FURTHER IMPROVEMENTS ON STABILITY ANALYSIS FOR UNCERTAIN TIME-DELAYED LINEAR SYSTEMS

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Abstract— This paper is concerned with stability analysis of uncertain systems with time-varying delay. It is presented an improved stability criterion derived within the framework of linear matrix inequalities (LMIs). The main idea consists in appropriately splitting into two parts a single integral term of a quadratic Lyapunov-Krasovskii functional recently proposed in the literature. Then we show that such division leads to conservatism reduction of the stability analysis criterion in the case when the delay is time-varying and/or the system is subject to uncertain parameters. Finally, numerical examples are given to illustrate the effectiveness of the proposed method.

Keywords— Time-varying delay, Uncertain system, Reciprocally convex lemma, Bessel-Legendre inequality, Lyapunov-Krasovskii functional.

Resumo— Este artigo trata da análise de estabilidade de sistemas incertos sujeitos a retardo variante no tempo. Um critério de estabilidade aprimorado é obtido no contexto de desigualdades matriciais lineares (LMIs, do inglês *Linear Matrix Inequalities*). A ideia principal consiste em dividir apropriadamente em duas partes um termo integral de um funcional de Lyapunov-Krasovskii proposto recentemente na literatura. Na sequência, é mostrado que essa divisão pode levar a redução do conservadorismo no critério para análise de estabilidade no caso em que o retardo é variante no tempo e/ou quando o sistema está sujeito a incertezas paramétricas. Por fim, exemplos númericos são apresentados para ilustrar a eficácia do método proposto.

Palavras-chave— Retardo variante no tempo, Sistema incerto, Lema da Reciprocidade Convexa, Desigualdade de Bessel-Legendre, Funcional de Lyapunov-Krasovskii.

1 Introduction

Stability analysis and stabilization of time-delay systems is a very active research field. The interest relies on the fact that time-delay is a common phenomenon in a variety of control systems, as chemical process, communication networks, vehicular traffic flows, population dynamics and epidemics (Fridman, 2014), to name a few. Besides that, the characteristic equations of time-delay systems have infinitely many roots, which makes the stability analysis and control design very complex.

The methods for stability analysis of time-delay systems are mainly classified into two categories: delay-independent and delay-dependent. A system is said to be delay-independent stable if its stability does not depend on the delay value. Otherwise, the system is said to be delay-dependent stable. The present paper focuses on the delay-dependent stability.

Among the stability analysis methodologies to deal with time-delay systems, the Lyapunov-Krasosvkii theory have received increasing attention since the resulting stability criteria can usually be expressed in terms of linear matrix inequalities (LMIs) conditions. On the other hand, finding a Lyapunov-Krasovskii functional (LKF) which provides nonconservative stability conditions is not an easy task. Strategies for the reduction of conservatism include the discretization/partition delay method (Gu et al., 2003; Gouaisbaut and Peaucelle, 2006; Fridman et al., 2009), new functional chooses (Fridman and Shaked, 2002; Sun et al., 2010) and the use of improved integral inequalities, as the Jensen (Gu et al., 2003), Wirtinger-based (Seuret and Gouaisbaut, 2013), Auxiliary Function-based (Park et al., 2015), and Bessel-Legendre (Seuret and Gouaisbaut, 2015) integral inequalities. This last approach is regarded as a convenient manner to relax the stability criteria due to the low complexity of the resulting LMIs when compared to the discretization methods.

Recently Zhang et al. (2017) have presented a new augmented LKF for stability analysis of time-varying delay systems, which is appropriate to explore the advantages of the Bessel-Legendre inequality and the Reciprocally Convex Lemma (Park et al., 2011), leading to more accurate results.

In this paper it is proposed a further change in the LKF described in Zhang et al. (2017). The change consists in splitting a single integral term in the LKF into two parts. Then it is shown that the LKF proposed is more suitable to be combined with the Reciprocally Convex Lemma allowing less conservative results mainly in the case of time-varying delay. Moreover, we extend the results of Zhang et al. (2017) to the stability analysis of uncertain linear systems with time-varying delay.

