POSITIVE INVARIANCE OF POLYHEDRAL SETS AND LINEAR CONSTRAINED REGULATION PROBLEM IN THE CONTEXT OF THE δ-OPERATOR

Matheus V. A. Pedrosa*, Eugênio B. Castelan*, Carlos E. T. Dórea†, Stephanie L. Brião*

*Department of Automation and Systems Engineering
Federal University of Santa Catarina
Florianópolis, SC, Brazil

†Department of Computing Engineering and Automation
Federal University of Rio Grande do Norte
Natal, RN, Brazil

Emails: `matheus.pedrosa@posgrad.ufsc.br`, `eugenio.castelan@ufsc.br`, `cetdorea@dca.ufrn.br`, `stephanie.loi@posgrad.ufsc.br`

Abstract— In the literature, methods are available to obtain positively invariant sets for continuous-time and discrete-time systems using the shift operator. However, there are not references showing how to develop methods using the delta operator. In this work, positive invariance relations of polyhedral sets are proposed in the context of the delta operator model for linear discrete-time systems. The delta operator approach is known to be of interest when using high sample rates and it also allows to unify discrete-time and continuous-time concepts and results. In this context, the proposed delta operator positive invariance relations which are obtained from the classical shift operator results, are also shown to recover the continuous-time invariance relations when the sample period tends to zero. Due to the interest of using the positive invariance property and polyhedral sets in constrained control, a linear programming optimization approach is also proposed in the context of the delta operator to solve a discrete-time linear constrained regulation problem. A numerical example is exploited to show that the proposed delta operator solution closely follows the continuous-time one.

Keywords— Delta operator, polyhedral sets, positive invariance, Linear Constrained Regulation Problem.

1 Introduction

Positively invariant sets are a well used object in the stability, control and preservation of dynamical system's constraints, playing a key role in the theory and applications of dynamical control systems (Horvath et al., 2017). A set of the space state is positively invariant if the state trajectories remain inside this set when the initial state belongs to it. Methods to characterize algebraically the positive invariance of non-symmetrical and symmetrical sets, named invariance relations, were developed for linear continuous-time systems (Castelan and Hennet, 1993; Kiendl et al., 1992) and linear discrete-time systems using the shift operator (Hennet, 1995; Dorea and Hennet, 1999).

If a system is subject to constraints on the state vector and/or on the control inputs, a state feedback regulation law can stabilize it by maintaining the state vector inside a positive invariant set included in an admissible domain of initial states. In this context, the Linear Constrained Regulation Problem (LCRP) consists in finding a stabilizing state feedback control law for linear systems under linear state and/or control constraints. This problem has been intensively studied, with classical results in e.g. (Vassilaki et al., 1988; Castelan and Hennet, 1993; Hennet, 1995). Nevertheless, recent results have been reported on, for example, convergence to an equilibrium situated on the boundary of the feasible region (Bitsoris and Olaru, 2013; Bitsoris et al., 2014), extensions to systems subject to information and physical constraints (Wang et al., 2016) and synthesis of state-feedback controllers based on delay-dependent positively invariant sets (Bensalah, 2015).

While the continuous-time systems approach is useful for a theoretical analysis, discrete-time systems allow for easier computational implementation (Yuz and Goodwin, 2014). However, discrete-time systems are usually obtained by discretization of a continuous-time system by the shift operator, which shows some disadvantages such as poles attracted to the border of the unit radio circle in high sampling frequencies, and truncation and rounding errors (Yang et al., 2012).

First proposed by (Middleton and Goodwin, 1986), the delta operator, also named δ-operator, tries to unify both approaches, continuous and discrete-time, presenting a discrete-time systems behavior that tends to the continuous-time systems one as the sampling time decreases. Also, in high sampling frequencies, the systems poles and zeros get close to the continuous-time systems poles and zeros, while the zeros introduced by the sampler tends to minus infinity (Yang et al., 2012). This makes the representation in this domain better approximate the physical model.

Algorithms are available to construct controllers in a positively invariant domain based on Linear Programming (LP), which allows the inclusion of constrains, or based on eigenstructure assignment (Hennet, 1995). These algorithms can be extended to discrete-time systems in the perspective of the δ-operator.

