UKF ON LIE GROUPS FOR RADAR TRACKING USING POLAR AND DOPPLER MEASUREMENTS

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Abstract— This paper proposes a construction of the Unscented Kalman FIlter (UKF) in which the system state propagates on a Lie Group. It is presented a parametrization for 2D-radar targets on Lie group to tackle the filtering tracking problem. By assuming concentrated Gaussian distributions on Lie groups rather than the conventional Gaussian distributions in Euclidean space, it is shown that the presented parametrization is a better approximation to the curved-shape distribution of the position of moving targets with noise in the linear and angular speeds. The parametrization is applied to the proposed UKF on Lie groups and to an Extended Kalman Filter (EKF) on Lie groups found in the literature. The conventional UKF and EKF, in Euclidean space, were also implemented. The considered system dynamics is a constant linear speed and constant turn rate model adapted to a Lie group structure. The Discrete Lie Group UKF presented the best performance among the implemented filters.

1 Introduction

The two most basic functions of a radar are detection and ranging. Over time, the target azimuth has been included as a basic function, and very often the Doppler velocity is also included (Frencl et al., 2017; Nathanson et al., 1999). Additional functions such as mapping, missile guidance and tracking are also incorporated to some radar systems, most on military radars.

In the context of target tracking, there are many techniques to filter the measurements in order to determine the target's trajectory, such as the Extended Kalman Filter, the Unscented Kalman Filter, the Particle Filter, the Minimum Mean Square Error Filter, etc. Many of theses techniques assume that both the system noise and the measurement noise are white Gaussian noise in the Euclidean space. However, previous work (Long et al., 2013) has shown that the pose of a moving body might present a curved-shape probability distribution, so-called *banana-distribution*, when subject to white Gaussian noise in translational and rotational speed. The authors in (Long et al., 2013) also verified that expressing the system in a Lie group structure is a better way to handle such systems as the noise in the coordinates of the Lie algebra are Gaussian.

Recently, a generalization of the Discrete Extended Kalman Filter was proposed on Lie groups (D-LG-EKF) (Bourmaud et al., 2013), which describes both the system state and measurements as elements of Lie groups. The filter is developed directly on Lie groups, so that the noise is considered Gaussian in the Lie algebra space and banana-shaped on the Lie group. Also, adaptations of the Unscented Kalman Filter (D-LG-UKF) were also proposed to work on Lie groups (Loianno et al., 2016; Brossard et al., 2017; Brossard et al., 2018). This work contributes with a slightly new approach, in which the mean is calculated in a way that the final distribution constructively obeys a concentrated Gaussian distribution.

The D-LG-EKF and the proposed D-LG-UKF are both applied to the tracking of radar targets in the 2D plane, which will also take advantage of the Doppler measurements provided by the radar system. The considered target's dynamic model is the constant linear speed and constant turn rate model in the original coordinate space. The performance of the proposed D-LG-UKF is compared to the D-LG-EKF and to the conventional D-EKF and D-UKF, in Euclidean space. It is expected a performance improvement than the D-LG-EKF and the D-EKF since the D-LG-UKF can provide a better approximation of the non-linear system model; and also a better performance than the D-UKF, brought by the better parametrization of the Lie group.

The ensuing sections are organized as follows: section 2 presents the mathematical background, in which the Lie Group theory is introduced in

2.1, the conventional Unscented Transform (UT) and Discrete Unscented Kalman Filter (D-UKF) are presented in 2.2 and 2.3, respectively, and the proposed UT on Lie groups (LG-UT) and UKF on Lie groups (D-LG-UKF) are shown in 2.4 and 2.5, repectively; section 3 applies the D-LG-UKF to the constant linear velocity and constant turn rate model (3.1) and illustrates how the concentrated Gaussian distribution is a better approximation to the system state than the Gaussian distribution in the Euclidean space (3.2); section 4 presents the simulated results and a comparison with the D-EKF, D-UKF and D-LG-EKF; and section 5 brings the conclusions.

2 Mathematical Background

2.1 Lie Group Theory

A Lie group is a group whose set G also has a structure of a smooth manifold, so that the product map

$$p : (g, h) \in G \times G \mapsto p(g, h) = gh \in G \quad (1)$$

is smooth (San Martin, 2016), where $\circ$ is the group operation.