The remainder of this paper is organized as follows. The problem formulation and preliminary concepts are presented in Section 2. In Section 3, the stability criteria for uncertain time-varying delay systems is obtained. Numerical examples taken from the literature are presented in Section 4 to demonstrate the effectiveness of the proposed method. Finally, Section 5 draws the conclusions.

Notation: The notations used throughout this paper are standard. $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real matrices. $*$ refers to symmetric terms in a symmetric matrix. M^T stands for transpose of the matrix M. For a symmetric matrix M, $M \succ 0$ ($M \prec 0$) means that M is positive (negative) definite. $\mathrm{sym}\{M\}$ denotes $M + M^T$. $\mathrm{col}\{a, b\}$ denotes a column vector whose elements are a, b. $\mathrm{diag}\{A, B\}$ stands for a diagonal matrix whose elements are A, B. $0_{n\times m}$ represents a $n \times m$ zero matrix.

2 Preliminaries and Problem Statement

Consider the class of uncertain linear systems with time-varying delay given by

$$\begin{cases} \dot{x}(t) = & (A+\Delta A)x(t) + (A_d + \Delta A_d)x(t-h(t)) \\ x(\theta) = & \varphi(\theta), \quad \theta : [-h_M,\ 0] \to \mathbb{R}^n \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ are the system states and $\varphi(\theta)$ is an initial condition. $h(t)$ is a continuous function used to describe the time-varying delay, that satisfies

$$0 \le h(t) \le h_M, \quad d_m \le \dot{h}(t) \le d_M < 1,$$

with h_M, d_m, and d_M being known constants.

The matrices ΔA and ΔA_d represent the uncertainties of the system and it is assumed that they satisfy the following condition

$$[\Delta A \ \ \Delta A_d] = D\Delta(t)[E_s \ \ E_d], \tag{2}$$

in which D, E_s, and E_d are known matrices and $\Delta(t)$ is a unknown norm-bounded function which verifies

$$\Delta^T(t)\Delta(t) \le I.$$

The aim of this paper is investigate the asymptotic stability of the system (1).

In the sequel, we present some useful lemmas which play an important role in the development of the main results.

Lemma 1 (Bessel-Legendre Inequality) *(Seuret and Gouaisbaut, 2015) For any $\mathcal{R} \succ 0 \in \mathbb{R}^{n\times n}$, any differentiable function x in $[a, b] \to \mathbb{R}^n$, the following inequality holds*

$$\int_a^b \dot{x}^T(u)\mathcal{R}\dot{x}(u)du \ge \frac{1}{b-a}\Omega^T \mathrm{diag}(\mathcal{R}, 3\mathcal{R}, 5\mathcal{R})\Omega,$$

where $\Omega = \mathrm{col}\{\Omega_1, \Omega_2, \Omega_3\}$, with $\Omega_1 = x(b) - x(a)$ and

$$\begin{aligned} \Omega_2 &= x(b) + x(a) - \frac{2}{b-a}\int_a^b x(u)du, \\ \Omega_3 &= \Omega_1 - \frac{6}{b-a}\int_a^b x(u)du \\ &\quad + \frac{12}{(b-a)^2}\int_a^b (b-u)x(u)du. \end{aligned}$$

Lemma 2 (Reciprocally Convex Lemma) *(Zhang et al., 2017) Let $\mathcal{R}_1 \succ 0$, $\mathcal{R}_2 \succ 0 \in \mathbb{R}^{m\times m}$; $\sigma_1, \sigma_2 \in \mathbb{R}^m$ and a scalar $\alpha \in (0, 1)$. Then, the following inequality holds for any $Y_1, Y_2 \in \mathbb{R}^{m\times m}$:*

$$\begin{aligned} &\frac{1}{\alpha}\sigma_1^T\mathcal{R}_1\sigma_1 + \frac{1}{1-\alpha}\sigma_2^T\mathcal{R}_2\sigma_2 \succeq \\ &\sigma_1^T[\mathcal{R}_1 + (1-\alpha)(\mathcal{R}_1 - Y_1\mathcal{R}_2^{-1}Y_1^T)]\sigma_1 \\ &\quad + \sigma_2^T[\mathcal{R}_2 + \alpha(\mathcal{R}_2 - Y_2^T\mathcal{R}_1^{-1}Y_2)]\sigma_2 \\ &\quad + 2\sigma_1^T[\alpha Y_1 + (1-\alpha)Y_2]\sigma_2. \end{aligned}$$