In the literature, studies about relations between the δ-operator and the shift operator (Neuman, 1993b; Kalman and Bertram, 1993), the formulation, properties and applications of the δ-operator in physical systems (Neuman, 1993a), and many other applications including systems subject to actuator saturation and model uncertainty (Soh, 1991; Yang et al., 2015) can be found. However, concise studies investigating the properties and the relations of the positive invariance concept in linear dynamic systems under the perspective of the δ-operator can not be found. Such a study would be useful for implementation in physical devices that need faster sampling rates.

Thus, the objective of this paper is to formulate the positive invariance properties of polyhedral sets for a linear system in the δ-operator representation and to show the relations with the associated continuous-time system and the shift operator discrete-time system model. In particular, the known LCRP will be formulated and analysed in the δ-operator context, showing how control and state constraints can also be treated in the δ-operator setting.

The paper is organized as follows: next section presents the description of a discrete-time system using δ-operator and introduces the positive invariance of polyhedral sets. The third section presents a theoretical review about the invariance relations for discrete-time systems with shift operator which is the basis to obtain the positive invariance relations for discrete-time δ-operator systems. Thus, in section 4, the proposed invariance relations for δ-operator discrete-time systems are formulated. Section 5 describes a LCRP in the δ-operator context with the associated Linear Programming problem. To show the effectiveness of the proposal when the sampling time is small, Section 6 shows a numerical example.

Notations: Vectors are represented by lowercase letters and matrices by capital ones. The elements of a vector $z \in \mathbb{R}^n$ are denoted by z_i, $\forall i \in \{1, \ldots, n\}$, and the elements of a matrix $Z \in \mathbb{R}^{n\times m}$ by Z_{ij}, $\forall i \in \{1, \ldots, n\}$ and $\forall j \in \{1, \ldots, m\}$. Equalities and inequalities between vectors and matrices are considered to be elementwise. The elements of matrix $|Z|$ are the absolute values of the elements of Z. A matrix Z is non-negative if $Z_{ij} \geq 0$ $\forall i \in \{1, \ldots, n\}$, $\forall j \in \{1, \ldots, m\}$, and it is essentially non-negative if $Z_{ij} \geq 0$ $\forall i \neq j$. The identity matrix with dimension $n \in \mathbb{N}$ is denoted by I_n. A dynamical variable $z \in \mathbb{R}$ is represented $z_k = z(kT)$, with $k \in \mathbb{N}$ and $T \in \mathbb{R}_+$ being the sampling time. Also, a 0 in a vector or matrix inequality means the null vector or matrix with appropriate dimension.

2 Problem Formulation

In this work, we consider linear discrete-time invariant systems represented by:

$$\begin{cases} \delta x_k = A_\delta x_k + B_\delta u_k \\ \ \ y_k = C_\delta x_k \end{cases} \tag{1}$$

where $k \in \mathbb{N}$ is the discrete-time instant, $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control vector and $y_k \in \mathbb{R}^p$ is the output vector, with associated matrices $A_\delta \in \mathbb{R}^{n\times n}, B_\delta \in \mathbb{R}^{n\times m}$ and $C_\delta \in \mathbb{R}^{p\times n}$, $m, n, p \in \mathbb{N}^*$ and δx_k is the delta operator, defined as follows (Middleton and Goodwin, 1986; Yuz and Goodwin, 2014):

$$\delta x_k = \frac{x_{k+1} - x_k}{T} \tag{2}$$

with $T \in \mathbb{R}_+$ being the sampling period (in seconds) used to discretize, using Zero-order Holder (ZoH), continuous-time systems represented by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3}$$

By construction, the system's matrices in (1) are related to the ones in (3) as follows:

$$A_\delta = \frac{e^{AT} - I_n}{T}, \quad B_\delta = \frac{1}{T}\int_0^T e^{A(T-s)} B ds, \quad \text{and } C_\delta = C. \tag{4}$$

Thus, when the sampling period tends to zero, the following holds true:

(i) $\lim_{T\to 0} \delta x_k = \dot{x}(t)$ since, by definition, $x_k = x(kT)$; and

(ii) $\lim_{T\to 0} A_\delta = A$ and $\lim_{T\to 0} B_\delta = B$.