This paper is based on a particular type of Lie group, the real matrix Lie group $G \subset GL(n, \mathbb{R})$, in which each element of the group is a square $n \times n$ invertible matrix with real elements and the group operation is the matrix multiplication (Chirikjian, 2010). As consequence, the identity element of the matrix Lie group is the identity matrix $Id_{n\times n}$.

Because of the structure of a smooth manifold, a Lie group locally resembles the Euclidean space. In particular, the vector space in an open neighbourhood of the tangent space at the identity element is called the Lie algebra $\mathfrak{g}$. The exponential map $exp_G : \mathfrak{g} \to G$ establishes a local diffeomorphism between an open neighbourhood of $0_{n\times n}$ in the Lie algebra and an open neighbourhood of $Id_{n\times n}$ in the Lie group, so that $g \in G$ can be written as $g = \exp_G(X)$ for some $X \in \mathfrak{g}$ with $||X|| \ll 1$. The Lie algebra can also be seen as the set af all matrices $\{X\}$ such that the exponential map of each X results in an element of the Lie group (Chirikjian, 2010). In the context of matrix Lie groups, the exponential map is the matrix exponential

$$\exp_G(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \quad (2)$$

The inverse of the exponential map is the logarithm map $\log_G : G \to \mathfrak{g}$.

As the Lie algebra $\mathfrak{g}$ is a vector space, it's possible to write every element $X \in \mathfrak{g}$ as a linear combination of an orthonormal basis $\{E_i\}$.

$$X = \sum_{i=1}^{p} x_i E_i \quad (3)$$

for a p-dimensional Lie algebra (associated to a p-dimensional Lie group). For matrix Lie algebra, the elements E_i of the basis are $n\times n$ real matrices.

Hence, there is a isomorphism between $\mathfrak{g}$ and $\mathbb{R}^p$ denoted as $[\cdot]_G^\vee : \mathfrak{g} \to \mathbb{R}^p$ and defined as (Chirikjian, 2010)

$$[X]_G^\vee = \left[\sum_{i=1}^{p} x_i E_i\right]_G^\vee = [x_1\, x_2\, \cdots\, x_p]^T \quad (4)$$

The isomorphism $[\cdot]_G^\wedge : \mathbb{R}^p \to \mathfrak{g}$ is the inverse map. The economic notations $\exp_G^\wedge(\mathbf{x}) = \exp_G\left([\mathbf{x}]_G^\wedge\right)$ and $\log_G^\vee(g) = [\log_G(g)]_G^\vee$ are also used.

2.2 The Unscented Transform

Let $\mathbf{x}$ be a random vector of dimension L, mean $\bar{\mathbf{x}}$ and covariance matrix $P_{\mathbf{xx}}$. Lef $f : \mathbf{x} \mapsto \mathbf{y} = f(\mathbf{x})$ be a nonlinear function. The Unscented Transform is a method for calculating the statistics of $\mathbf{y}$ based on the know statistics of $\mathbf{x}$. In order to do this, the following sigma points (and the corresponding weights) are chosen as (Julier and Uhlmann, 1997; Julier, 2002; Wan and Van Der Merwe, 2000):

$$\begin{aligned}
\boldsymbol{\mathcal{X}}^{(0)} &= \bar{\mathbf{x}} && (5)\\
\boldsymbol{\mathcal{X}}^{(i)} &= \bar{\mathbf{x}} + \left(\sqrt{(L+\lambda)P_{\mathbf{xx}}}\right)_i,\ i = 1, \ldots, L\\
\boldsymbol{\mathcal{X}}^{(i+L)} &= \bar{\mathbf{x}} - \left(\sqrt{(L+\lambda)P_{\mathbf{xx}}}\right)_i,\ i = 1, \ldots, L\\
W_m^{(0)} &= \lambda/(L+\lambda) && (6)\\
W_c^{(0)} &= \lambda/(L+\lambda) + (1 - \alpha^2 + \beta)\\
W_m^{(i)} &= W_c^{(i)} = 1/[2(L+\lambda)],\ i = 1, \ldots, 2L
\end{aligned}$$

where $L = q + r$, $\lambda = \alpha^2(L+\kappa) - L$ is a scaling parameter, α determines the spread of the sigma points around the mean ($0 \le \alpha \le 1$), κ is a secondary scaling parameter usually set to 0, and β incorporate prior knowledge about the distribution of $\mathbf{x}$ (for Gaussian distribution, $\beta = 2$ is optimal). The operator $(\sqrt{A})_i$ is the i-th column of the matrix square root of A.