Lemma 3 *(Lee et al., 2001) Let $D \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{m\times n}$ and $\Delta(t) \in \mathbb{R}^{m\times m}$, and assume that $\Delta(t)$ satisfies the condition $\Delta^T(t)\Delta(t) < I$. Then, for any diagonal matrix $\Theta \succ 0 \in \mathbb{R}^{m\times m}$, the following inequality holds*

$$DF(t)E + E^TF^T(t)D^T \succeq E^T\Theta E + D\Theta^{-1}D^T.$$

Lemma 4 *(Kim, 2016) Let $f(\ell) = a_2\ell^2 + a_1\ell + a_0$, where $a_2, a_1, a_0 \in \mathbb{R}$. If*

$$(i) f(0) < 0, \quad (ii) f(\tau) < 0, \quad (iii) -\tau^2 a_2 + f(0) < 0,$$

then $f(\ell) < 0, \forall \ell \in [0, \tau]$.

Lemma 5 (Finsler's Lemma) *(de Oliveira and Skelton, 2001) Let $\zeta \in \mathbb{R}^n, \Phi = \Phi^T$, and $\mathcal{B} \in \mathbb{R}^{m\times n}$ such that $rank(\mathcal{B}) < 0$. Then, the following statements are equivalent:*

(i) $\zeta^T\Phi\zeta \prec 0, \forall \mathcal{B}\zeta = 0, \zeta \ne 0,$

(ii) ${\mathcal{B}^{\perp}}^T\Phi\mathcal{B}^{\perp} \prec 0,$

(iii) $\exists \mu \in \mathbb{R} : \Phi - \mu\mathcal{B}^T\mathcal{B} \prec 0,$

(iv) $\exists \mathcal{X} \in \mathbb{R}^{n\times m} : \Phi + \mathcal{X}\mathcal{B} + \mathcal{B}^T\mathcal{X}^T \prec 0.$

3 Main Results

In this section, we derive delay dependent stability conditions for system (1) within the framework of LMIs. The main result is stated in the following Theorem.

Theorem 6 *For given scalars d_m, d_M, and h_M, system (1) is asymptotically stable if there exist positive definite matrices $Q_1 \in \mathbb{R}^{6n\times 6n}$, $Q_2 \in \mathbb{R}^{6n\times 6n}$, $R_1 \in \mathbb{R}^{n\times n}$, $R_2 \in \mathbb{R}^{n\times n}$, positive definite diagonal matrix $\Theta \in \mathbb{R}^{n\times n}$, and matrices $Y_1 \in \mathbb{R}^{3n\times 3n}$, $Y_2 \in \mathbb{R}^{3n\times 3n}$ and $X_i \in \mathbb{R}^{n\times n}$, for $i = 1, 2, 3$, such that the following conditions hold for $d = d_m$ and $d = d_M$.*

$$\begin{bmatrix} \tilde{\Upsilon}(0,d) & \Gamma_2^T Y_2^T & \bar{e}^T\bar{D} \\ * & -\tilde{R}_1 & 0_{3n\times n} \\ * & * & -\Theta \end{bmatrix} \prec 0, \tag{3}$$

$$\begin{bmatrix} \tilde{\Upsilon}(h_M,d) & \Gamma_1^T Y_1 & \bar{e}^T\bar{D} \\ * & -\tilde{R}_2 & 0_{3n\times n} \\ * & * & -\Theta \end{bmatrix} \prec 0, \tag{4}$$