Furthermore, if the spectrum of matrices A and A_δ are, respectively, $\sigma(A) = \{\lambda_i, i = 1, ..., n\}$ and $\sigma(A_\delta) = \{\lambda_i^\delta, i = 1, ..., n\}$, then:

$$\lambda_i^\delta = \frac{e^{\lambda_i T} - 1}{T}, \ \forall i = 1, ..., n \tag{5}$$

and it also implies that $lim_{T\to 0}\lambda_i^\delta = \lambda_i$.

Equation (5) implies that the eigenvalues of A_δ belong to a circle with radius $\frac{1}{T}$ and centered (in every case) in $-\frac{1}{T}$ (see (Yuz and Goodwin, 2014; Neuman, 1993a; Middleton and Goodwin, 1986)). Figure 1 depicts the location of the eigenvalues of A_δ obtained from $A = \begin{bmatrix} -0.3 & 2 \\ -2 & -0.3 \end{bmatrix}$ for different sampling periods ($T = 1$, $T = 0.3$ and $T = 0.1$), where the eigenvalues of A are depicted by dots.

In the present work we are primarily interested in deriving algebraic conditions that guarantee the positive invariance property of given polyhedral sets with respect to δ-operator systems and

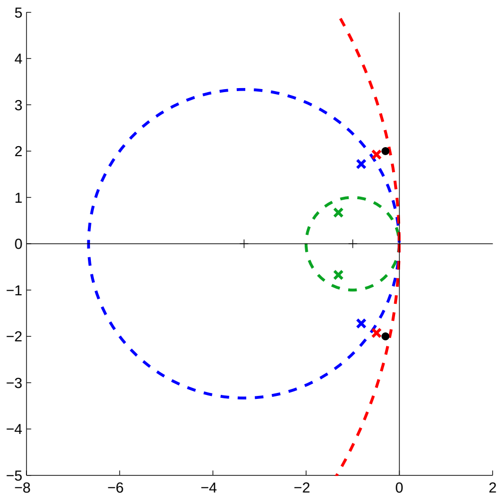

Figure 1: Spacial distribution on complex plane of the spectrum of A_δ for sampling times $T = 1$, $T = 0.3$ and $T = 0.1$. The dots represents the continuous-time eigenvalues of matrix A.

to show a way of using the proposed invariance relations to deal with a linear constrained regulation problem. For this purpose, consider a convex polyhedron in the state space, defined as:

Definition 1 *A nonempty convex polyhedron of $\mathbb{R}^n$ is defined by:*

$$R[Q, \rho] = \{x \in \mathbb{R}^n; Qx \leq \rho\} \tag{6}$$

with $Q \in \mathbb{R}^{r\times n}$, $\rho \in \mathbb{R}^r$, and $r, n \in \mathbb{N}^*$.

In particular, we can consider the case when a convex polyhedral domain is symmetrical around the origin point and defined by:

Definition 2 *A convex polyhedron symmetrical around the origin $S(G, \omega)$ is defined by:*

$$S(G, \omega) = \{x \in \mathbb{R}^n; -\omega \leq Gx \leq \omega\} \tag{7}$$

with $G \in \mathbb{R}^{s\times n}$, $s \leq n$, $\omega \in \mathbb{R}^s$, $\omega_i > 0$ for $i = 1, \ldots, s$.

By definition (Hennet, 1995), positive invariance is a property characterizing some function of time generated by a dynamical system such that any trajectory of this function starting in a region of space always remains in that region along the system evolution. In the case of discrete-time systems, the positive invariance of any set $\Omega \subset \mathbb{R}^n$ is obtained if:

$$x_k \in \Omega \implies x_{k+1} \in \Omega \tag{8}$$

The algebraic relations that characterize the positive invariance of the polyhedral sets (6) and (7) for classical linear discrete-time systems can be obtained by applying, for instance, Farka's Lemma to relation (8) and are firstly recalled in the next section. Thus, the algebraic relations for δ-operator discrete-time systems are proposed in section 4.