The sigma points are propagated through f as $\boldsymbol{\mathcal{Y}}^{(i)} = f(\boldsymbol{\mathcal{X}}^{(i)})$. The mean and covariance of $\mathbf{y}$ and cross-covariace of $\mathbf{x}$ abd $\mathbf{y}$ are approximated by a weighted sample mean as follows:

$$\bar{\mathbf{y}} = \sum_{i=0}^{2L+1} W_m^{(i)} \boldsymbol{\mathcal{Y}}^{(i)} \quad (7)$$

$$P_{\mathbf{yy}} = \sum_{i=0}^{2L+1} W_c^{(i)} \left[\boldsymbol{\mathcal{Y}}^{(i)} - \bar{\mathbf{y}}\right]\left[\boldsymbol{\mathcal{Y}}^{(i)} - \bar{\mathbf{y}}\right]^T \quad (8)$$

$$P_{\mathbf{xy}} = \sum_{i=0}^{2L+1} W_c^{(i)} \left[\boldsymbol{\mathcal{X}}^{(i)} - \bar{\mathbf{x}}\right]\left[\boldsymbol{\mathcal{Y}}^{(i)} - \bar{\mathbf{y}}\right]^T \quad (9)$$

2.3 Unscented Kalman Filter

The Unscented Kalman Filter performs the state filtering by appling the Unscented Transform in the prediction and update steps of the Kalman Filter.

The state at a timestep t is the p-dimensional vector $\mathbf{x}_t$, and the measurements are q-dimensional vectors $\mathbf{y}_t$. The system and measurement model are as follows:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{q}_t) \tag{10}$$

$$\mathbf{y}_t = h(\mathbf{x}_t, \mathbf{r}_t) \tag{11}$$

with $\mathbf{q}_t \sim \mathcal{N}_{\mathbb{R}^p}(\mathbf{0}, P_{\mathbf{qq}})$ and $\mathbf{r}_t \sim \mathcal{N}_{\mathbb{R}^q}(\mathbf{0}, P_{\mathbf{rr}})$ white Gaussian noises.

The filter is initiated with:

$$\mathbf{x}_{0|0} = E[\mathbf{x}_0] \tag{12}$$

$$P_{0|0} = E[(\mathbf{x}_0 - \mathbf{x}_{0|0})(\mathbf{x}_0 - \mathbf{x}_{0|0})^T] \tag{13}$$

$$\mathbf{x}^a_{0|0} = [\mathbf{x}^T_{0|0} \quad \mathbf{0}_{1\times p} \quad \mathbf{0}_{1\times q}]^T \tag{14}$$

$$P^a_{0|0} = \begin{bmatrix} P_{0|0} & & \\ & P_{\mathbf{qq}} & \\ & & P_{\mathbf{rr}} \end{bmatrix} \tag{15}$$

At each step, the sigma points $\boldsymbol{\mathcal{X}}^{a,(i)}_{t-1|t-1} = \left[\left(\boldsymbol{\mathcal{X}}^{\mathbf{x},(i)}_{t-1|t-1}\right)^T \left(\boldsymbol{\mathcal{X}}^{\mathbf{q},(i)}_{t-1|t-1}\right)^T \left(\boldsymbol{\mathcal{X}}^{\mathbf{r},(i)}_{t-1|t-1}\right)^T\right]^T$ for $\mathbf{x}^a$ and weights are created as in (5) and (6) and the prediction to the time step t is performed as:

$$\boldsymbol{\mathcal{X}}^{\mathbf{x},(i)}_{t|t-1} = f\left(\boldsymbol{\mathcal{X}}^{\mathbf{x},(i)}_{t-1|t-1}, \boldsymbol{\mathcal{X}}^{\mathbf{q},(i)}_{t-1|t-1}\right) \tag{16}$$