$$\begin{bmatrix} -h_M^2\mathcal{G}_2(d) + \tilde{\Upsilon}(0,d) & \Gamma_2^T Y_2^T & \bar{e}^T\bar{D} \\ * & -\tilde{R}_1 & 0_{3n\times n} \\ * & * & -\Theta \end{bmatrix} \prec 0, \tag{5}$$

where $\tilde{R}_i = \mathrm{diag}\{R_i, 3R_i, 5R_i\}$ *for* $i = 1, 2$,

$$\tilde{\Upsilon}(h(t),\dot{h}(t)) = \Upsilon(h(t),\dot{h}(t)) + \mathrm{sym}\{M_1\} + \bar{e}^T\bar{E}^T\Theta\bar{E}\bar{e},$$

$$\Upsilon(h(t),\dot{h}(t)) = \Upsilon_0(h(t),\dot{h}(t)) - \Upsilon_1(h(t)), \tag{6}$$

$$\begin{aligned}\Upsilon_0(h(t),\dot{h}(t)) &= -(1-\dot{h}(t))\mathcal{C}_2^T Q_1\mathcal{C}_2 - \mathcal{C}_5^T Q_2\mathcal{C}_5 \\ &+ (\mathcal{C}_{11} + h(t)C_{12})^T Q_1(\mathcal{C}_{11} + h(t)C_{12}) \qquad (7)\\ &+ \mathrm{sym}\{(\mathcal{C}_{30} + h(t)\mathcal{C}_{31} + h^2(t)\mathcal{C}_{32})^T Q_1\mathcal{D}_1\} \\ &+ (1-\dot{h}(t))(\mathcal{C}_{41} + \alpha h_M\mathcal{C}_{42})^T Q_2(\mathcal{C}_{41} + \alpha h_M\mathcal{C}_{42}) \\ &+ \mathrm{sym}\{(\mathcal{C}_{60} + \alpha h_M\mathcal{C}_{61} + (\alpha h_M)^2\mathcal{C}_{62})^T Q_2\mathcal{D}_2\} \\ &+ h_M^2 e_8^T R_2 e_8 + \alpha h_M^2(1-\dot{h}(t))e_9^T(R_1 - R_2)e_9,\end{aligned}$$

$$\begin{aligned}\Upsilon_1(h(t)) &= (2-\alpha)\Gamma_1^T\tilde{R}_1\Gamma_1 + (1+\alpha)\Gamma_2^T\tilde{R}_2\Gamma_2 \\ &+ \mathrm{sym}\{\Gamma_1^T[\alpha Y_1 + (1-\alpha)Y_2]\Gamma_2\},\end{aligned} \tag{8}$$

$$M_1 = \left[\begin{array}{cc|c|c|c} X_1A & X_1A_d & & -X_1 & \\ X_2A & X_2A_d & & -X_2 & \\ 0_{5n\times n} & 0_{5n\times n} & 0_{10n\times 5n} & 0_{5n\times n} & 0_{10n\times 2n} \\ X_3A & X_3A_d & & -X_3 & \\ 0_{2n\times n} & 0_{2n\times n} & & 0_{2n\times n} & \end{array}\right],$$

$$\bar{e} = \mathrm{col}\{e_1, e_2, e_8\},$$

$$\bar{D} = \mathrm{col}\{X_1D, X_2D, X_3D\},$$

$$\bar{E} = \mathrm{col}\{E_s^T, E_d^T, 0_{n\times n}\},$$

$$\begin{aligned}\mathcal{G}_2(\dot{h}(t)) &= \mathcal{C}_{12}^T Q_1\mathcal{C}_{12} + (1-\dot{h}(t))C_{42}^T Q_2\mathcal{C}_{42} \\ &+ \mathrm{sym}\{\mathcal{D}_1^T Q_1\mathcal{C}_{32} + \mathcal{D}_2^T Q_2\mathcal{C}_{62}\},\end{aligned} \tag{9}$$

$$\Gamma_1 = \mathrm{col}\{e_2 - e_3, e_2 + e_3 - 2e_4, e_2 - e_3 - 6e_4 + 12e_5\}, \tag{10}$$