3 Preliminary Results

From the previous presentation, we can verify that, when using high sampling rates, the δ-operator representation (2) is more appropriate to represent discretized linear systems than the classical shift operator representation given by:

$$\begin{cases} x_{k+1} = A_d x_k + B_d u_k \\ \quad y_k = C_d x_k \end{cases} \tag{9}$$

where $A_d = e^{AT}$, $B_d = \int_0^T e^{A(T-s)} B ds$ and $C_d = C$. In this case, one has: $\lim\limits_{T\to 0} A_d = I$, $\lim\limits_{T\to 0} B_d = 0$ and, by considering $\sigma(A_d) = \{\lambda_i^d, i = 1, ..., n\}$:

$$\lambda_i^d = e^{\lambda_i T}, \ \forall i = 1, ..., n$$

and, in consequence, $\lim\limits_{T\to 0} \lambda_i^d = 1$.

It is important to recall that (1) and (9) represent the same discretized system obtained from continuous-time system (3). These two discrete-time models are related by the following relations:

$$\begin{aligned} &A_d = I_n + TA_\delta, \quad B_d = TB_\delta, \\ &\text{and } \lambda_i^d = 1 + T\lambda_i^\delta, \ \forall i = 1, ..., n. \end{aligned} \tag{10}$$

Besides the interest of using the δ-operator model to deal with small sampling time periods, it also allows to present continuous and discrete-time concepts and results in a unified way, since:

$$(A_\delta, B_\delta, \lambda_i^\delta) = \begin{cases} (A, B, \lambda_i), & \text{when } T \to 0 \\ (A_d - I_n, B_d, \lambda_i^d - 1), & \text{when } T \to 1 \end{cases}$$

The following two results related to an autonomous shift operator system (see (Hennet, 1995)):

$$x_{k+1} = A_d x_k \tag{11}$$

will be used to obtain the positive invariance relations for δ-operator systems.

Proposition 1 *The polyhedral set $R[Q, \rho]$ is a positively invariant set of (11) if and only if there exists a non-negative matrix $\mathscr{H}_d \in \mathbb{R}^{r\times r}$ such that:*

$$\mathscr{H}_d Q = QA_d \tag{12}$$

$$\mathscr{H}_d \rho \leq \rho \tag{13}$$

Also, for the symmetrical polyhedral set $S(G, \omega)$ to be invariant for (11), there is the following proposition.

Proposition 2 *A necessary and sufficient condition for the symmetrical polyhedral set $S(G, \omega)$ to be invariant for (11) is the existence of a matrix $H_d \in \mathbb{R}^{s \times s}$ such that:*

$$H_d G = G A_d \tag{14}$$
$$|H_d| \omega \leq \omega \tag{15}$$

4 Invariance Relations with δ-Operator

The theorems formulated in this section present the proposed invariance relations for an autonomous δ-operator system:

$$\delta x_k = A_\delta x_k \tag{16}$$

Theorem 1 *The polyhedral set $R[Q, \rho]$ is a positively invariant set of (16) if and only if there exists an essentially non-negative matrix $\mathscr{H}_\delta \in \mathbb{R}^{r \times r}$, with $-\frac{1}{T} \leq \mathscr{H}_{\delta ii} \leq 0 \ \forall i = 1, ..., r$, such that:*

$$\mathscr{H}_\delta Q = Q A_\delta \tag{17}$$
$$\mathscr{H}_\delta \rho \leq 0 \tag{18}$$

Proof: From (10), we can rewrite (12) as $\mathscr{H}_d Q = Q(TA_\delta + I_r)$. Then, $\left(\frac{\mathscr{H}_d - I_r}{T}\right) Q = Q A_\delta$ and we obtain (17) by defining:

$$\mathscr{H}_\delta = \frac{\mathscr{H}_d - I_r}{T} \tag{19}$$

From this definition, we can also rewrite (13) as $(T\mathscr{H}_\delta + I_r)\rho \leq \rho \Leftrightarrow \mathscr{H}_\delta \rho \leq 0$.