$$\mathbf{x}_{t|t-1} = \sum_{i=0}^{2L+1} W^{(i)}_m \boldsymbol{\mathcal{X}}^{\mathbf{x},(i)}_{t|t-1} \tag{17}$$

$$P_{t|t-1} = \sum_{i=0}^{2L+1} W^{(i)}_c \left[\boldsymbol{\mathcal{X}}^{\mathbf{x},(i)}_{t|t-1} - \mathbf{x}_{t|t-1}\right] \tag{18}$$

$$\cdot \left[\boldsymbol{\mathcal{X}}^{\mathbf{x},(i)}_{t|t-1} - \mathbf{x}_{t|t-1}\right]^T \tag{19}$$

And the update step at the arrival of the measurement $\mathbf{y}_t$:

$$\boldsymbol{\mathcal{Y}}^{(i)}_{t|t-1} = h\left(\boldsymbol{\mathcal{X}}^{\mathbf{x},(i)}_{t|t-1}, \boldsymbol{\mathcal{X}}^{\mathbf{r},(i)}_{t-1|t-1}\right) \tag{20}$$

$$\hat{\mathbf{y}}_{t|t-1} = \sum_{i=0}^{2L+1} W^{(i)}_m \boldsymbol{\mathcal{Y}}^{(i)}_{t-1|t} \tag{21}$$

$$P_{\mathbf{yy}} = \sum_{i=0}^{2L+1} W^{(i)}_c \left[\boldsymbol{\mathcal{Y}}^{(i)}_{t|t-1} - \hat{\mathbf{y}}_{t|t-1}\right] \tag{22}$$

$$\cdot \left[\boldsymbol{\mathcal{Y}}^{(i)}_{t|t-1} - \hat{\mathbf{y}}_{t|t-1}\right]^T \tag{23}$$

$$P_{\mathbf{xy}} = \sum_{i=0}^{2L+1} W^{(i)}_c \left[\boldsymbol{\mathcal{X}}^{\mathbf{x},(i)}_{t|t-1} - \mathbf{x}_{t|t-1}\right] \tag{24}$$

$$\cdot \left[\boldsymbol{\mathcal{Y}}^{(i)}_{t|t-1} - \hat{\mathbf{y}}_{t|t-1}\right]^T \tag{25}$$

$$\mathbf{x}_{t|t} = \mathbf{x}_{t|t-1} + P_{\mathbf{xy}} P^{-1}_{\mathbf{yy}} (\mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1}) \tag{26}$$

$$P_{t|t} = P_{t|t-1} - P_{\mathbf{xy}} P^{-1}_{\mathbf{yy}} P^T_{\mathbf{xy}} \tag{27}$$

2.4 The Unscented Transform on Lie Groups

This section shows how the Unscented Transform can be generalized to Lie groups.

Let G and H be matrix Lie groups with dimensions p and q, repectively, and let

$$\begin{aligned} f : H \times \mathbb{R}^r &\to G \\ (h, \mathbf{v}) &\mapsto g = f(h, \mathbf{v}) \end{aligned} \tag{28}$$

be a continuous map.

Suppose that $\mathbf{v} \sim \mathcal{N}_{\mathbb{R}^r}(\bar{\mathbf{v}}, P_{\mathbf{vv}})$ and that h is a random variable on H with a concentrated Gaussian distribution, i.e., $h \sim \mathcal{N}_H(\bar{h}, P_{hh})$ and, thus, h can be written as $h = \bar{h} \exp^\wedge_H(\boldsymbol{\delta})$ with $\boldsymbol{\delta} \sim \mathcal{N}_{\mathbb{R}^q}(\mathbf{0}_{q\times 1}, P_{hh})$. The goal of the proposed Unscented Transform is to provide means for estimation of the statistics of the resulting random variable $g = f(h, \mathbf{v})$ based on the statistics of h and $\mathbf{v}$.

The distribution of g is also assumed to be a concentrated Gaussian distribution on G: $g \sim \mathcal{N}_G(\bar{g}, P_{gg})$; and thus: $g = \bar{g} \exp^\wedge_G(\boldsymbol{\epsilon})$ with $\boldsymbol{\epsilon} \sim \mathcal{N}_{\mathbb{R}^p}(\mathbf{0}_{p\times 1}, P_{gg})$, with $\bar{g}$ and P_{gg} to be determined. We also want to determine the cross-covariances P_{hg} and $P_{\mathbf{v}g}$.