$$\Gamma_2 = \mathrm{col}\{e_1 - e_2, e_2 + e_1 - 2e_6, e_1 - e_2 - 6e_6 + 12e_7\}, \tag{11}$$

where $\alpha = (h_M - h(t))/h_M$ *and*

$$\begin{aligned}
\mathcal{C}_{11} &= \mathrm{col}\{e_8, e_1, e_1, e_2, e_3, 0_{n\times 10n}\}\\
\mathcal{C}_{12} &= \mathrm{col}\{0_{5n\times 10n}, e_6\}\\
\mathcal{C}_{2} &= \mathrm{col}\{e_9, e_2, e_1, e_2, e_3, 0_{n\times 10n}\}\\
\mathcal{C}_{30} &= \mathrm{col}\{e_1 - e_2, 0_{5n\times 10n}\}\\
\mathcal{C}_{31} &= \mathrm{col}\{0_{n\times 10n}, e_6, e_1, e_2, e_3, 0_{n\times 10n}\}\\
\mathcal{C}_{32} &= \mathrm{col}\{0_{5n\times 10n}, e_7\}\\
\mathcal{C}_{41} &= \mathrm{col}\{e_9, e_2, e_1, e_2, e_3, 0_{n\times 10n}\}\\
\mathcal{C}_{42} &= \mathrm{col}\{0_{5n\times 10n}, e_4\}\\
\mathcal{C}_{5} &= \mathrm{col}\{e_{10}, e_3, e_1, e_2, e_3, 0_{n\times 10n}\}\\
\mathcal{C}_{60} &= \mathrm{col}\{e_2 - e_3, 0_{5n\times 10n}\}\\
\mathcal{C}_{61} &= \mathrm{col}\{0_{n\times 10n}, e_4, e_1, e_2, e_3, 0_{n\times 10n}\}\\
\mathcal{C}_{62} &= \mathrm{col}\{0_{5n\times 10n}, e_5\}\\
\mathcal{D}_1 &= \mathrm{col}\{0_{2n\times 10n}, e_8, (1-\dot{h}(t))e_9, e_{10}, (\dot{h}(t)-1)e_2\}\\
\mathcal{D}_2 &= \mathrm{col}\{0_{2n\times 10n}, e_8, (1-\dot{h}(t))e_9, e_{10}, -e_3\}
\end{aligned}$$

with $e_i = [0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(10-i)n}]$, *for* $i = 1, 2, ..., 10$.

Proof: The proof follows similar steps as in Zhang et al. (2017) and it will be only sketched.

Consider the Lyapunov-Krasovskii functional candidate:

$$V(x_t,\dot{x}_t) = V_1(x_t,\dot{x}_t) + h_M V_2(\dot{x}_t), \tag{12}$$

where $x_t = x(t+\theta)$, $\theta \in [-h_M,\ 0]$ and

$$\begin{aligned} V_1(x_t,\dot{x}_t) &= \int_{t-h(t)}^{t}\eta_1^T(t,s)Q_1\eta_1(t,s)ds \\ &+ \int_{t-h_M}^{t-h(t)}\eta_2^T(t,s)Q_2\eta_2(t,s)ds, \end{aligned}$$

$$\begin{aligned} V_2(\dot{x}_t) &= \int_{t-h_M}^{t-h(t)}(h_M - t + s)\dot{x}^T(s)R_1\dot{x}(s)ds \\ &+ \int_{t-h(t)}^{t}(h_M - t + s)\dot{x}^T(s)R_2\dot{x}(s)ds, \end{aligned}$$

and

$$\eta_0^T(t) = [x^T(t) \quad x^T(t-h(t)) \quad x^T(t-h_M)],$$

$$\eta_1^T(t,s) = [\dot{x}^T(s) \quad x^T(s) \quad \eta_0^T(t) \quad \int_{t-h(t)}^{s}x^T(\theta)d\theta],$$

$$\eta_2^T(t,s) = [\dot{x}^T(s) \quad x^T(s) \quad \eta_0^T(t) \quad \int_{t-h_M}^{s}x^T(\theta)d\theta].$$