From Proposition 1 and (19), we must also have:

$$\mathscr{H}_d = I_r + T\mathscr{H}_\delta \geq 0 \Leftrightarrow \mathscr{H}_\delta \geq -\frac{I_r}{T}$$

Since, by definition, $\mathscr{H}_{\delta ij} = \frac{\mathscr{H}_{dij}}{T}$, $\forall i \neq j$, it is then required from (18) that:

$$-\frac{1}{T} \leq \mathscr{H}_{\delta ii} \leq 0$$

□

From the above Theorem and its proof, when $T \to 0$ we get the positive invariance relations for continuous-time system, since $A_\delta \to A$ and $\mathscr{H}_\delta$ tends to an essentially non-negative matrix without any restriction on its diagonal elements: $\mathscr{H}_\delta \to \mathscr{H}$. Then, we have (see (Castelan and Hennet, 1993)): $R[Q, \rho]$ is positively invariant for $\dot{x}(t) = Ax(t) \Leftrightarrow \exists$ an essentially non-negative $\mathscr{H} \in \mathbb{R}^{r \times r}$ such that:

$$\begin{cases} \mathscr{H} G = GA \\ \mathscr{H} \rho \leq 0 \end{cases}.$$

The positive invariance relations for the symmetrical polyhedral set $S(G, \omega)$ is given by the following Theorem:

Theorem 2 *A necessary and sufficient condition for the symmetrical polyhedral set $S(G, \omega)$ to be positively invariant for (16) is the existence of matrix $H_\delta \in \mathbb{R}^{s \times s}$ such that:*

$$H_\delta G = G A_\delta \tag{20}$$
$$\hat{H}_\delta \omega \leq 0 \tag{21}$$
$$\bar{H}_\delta \omega \geq -\frac{2}{T}\omega \tag{22}$$

with $\hat{H}_\delta = \begin{cases} H_{\delta ij}, & \text{if } i = j \\ |H_{ij}|, & \text{if } i \neq j \end{cases}$, $i, j = 1, ..., s$

and $\bar{H}_\delta = \begin{cases} H_{\delta ij}, & \text{if } i = j \\ -|H_{ij}|, & \text{if } i \neq j \end{cases}$, $i, j = 1, ..., s$

Proof: From (10) and (14), we have $H_d G = G(TA_\delta + I_s)$, which is equivalent to $\left(\frac{H_d + I_s}{T}\right) G = G A_\delta$. Thus, by defining

$$H_\delta = \frac{H_d - I_s}{T} \tag{23}$$

we get (20).

Also, from (23), inequality (15) can be written as $|TH_\delta + I_s|\omega \leq \omega \Leftrightarrow |H_\delta + \frac{I_s}{T}|\omega \leq \frac{1}{T}\omega$. Writing this inequality for each row, we have:

$$|H_{\delta ii} + \frac{1}{T}|\omega_i \leq \frac{1}{T}\omega_i - \sum_{j \neq i} |H_{\delta ij}|\omega_j$$

which is equivalent to:

$$-\frac{1}{T}\omega_i + \sum_{j \neq i} |H_{\delta ij}|\omega_j \leq H_{\delta ii}\omega_i + \frac{1}{T}\omega_i \leq \frac{1}{T}\omega_i - \sum_{j \neq i} |H_{\delta ij}|\omega_j$$

For the right inequality, we can see that $H_{\delta ii}\omega_i + \sum_{j \neq i} |H_{\delta ij}|\omega_j \leq 0$, thus obtaining (21); and for the left one, $-\frac{2}{T}\omega_i \leq H_{\delta ii}\omega_i - \sum_{j \neq i} |H_{\delta ij}|\omega_j$, thus obtaining (22).

□

The positive invariance relations for continuous-time system are also verified from the Theorem 2 when $T \to 0$, since $A_\delta \to A$ and the constraint (22) on H_δ disappears and only (21) remains active. Thus, $H_\delta \to H$ and the following known result for continuous-time systems is encountered (see (Castelan and Hennet, 1993))(see (Castelan and Hennet, 1993)): $S(G, \omega)$ is positively invariant for $\dot{x}(t) = Ax(t) \Leftrightarrow \exists$ a matrix $H \in \mathbb{R}^{s \times s}$ such that:

$$\begin{cases} HG = GA \\ \hat{H}\omega \leq 0 \end{cases}$$

where $\hat{H} = \begin{cases} H_{ij}, & \text{if } i = j \\ |H_{ij}|, & \text{if } i \neq j \end{cases}$, $i, j = 1, ..., s$.