Note that it is not possible to apply the conventional unscented transform directly on f since g and h are not in an Euclidean space. However, based on the assumptions, and from $g = f(h, \mathbf{v})$, it is possible to write:

$$\boldsymbol{\epsilon} = \log^\vee_G\left(\bar{g}^{-1} f\left(\bar{h} \exp^\wedge_H(\boldsymbol{\delta}), \mathbf{v}\right)\right) = f^*(\boldsymbol{\xi}) \tag{29}$$

where $\boldsymbol{\xi} = \left[\boldsymbol{\delta}^T \, \mathbf{v}^T\right]^T \in \mathbb{R}^{q+r}$ is and augumented random vector with Gaussian distribution $\mathcal{N}_{\mathbb{R}^{q+r}}(\bar{\boldsymbol{\xi}}, P_{\boldsymbol{\xi}})$ and $f^* : \mathbb{R}^{q+r} \to \mathbb{R}^p$ is a continuous map (because it is a composition of continuous maps) in Euclidean vector spaces. This way, it is possible to apply the conventional unscented transform to estimate the desired statistics and then an Unscented Transform can be constucted for Lie groups.

The sigma points $\boldsymbol{\Xi}^{(i)} = \left[\left(\boldsymbol{\Delta}^{(i)}\right)^T \left(\boldsymbol{\mathcal{V}}^{(i)}\right)^T\right]^T$ for $\boldsymbol{\xi}$ (with the corresponding weights) are created as in (5) and (6) and propagated through f^* with (29), so that we obtain the following sigma points $\boldsymbol{\mathcal{E}}^{(i)}$ for $\boldsymbol{\epsilon}$:

$$\begin{aligned} \boldsymbol{\mathcal{E}}^{(i)} &= \log^\vee_G\left(\bar{g}^{-1} \mathcal{G}^{(i)}\right) \\ \mathcal{G}^{(i)} &= f\left(\mathcal{H}^{(i)}, \boldsymbol{\mathcal{V}}^{(i)}\right) \\ \mathcal{H}^{(i)} &= \bar{h} \exp^\wedge_H\left(\boldsymbol{\Delta}^{(i)}\right) \end{aligned} \tag{30}$$

The mean of $\boldsymbol{\epsilon}$ is approximated using the weighted sample mean:

$$\bar{\boldsymbol{\epsilon}} = \sum_{i=0}^{2L+1} W^{(i)}_m \boldsymbol{\mathcal{E}}^{(i)} = \mathbf{0} \tag{31}$$

Note that we want $\bar{\boldsymbol{\epsilon}}_g = 0$ for g obeys a concentrated Gaussian distribution. Thus we need a value of $\bar{g}$ that satisfies (31).

Let the sequence $\{\bar{g}_k\}_{k=0,1,\dots}$ be defined as:

$$\begin{aligned}\bar{g}_{k+1} &= \bar{g}_k \exp_G\left(\sum_{i=0}^{2L+1} \alpha^2 W_m^{(i)} \log_G\left(\bar{g}_k^{-1}\mathcal{G}^{(i)}\right)\right)\\ &= m\left(\bar{g}_k\right) \qquad (32)\\ \bar{g}_0 &= \mathcal{G}^{(0)}\end{aligned}$$

Note that if $\{\bar{g}_k\}_{k=0,1,\dots}$ converges to a value $\bar{g}$ as $k \to \infty$ so that $\bar{g} = m\left(\bar{g}\right)$, the condition (31) is satisfied. Hence $\bar{g}$ is determined via a fixed-point iteration of $m\left(\bar{g}_k\right)$, i.e., $\bar{g} = m\left(\bar{g}_k\right)|_{k\to\infty}$. The factor α^2 does not change the mean value and it was added because the scaled UT scales the weights by a factor of $1/\alpha^2$ and this can jeopardize the condition of working in the region where the exponential and logarithm maps are bijective.