The difference between this functional and the one in Zhang et al. (2017) is the term $V_2(\dot{x}_t)$, where if we set $R_1 = R_2$ we recover the functional in Zhang et al. (2017). In this paper, we divide the integral interval $[t, t-h_M]$ into two subintervals and for each subinterval we consider a different matrix variable. This choice is more appropriate to use with Reciprocally Convex Lemma and it can leads less conservative results, as shown in Section 4.

The functional candidate in (12) is guaranteed positive by imposing $Q_i \succ 0$ and $R_i \succ 0$ for $i = 1, 2$.

Initially, to obtain conditions that guarantee $\dot{V}(x_t, \dot{x}_t) < 0$ the augmented vector is defined

$$\xi(t) = \mathrm{col}\{x(t), x(t-h(t)), x(t-h_M), \rho_1(t), \rho_2(t) \\ \rho_3(t), \rho_4(t), \dot{x}(t), \dot{x}(t-h(t)), \dot{x}(t-h_M)\},$$

with

$$\rho_1(t) = \int_{t-h_M}^{t-h(t)} \frac{x(s)}{h_M - h(t)} ds,$$
$$\rho_2(t) = \int_{t-h_M}^{t-h(t)} \frac{(t-h(t)-s)x(s)}{(h_M - h(t))^2} ds,$$
$$\rho_3(t) = \int_{t-h(t)}^{t} \frac{x(s)}{h(t)} ds,$$
$$\rho_4(t) = \int_{t-h(t)}^{t} \frac{(t-s)x(s)}{h^2(t)} ds.$$

Then the time derivative of (12) can be written as

$$\begin{aligned} \dot{V}(x_t, \dot{x}_t) =& \xi^T(t)\Upsilon_0(h(t), \dot{h}(t))\xi(t) \\ &- h_M \int_{t-h_M}^{t-h(t)} \dot{x}^T(s) R_1 \dot{x}(s) ds \\ &- h_M \int_{t-h(t)}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds, \end{aligned} \quad (13)$$

where $\Upsilon_0(h(t), \dot{h}(t))$ is given in (7).

Assuming $R_1 \succ 0$ and $R_2 \succ 0$, an upper bound for the integral terms in (13) can be obtained by applying Lemma 1, which leads to

$$h_M \left(\int_{t-h_M}^{t-h(t)} \dot{x}^T(s) R_1 \dot{x}(s) ds + \int_{t-h(t)}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds \right) \\ \succeq \frac{1}{\alpha} \xi^T(t) \Gamma_1^T \tilde{R}_1 \Gamma_1 \xi(t) + \frac{1}{1-\alpha} \xi^T(t) \Gamma_2^T \tilde{R}_2 \Gamma_2 \xi(t),$$

where $\tilde{R}_i = \mathrm{diag}\{R_i, 3R_i, 5R_i\}$ for $i = 1, 2$, $\alpha = (h_M - h(t))/h_M$ and Γ_1, Γ_2 are given in (10) and (11), respectively. Hence, we can apply Lemma 2 to obtain

$$\begin{aligned} &\frac{1}{\alpha} \xi^T(t) \Gamma_1^T \tilde{R}_1 \Gamma_1 \xi(t) + \frac{1}{1-\alpha} \xi^T(t) \Gamma_2^T \tilde{R}_2 \Gamma_2 \xi(t) \\ &\succeq \xi^T(t)[\Upsilon_1(h(t)) - (1-\alpha)\Gamma_1^T Y_1 \tilde{R}_2^{-1} Y_1^T \Gamma_1 \\ &\quad - \alpha \Gamma_2^T Y_2^T \tilde{R}_1^{-1} Y_2 \Gamma_2]\xi(t), \end{aligned} \quad (14)$$

where

$$\Upsilon_1(h(t)) = (2-\alpha)\Gamma_1^T \tilde{R}_1 \Gamma_1 + (1+\alpha)\Gamma_2^T \tilde{R}_2 \Gamma_2 \\ + \mathrm{sym}\{\Gamma_1^T [\alpha Y_1 + (1-\alpha) Y_2] \Gamma_2\}.$$