Some remarks about Theorem 2 are of interest at this point:

(i) From the above proof, we have that (21) and (22) are equivalent to $|H_\delta + \frac{I_s}{T}|\omega \leq \frac{1}{T}\omega$, which, by defining $W = \text{diag}(w_i)$ (a positive diagonal matrix which diagonal elements are the elements of vector $\omega > 0$), can be rewritten as follows, for some positive scalar $\epsilon_\delta \leq 1$:

$$\|W^{-1}H_\delta W + \frac{I_s}{T}\|_\infty \leq \frac{\epsilon_\delta}{T} \tag{24}$$

Thus, from (24) we have that the eigenvalues of matrix H_δ belong to a circle centered in $-\frac{1}{T}$ with radius equals to $\frac{\epsilon_\delta}{T}$.

(ii) When $\epsilon_\delta < 1$, the following Lyapunov function associated to the positively invariant set $S(G, \omega)$ is decreasing along the trajectories of the discrete-time system (11) or (16):

$$V(x_k) = ||Gx_k||_\infty = \max_{i=1,\ldots,s} |G_{(i)}x_k| \tag{25}$$

where $G_{(i)}$ stands for the ith row of matrix G (Hennet, 1995).

(iii) When $T \to 0$, (20) and (22) or, equivalently, (24) reduce to $\hat{H}\omega \leq 0$, which can be rewritten as follows, for some non-negative scalar ϵ:

$$\hat{H}\omega \leq -\epsilon\omega \Leftrightarrow \mu_\infty(W^{-1}HW) \leq -\epsilon$$

where $\mu(.)$ stands for the infinity matrix measure (Kiendl et al., 1992). Thus, for $\epsilon > 0$, it can be deduced that eigenvalues of H will have real parts less than or equal to ϵ.

(iv) For $0 \leq \epsilon_d \leq 1$, in the shift operator case we have:

$$|H|\omega \leq \epsilon_d\omega \Leftrightarrow ||W^{-1}H_dW||_\infty \leq \epsilon_d$$

and, hence, the spectrum of H_d belongs to the classical unit circle centered in the origin.

5 LCRP Using δ-Operator

Aiming at illustrating the use of proposed results for discrete-time control synthesis, let us consider the classical linear constrained regulation problem, here formulated to be applied to the δ-operator discrete-time model.

For system (1), let us consider that the states are all measurable ($C_\delta = I_n$), the states are constrained to belong to a given symmetrical polyhedron $S(G, \omega)$ and that the control inputs $u_k \in \mathcal{U} \subset \mathbb{R}^m$ are symmetrically bounded, as follows:

$$\mathcal{U} = \{u_k \in \mathbb{R}^m; -\gamma \leq u_k \leq \gamma\} \tag{26}$$

with $\gamma \in \mathbb{R}^m$ and $\gamma > 0$.

Thus, the following Linear Constrained Regulation Problem can be stated: *Find a state feedback control law*

$$u_k = F_\delta x_k, \; F_\delta \in \mathbb{R}^{m\times n}$$

such that the trajectories of the closed-loop system

$$\delta x_k = (A_\delta + B_\delta F_\delta)x_k \tag{27}$$

starting from $S(G, \omega)$ converge to the origin while respecting control constraints (26).

Thus, as in (Hennet, 1995; Vassilaki et al., 1988), the following linear programming optimization problem can be set to find solutions to the stated LCRP by imposing the positive invariance of $S(G, \omega)$ with respect to closed-loop system (27) and the inclusion $S(G, \omega) \subseteq S(F_\delta, \gamma) = \{x_k \in \mathbb{R}^n; -\gamma \leq F_\delta x_k \leq \gamma\}$ to respect the control constraint:

$$\begin{aligned}
&\underset{H_\delta, F_\delta, M}{\text{minimize}} && \epsilon_\delta && \text{(28a)}\\
&\text{subject to} && H_\delta G = GA_\delta + GB_\delta F_\delta, && \text{(28b)}\\
& && |H_\delta + \frac{I_s}{T}|\omega \leq \frac{1}{T}\epsilon_\delta\omega, && \text{(28c)}\\
& && |M|\omega = \gamma, && \text{(28d)}\\
& && MG = F_\delta, && \text{(28e)}\\
& && 0 \leq \epsilon_\delta < 1 && \text{(28f)}
\end{aligned}$$

In the above LP optimization problem, constraints (28b)-(28c) guarantee the closed-loop positive invariance of $S(G, \omega)$ and (28d)-(28e) guarantee the required set inclusions.