We stress that this method for estimating $\bar{g}$ is different from the method presented in previous derivations of the Unscented Kalman Filter on Lie groups, (Brossard et al., 2017; Loianno et al., 2016), in which the authors approximate $\bar{g}$ to $\mathcal{G}^{(0)}$, what allows $\bar{\boldsymbol{\epsilon}}_g \neq 0$.

The covariance P_{gg} and cross-covariances P_{hg} and $P_{\mathbf{v}g}$ are also approximated using the weighted sample mean:

$$\begin{aligned}P_{gg} &= P_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}} = \sum_{i=0}^{2L+1} W_c^{(i)} \boldsymbol{\mathcal{E}}^{(i)} \boldsymbol{\mathcal{E}}^{(i)T}\\ &= \sum_{i=0}^{2L+1} W_c^{(i)} \log_G^{\vee}\left(\bar{g}^{-1}\mathcal{G}^{(i)}\right) \log_G^{\vee}\left(\bar{g}^{-1}\mathcal{G}^{(i)}\right)^T\end{aligned} \qquad (33)$$

$$\begin{bmatrix} P_{hg} \\ P_{\mathbf{v}g}\end{bmatrix} = P_{\boldsymbol{\xi}\boldsymbol{\epsilon}} = \sum_{i=0}^{2L+1} W_c^{(i)} \left(\boldsymbol{\Xi}^{(i)} - \bar{\boldsymbol{\xi}}\right) \boldsymbol{\mathcal{E}}^{(i)T} \qquad (34)$$

Evaluating P_{hg} and $P_{\mathbf{v}g}$:

$$P_{hg} = \sum_{i=0}^{2L+1} W_c^{(i)} \log_H^{\vee}\left(\bar{h}^{-1}\mathcal{H}^{(i)}\right) \log_G^{\vee}\left(\bar{g}^{-1}\mathcal{G}^{(i)}\right)^T \qquad (35)$$

$$P_{\mathbf{v}g} = \sum_{i=0}^{2L+1} W_c^{(i)} \left(\boldsymbol{\mathcal{V}}^{(i)} - \bar{\mathbf{v}}\right) \log_G^{\vee}\left(\bar{g}^{-1}\mathcal{G}^{(i)}\right)^T \qquad (36)$$

Summarizing, the constructed Unscented Trasform for Lie groups (LG-UT) can approximate $\bar{g}$, P_{gg}, P_{hg} and $P_{\mathbf{v}g}$, given f defined as in (28), $h\sim\mathcal{N}_H(\bar{h}, P_{hh})$ and $\mathbf{v}\sim\mathcal{N}_{\mathbb{R}^r}(\bar{\mathbf{v}}, P_{\mathbf{v}\mathbf{v}})$, with the following steps:

1. Let the augumented vector $\boldsymbol{\xi}$ be distributed as

$$\boldsymbol{\xi}\sim\mathcal{N}_{\mathbb{R}^{q+r}}\left(\begin{bmatrix}\mathbf{0}_{q\times 1}\\ \bar{\mathbf{v}}\end{bmatrix}, \begin{bmatrix}P_{hh} & P_{h\mathbf{v}}\\ P_{\mathbf{v}h} & P_{\mathbf{v}\mathbf{v}}\end{bmatrix}\right) \qquad (37)$$

2. Choose α, κ, and β and calculate $\lambda = \alpha^2(L+\kappa) - L$, with $L = q + r$;
3. Create the sigma points $\boldsymbol{\Xi}^{(i)} = \left[\left(\boldsymbol{\Delta}^{(i)}\right)^T \left(\boldsymbol{\mathcal{V}}^{(i)}\right)^T\right]^T$ and corresponging weights with (5) and (6);
4. Propagate the sigma points through f as in (30) and obtain $\mathcal{G}^{(i)}$ and $\mathcal{H}^{(i)}$;
5. Obtain the approximation of the mean $\bar{g} = \bar{g}_k|_{k\to\infty}$, with $\bar{g}_k = m\left(\bar{g}_{k-1}\right)$ defined in (32);
6. Approximate the remaining desired quantities with (33), (35) and (36), obtaining P_{gg}, P_{hg} and $P_{\mathbf{v}g}$.