By using (14), $\dot{V}(x_t, \dot{x}_t)$ can be bounded by

$$\dot{V}(x_t, \dot{x}_t) \leq \xi^T(t)[\Upsilon(h(t), \dot{h}(t)) + \Upsilon_2(h(t))]\xi(t),$$

with $\Upsilon(h(t), \dot{h}(t))$ given in (6) and

$$\Upsilon_2(h(t)) = (1-\alpha)\Gamma_1^T Y_1 \tilde{R}_2^{-1} Y_1^T \Gamma_1 \\ + \alpha \Gamma_2^T Y_2^T \tilde{R}_1^{-1} Y_2 \Gamma_2.$$

Therefore, the derivative of the functional candidate is negative if

$$\xi^T(t)[\Upsilon(h(t), \dot{h}(t)) + \Upsilon_2(h(t))]\xi(t) \prec 0.$$

By applying Finsler's Lemma, with $\zeta = \xi(t)$ and $\mathcal{B}^T = \mathrm{col}\{\tilde{A}^T, \tilde{A}_d^T, 0_{5n\times n}, -I_n, 0_{2n\times n}\}$, where $\tilde{A} = A + \Delta A$ and $\tilde{A}_d = A_d + \Delta A_d$, the previous condition is equivalent to

$$\Upsilon(h(t), \dot{h}(t)) + \Upsilon_2(h(t)) + \mathcal{X}\mathcal{B} + \mathcal{B}^T \mathcal{X}^T \prec 0. \quad (15)$$

Let $\mathcal{X} = \mathrm{col}\{X_1, X_2, 0_{5n\times n}, X_3, 0_{2n\times n}\}$. Thus, by condition in (2) and Lemma 3

$$\begin{aligned} \mathrm{sym}\{\mathcal{X}\mathcal{B}\} &= \mathrm{sym}\{M_1\} + 2\bar{e}^T \bar{E} \Delta(t) \bar{D}^T \bar{e} \\ &\preceq \mathrm{sym}\{M_1\} + \bar{e}^T \bar{E}^T \Theta \bar{E} \bar{e} \\ &\quad + \bar{e}^T \bar{D} \Theta^{-1} \bar{D}^T \bar{e}, \end{aligned}$$

with M_1, $\bar{e}$, $\bar{D}$, and $\bar{E}$ defined in Theorem 6. If

$$\Upsilon(h(t), \dot{h}(t)) + \Upsilon_2(h(t)) + \mathrm{sym}\{M_1\} \\ + \bar{e}^T \bar{E}^T \Theta \bar{E} \bar{e} + \bar{e}^T \bar{D} \Theta^{-1} \bar{D}^T \bar{e} \prec 0, \quad (16)$$

is verified, the inequality in (15) is also satisfied.

On the other hand, note that the left side of (16) can be rewritten as

$$h^2(t)\mathcal{G}_2(\dot{h}(t)) + h(t)\mathcal{G}_1(\dot{h}(t)) + \mathcal{G}(\dot{h}(t)),$$

with $\mathcal{G}_2(\dot{h}(t))$ given in (9) and $\mathcal{G}_1(\dot{h}(t))$ and $\mathcal{G}(\dot{h}(t))$ being proper real symmetric matrices. Therefore, by Lemma 4, with $a_2 = \xi^T(t)\mathcal{G}_2(\dot{h}(t))\xi(t)$, $a_1 = \xi^T(t)\mathcal{G}_1(\dot{h}(t))\xi(t)$, and $a_0 = \xi^T(t)\mathcal{G}(\dot{h}(t))\xi(t)$, and Schur complement, equation (16) is satisfied if LMI conditions (3), (4) and (5) hold, which concludes the proof. □