The objective function (28a) is such that the eigenvalues of $(A_\delta + B_\delta F_\delta)$ are placed in the smallest circle centered at $-\frac{1}{T}$. From the remarks (i) and (iii) at the end of previous section, it is important to stress that: *i*) by setting $T = 1$, the LP above correspond to the shift operator solution proposed in (Vassilaki et al., 1988), but applied to a shifted system $x_{k+1} = (A_d - I_n)x_k + B_d u_k$; and *ii*) when $T \to 0$, we can retrieve a solution for the continuous-time LCRP (see, for instance, (Bitsoris et al., 2014)).

6 Numerical Example

Consider the following continuous-time linear system borrowed and adapted from (Castelan and Hennet, 1993):

$$A = \begin{bmatrix} 9.10 & 0.47 & -6.33 \\ 7.62 & 0.00 & 7.56 \\ 2.62 & -3.28 & 9.91 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.82 & 3.61 \\ 1.24 & -3.77 \\ -4.91 & 0.00 \end{bmatrix}.$$

This unstable open-loop system has eigenvalues:

$$\begin{cases} 9.5036 \\ 4.7532 \pm j3.8786 \end{cases}$$

The symmetrical constraints on the state vector and the vector of control bounds are defined by:

$$G = \begin{bmatrix} 5.69 & 1.97 & -1.68 \\ 2.24 & -1.68 & 5.59 \\ 2.00 & 0.00 & 0.00 \end{bmatrix},$$

$$\omega = \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} \text{ and } \quad \gamma = \begin{bmatrix} 1.50 \\ 5.00 \end{bmatrix}.$$

Solutions for the LCRP by using the LP optimization problem (28) were obtained for three different sampling periods: $T = 10^{-1}$, $T = 10^{-3}$ and $T = 10^{-5}$. In Table 1, we can compare the eigenvalues obtained with the δ-operator approach (28) to those obtained using the same optimization problem with equations (28b) and (28c) related to the invariance relations for shift operator discrete-time systems and for continuous-time systems commented before. It can be observed that the eigenvalues obtained with the δ-operator tend to the continuous-time ones while the shift operator ones tend to one.

T	$\sigma(A+BF)$	$\sigma(A_d+B_dF_d)$	$\sigma(A_\delta+B_\delta F_\delta)$
10^{-1}	-1.4405 -9.9005 -19.2494	0.8588 0.1974 0.1974	-1.4118 -7.8252 -8.5175
10^{-3}	-1.4405 -9.9005 -19.2494	0.9986 0.9920 0.9828	-1.4395 -8.2598 -15.7579
10^{-5}	-1.4405 -9.9005 -19.2494	1.0000 0.9999 0.9998	-1.4406 -8.4917 -18.7951

Table 1: Eigenvalues found solving LCRP for different sampling periods.

The result obtained for $T = 10^{-5}$ is compared to the continuous-time result through the following matrices:

- δ-operator:

$$F_\delta = \begin{bmatrix} 0.6400 & -1.5404 & 3.9148 \\ -6.4153 & 0.6047 & -1.5922 \end{bmatrix},$$

$$M = \begin{bmatrix} -0.2483 & 0.6257 & -0.3143 \\ 0.0862 & -0.2589 & -3.1628 \end{bmatrix},$$

$$H_\delta = \begin{bmatrix} -6.2591 & 2.3665 & -2.4520 \\ 1.6166 & -16.3461 & -13.2880 \\ -2.2373 & -2.4444 & -6.1220 \end{bmatrix}.$$

- continuous-time:

$$F = \begin{bmatrix} -0.6193 & -1.6387 & 4.1544 \\ -6.6861 & 0.5058 & -1.4632 \end{bmatrix},$$

$$M = \begin{bmatrix} -0.2663 & 0.6632 & -0.2948 \\ 0.0451 & -0.2482 & -3.1932 \end{bmatrix},$$

$$H = \begin{bmatrix} -7.1756 & 3.2956 & -2.4395 \\ 1.4810 & -17.1445 & -14.2229 \\ -2.5993 & -2.2306 & -6.2704 \end{bmatrix}.$$

Figures 2 and 3 show the comparative state trajectories and control actions trajectories, starting from $x_0 = \{0.1000, -0.0100, 0.0400\}$, for the three closed-loop systems when $T = 10^{-5}$. The δ-operator closed-loop system showed to be effectively closest to the continuous-time one when compared to the shift operator system's response. Also, the constraints were respected.

