & 0 & 5.1546 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad (27)$$

and

$$B_1 = \begin{bmatrix} 0 \\ 0.4082 \\ 0 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.6873 \\ 0 \\ -1 \end{bmatrix}. \quad (28)$$

The range of the states are: $-0.02 \leq x_1, x_3 \leq 0.02$ (m) and $-0.15 \leq x_2, x_4 \leq 0.15$ (m/s). Therefore, the initial conditions polytope has sixteen vertices ($p = 16$).

To solve the LMIs was used MatLab® software, together with "LMILab" solver and the YALMIP interface (Lofberg, 2004). The weighting matrices and the γ parameter chosen were: $\mathcal{Q} = diag(10 \quad 0.1 \quad 10 \quad 0.1)$, $\mathcal{R} = 0.1$, $r = 60$ and $\gamma = 0.75$.

The following robust controller was designed:

$$K = \begin{bmatrix} 74.5819 & 3.1355 & -70.1719 & -0.2442 \end{bmatrix}, \quad (29)$$

As reference signal (z_r) for practical implementation a square wave signal was adopted. Such signal has an amplitude of 0.02 m, frequency of $\frac{1}{3}$ Hz and pulse width of 50%. The sampling period was 1 ms. For all the implementations a time interval of 0 to 12 seconds was considered, and until 5.99 seconds the system is in open-loop, in 6 seconds the system is in closed-loop with control law $u(t) = -K\dot{x}(t)$.

Two implementations are performed. The first one considers the robust controller K with mass $M_s = 1.455$ kg. The other one considers the robust controller K with mass $M_s = 2.45$ kg. The result of the implementations can be seen in Figures 4 and 5. The system is naturally stable, but it is possible to observe that its settling time is high and have abrupt oscillations. These abrupt oscillations or vibrations can cause damage to the system and discomfort to the driver and passengers inside the vehicle. Using the force F_c, an active suspension system is capable of to decrease the oscillations/vibrations.

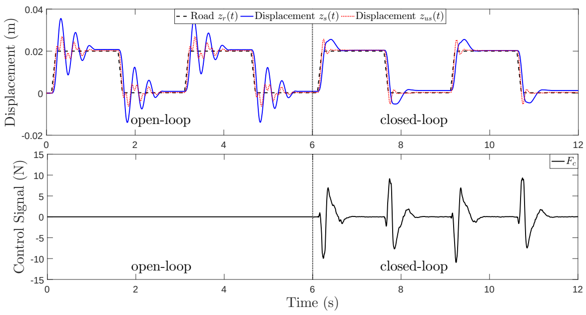

Figure 4: Real system behavior with mass $M_s = 1.455$ kg.

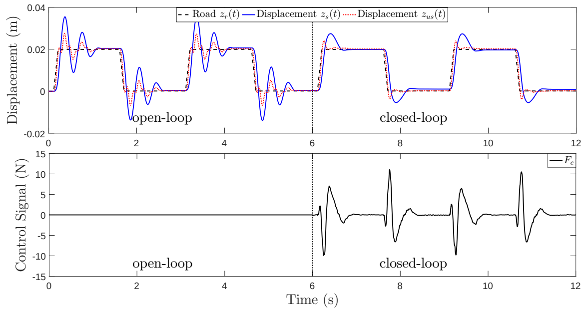

Figure 5: Real system behavior with mass $M_s = 2.45$ kg.

Figures 4 and 5 show that the designed controller was able to mitigate the vibrations (attenuate the oscillations) considering the mass M_s uncertain. Observe that the control signal has a small amplitude, which is an advantage in practical implementations. Lower amplitudes from control signal is a characteristic of state derivative feedback.

In Figure 6 the closed-loop eigenvalues are showed. Note that the all eigenvalues are inside of the desired region $\mathcal{D}$.

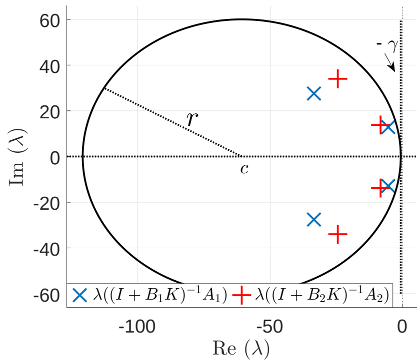

Figure 6: Location of the eigenvalues of the closed-loop system.

7 Conclusions

In this paper, new conditions for the $\mathcal{D}$-stabilization of linear and time-invariant systems are proposed. The advantage of the proposed technique is the possibility to include uncertainties in the LQR problem, as well as to achieve project requirements through the pole placement constraints. Besides, the state derivative feedback plus LQR is an important technique in control of mechanical systems, since that the signal of the second derivative it is available due to accelerometers sensors and within the method, it is possible to weight the control input and the state derivative vector. For the practical implementation of the active suspension system, the designed robust controller was efficient to mitigate the vibrations even when the mass M_s is considered uncertain. For future works, the study of the flexibilization of constraints of the problem and how the matrices $\mathcal{Q}$ and $\mathcal{R}$ behave with restrictions less conservative are intended to.

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