2.5 Discrete Unscented Kalman Filter on Lie Groups

In this section, we propose a generalization of the Unscented Kalman Filter on Lie groups, using the Unscented Transform constructed in 2.4 and analogous to the UKF presentend in 2.3.

Let G be a p-dimensional matrix Lie group. The system dynamics is modeled as

$$g_{t+1} = g_t \exp_G^{\wedge}(\Omega(g_t) + \mathbf{q}_t) \qquad (38)$$

where $g_t \in G$ is the state to be estimated, $\mathbf{q}_t\sim\mathcal{N}_{\mathbb{R}^p}(\mathbf{0}, P_{\mathbf{q}\mathbf{q}})$ is a white Gaussian noise, and $\Omega : G \to \mathbb{R}^p$ is a non-linear C^2 function describing the system dynamics.

The measurements are considered to be elements of a q-dimensional matrix Lie group H:

$$h_t = \Phi(g_t) \exp_H^{\wedge}(\mathbf{r}_t) \qquad (39)$$

where $h_t \in H$ is a measurement arriving at the time step t, $\Phi : G \to H$ describes the measurement map, and $\mathbf{r}_t\sim\mathcal{N}_{\mathbb{R}^q}(\mathbf{0}, P_{\mathbf{r}\mathbf{r}})$ is a white Gaussian noise.

2.5.1 Prediction

It is assumed that the posterior state distribution after the arrival of $(t-1)$-th measurement is a concentrated Gaussian distribution on Lie groups, i.e., $g_{t-1}\sim\mathcal{N}_G(g_{t-1|t-1}, P_{t-1|t-1})$ and thus $g_{t-1} = g_{t-1|t-1}\exp_G^{\wedge}(\boldsymbol{\epsilon}_{t-1})$, where $\boldsymbol{\epsilon}_{t-1}\sim\mathcal{N}_{\mathbb{R}^p}(\mathbf{0}, P_{t-1|t-1})$.

The prediction step is performed as a direct application of the Unscented Transform on Lie groups. Let the random vector $\boldsymbol{\xi}_{t-1} = \left[\boldsymbol{\epsilon}_{t-1}^T\, \mathbf{q}_{t-1}^T, \mathbf{r}_{t-1}^T\right]^T$ be distibuted as:

$$\boldsymbol{\xi}_{t-1}\sim\mathcal{N}_{\mathbb{R}^{2p+q}}\left(\mathbf{0}, \begin{bmatrix}P_{t-t|t-1} & & \\ & P_{\mathbf{q}\mathbf{q}} & \\ & & P_{\mathbf{r}\mathbf{r}}\end{bmatrix}\right) \qquad (40)$$

The sigma points $\boldsymbol{\Xi}_{t-1|t-1}^{(i)} = \left[\left(\boldsymbol{\mathcal{E}}_{t-1|t-1}^{(i)}\right)^T \left(\boldsymbol{\mathcal{Q}}_{t-1|t-1}^{(i)}\right)^T \left(\boldsymbol{\mathcal{R}}_{t-1|t-1}^{(i)}\right)^T\right]^T$

(and weights) are created as in (5) and (6). and progapated through (38) as shown in (30), resulting in the following sigma points:

$$\mathcal{G}^{(i)}_{t-1|t-1} = g_{t-1|t-1} \exp^{\wedge}_{G}\left(\boldsymbol{\mathcal{E}}^{(i)}_{t-1|t-1}\right) \tag{41}$$

$$\mathcal{G}^{(i)}_{t|t-1} = \mathcal{G}^{(i)}_{t-1|t-1} \exp^{\wedge}_{G}\left(\Omega\left(\mathcal{G}^{(i)}_{t-1|t-1}\right) + \boldsymbol{\mathcal{Q}}^{(i)}_{t-1|t-1}\right) \tag{42}$$

The propagation of this state will result in the predicted state $g_t \sim \mathcal{N}_G(g_{t|t-1}, P_{t|t-1})$ with mean and covariance approximated as:

$$g_{t|t-1} = g_{t|t-1,k}\big|_{k\to\infty} \tag{43}$$

$$P_{t|t-1} = \sum_{i=0}^{2L+1} W_c^{(i)} \log^{\vee}_{G}\left(g^{-1}_{t|t-1}\mathcal{G}^{(i)}_{t|t-1}\right) \cdot \log^{\vee}_{G}\left(g^{-1}_{t|t-1}\mathcal{G}^{(i)}_{t|t-1}\right)^T \tag{44}$$

with $g_{t|t-1,k} = m(g_{t|t-1,k-1})$ and $g_{t|t-1,0} = \mathcal{G}^{(0)}_{t|t-1}$, m defined in (32).