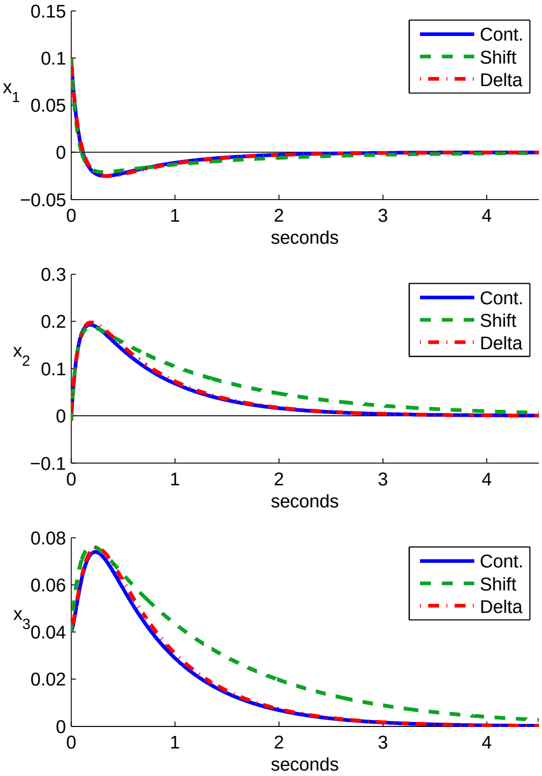

Figure 2: States trajectories for $T = 10^{-5}$.

7 Conclusions

In this work, the positive invariance of polyhedral sets was considered by using the δ-operator representation of discrete-time systems. The first interest in deducing the associated invariance relations is that the δ-operator approach allows to unify the discrete-time and the continuous-time ones when the sample period tends to zero.

Besides showing that the proposed δ-operator positive invariance relations tends to the continuous-time ones, they also have been applied to propose a LP optimization problem to solve a linear constrained regulation problem which considers state and control constraints. The numerical results showed that the δ-operator solution closer approaches the continuous-time one than the classical shift operator solution when the sampling rate is high.

Further studies are in development for a better understanding of the proposed δ-operator approach aiming at using it in a broader class of constrained control problems and in practical applications.

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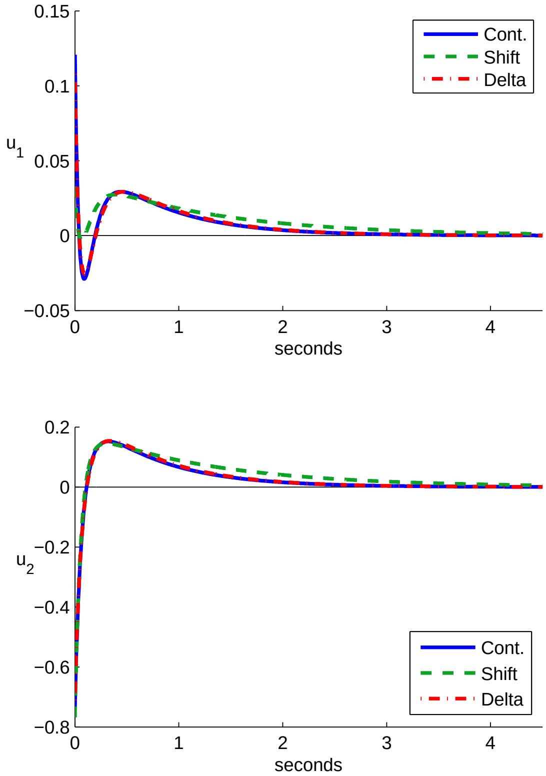

Figure 3: Control actions trajectories for $T = 10^{-5}$.

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