Remark 1 *Theorem 6 can be used for stability analysis of nominal systems by eliminating the third line and the third column of the LMIs in* (3), (4), *and* (5), *and setting* $\bar{e}^T \bar{E}^T \Theta \bar{E} \bar{e} = 0$. *If, additionally, we consider* $R_1 = R_2$ *the results of Zhang et al. (2017) are recovered. It is important to point out that, different from Zhang et al. (2017), in the proposed conditions there is no products between the system matrices, which makes the controller synthesis simpler.*

4 Numerical Examples

The purpose of this section is to illustrate the improvements of our result. First, we present examples to illustrate the application of Theorem 6 for stability analysis of nominal systems and then, the stability analysis of systems with uncertain parameters is treated.

Example 1 *Consider system* (1) *with*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

It is well known that this system is stable for the maximum constant delay $h^* = 6.1725$, *i.e.* $h_M = 6.1725$ *and* $\dot{h}(t) = 0$ *(Gu et al., 2003). Applying the proposed method with* $\dot{h}(t) = 0$ *(*$d_m = d_M = d = 0$*) it is verified that the system is stable for the maximum delay* $h_M = 6.168$*. Thus in the case of constant delay one can certify that the proposed approach provides the same result of the recent methods from the literature listed in Table 1. On the other hand, Table 1 shows that the proposed method provides less conservative results in the case of time-varying delay, i.e.* $\dot{h}(t) \neq 0$.

Table 1: Maximum admissible upper bound h_M of $h(t)$ for $-d \leq \dot{h}(t) \leq d$ (Example 1).

	d		
Method	0.1	0.5	0.8
Kim (2016)	4.753	2.429	2.183
Kwon and Park (2017)	4.757	2.483	2.239
Lee and Park (2017)	4.829	3.155	2.730
Zhang et al. (2017)	4.910	3.233	2.789
Theorem 6	**4.915**	**3.241**	**2.798**

Example 2 *Consider system* (1) *with* $D = I_2$ *and*

- *Case 1:*

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$
$$E_s = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.05 \end{bmatrix}, E_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

- *Case 2:*

$$A = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, A_d = \begin{bmatrix} -0.5 & -1 \\ 0 & 0.6 \end{bmatrix},$$
$$E_s = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, E_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Here we repeat the procedure in the previous example, for given bounds of $\dot{h}(t)$*, we seek for the largest value of* h_M *that the proposed method asserts the system stability. Tables 2 and 3 give the comparative results obtained by applying Theorem 6 and some recent methods from the literature. It is worth to note that for the uncertain system case, the proposed method improved significantly the existing results.*

Table 2: Maximum admissible upper bound h_M of $h(t)$ for $-d \leq \dot{h}(t) \leq d$ (Example 2, Case 1).

	d	
Method	0.5	0.9
Wang and Shen (2011) Theorem 1	0.9561	0.8919
Kwon and Park (2017) Theorem 3.1	1.2288	1.2050
Theorem 6	**1.3263**	**1.2649**

Table 3: Maximum admissible upper bound h_M of $h(t)$ for $-d \leq \dot{h}(t) \leq d$ (Example 2, Case 2).

	d	
Method	0.5	0.9
Ko and Park (2011) Theorem 1, $N = 3$	0.737	0.534
Kwon and Park (2017) Theorem 3.1	0.7768	0.6351
Theorem 6	**0.9105**	**0.7814**

5 Conclusions

In this paper we have addressed the problem of stability analysis of uncertain linear systems subject to time-varying delays. The proposed approach was based on the construction of a new Lyapunov-Krasovskii functional by adequately modifying one of such functional that was recently presented in the literature. As result, the proposed functional is more suitable to be combined with the Reciprocally Convex Lemma allowing we obtain a less conservative criterion of delay-dependent stability analysis. Numerical examples have shown that the proposed method can improve existing results, mainly in the case of systems subject to time-varying delay and/or uncertain parameters. Possible future work includes extensions for systems subject to multiple time-delays and conditions to the synthesis of stabilizing controllers.

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