This finishes the prediction step.

2.5.2 Update step

In the update step, we apply the Unscented Transform to the measurement model in (39) in order to approximate the measurement statistics. The propagation of the sigma points results in:

$$\mathcal{H}^{(i)}_{t|t-1} = \Phi\left(\mathcal{G}^{(i)}_{t|t-1}\right) \exp^{\wedge}_{H}\left(\boldsymbol{\mathcal{R}}^{(i)}_{t-1|t-1}\right) \tag{45}$$

Note that instead of creating a new set os sigma points for g_t, the propagated sigma points $\mathcal{G}^{(i)}_{t|t-1}$ are used.

We assume that the measurement has a concentrated Gaussian distribution $h_t \sim \mathcal{N}_H(h_{t|t-1}, P_{hh})$ and can be written as $h_t = h_{t|t-1} \exp^{\wedge}_{H}(\boldsymbol{\delta}_t)$, with $\boldsymbol{\delta}_t \sim \mathcal{N}_{\mathbb{R}^q}(\mathbf{0}, P_{hh})$. The statistics, according to the Unscented Transform on Lie groups, are approximated as:

$$h_{t|t-1} = h_{t|t-1,k}\big|_{k\to\infty} \tag{46}$$

$$P_{hh} = \sum_{i=0}^{2L+1} W_c^{(i)} \log^{\vee}_{H}\left(h^{-1}_{t|t-1}\mathcal{H}^{(i)}_{t|t-1}\right) \cdot \log^{\vee}_{H}\left(h^{-1}_{t|t-1}\mathcal{H}^{(i)}_{t|t-1}\right)^T \tag{47}$$

$$P_{gh} = \sum_{i=0}^{2L+1} W_c^{(i)} \log^{\vee}_{G}\left(g^{-1}_{t|t-1}\mathcal{G}^{(i)}_{t|t-1}\right) \cdot \log^{\vee}_{H}\left(h^{-1}_{t|t-1}\mathcal{H}^{(i)}_{t|t-1}\right)^T \tag{48}$$

with $h_{t|t-1,k} = m(h_{t|t-1,k-1})$ and $h_{t|t-1,0} = \mathcal{H}^{(0)}_{t|t-1}$, m defined in (32).

We can not apply the filter update directly on g_t based on the measurement h_t, instead we update $\boldsymbol{\epsilon}_t$ based on $\boldsymbol{\delta}_t$, which is obtained from the arrived measurement h_t as $\boldsymbol{\delta}_t = \log^{\vee}_{H}\left(h^{-1}_{t|t-1}h_t\right)$.

$$\begin{aligned}\boldsymbol{\epsilon}^*_{t|t} &= \boldsymbol{\epsilon}_{t|t-1} + P_{\boldsymbol{\epsilon\delta}} P^{-1}_{\boldsymbol{\delta\delta}}\left(\boldsymbol{\delta}_k - \bar{\boldsymbol{\delta}}_k\right) \\ &= P_{gh}P^{-1}_{hh}\log^{\vee}_{H}\left(h^{-1}_{t|t-1}h_t\right)\end{aligned} \tag{49}$$

$$\begin{aligned}P_{t|t} &= P_{t|t-1} - P_{\boldsymbol{\epsilon\delta}} P^{-1}_{\boldsymbol{\delta\delta}} P^T_{\boldsymbol{\epsilon\delta}} \\ &= P_{t|t-1} - P_{gh}P^{-1}_{hh}P^T_{gh}\end{aligned} \tag{50}$$

Note that $\boldsymbol{\epsilon}^*_t \sim \mathcal{N}_{\mathbb{R}^p}(\boldsymbol{\epsilon}^*_{t|t}, P_{t|t})$ has non-zero mean, thus we write it as $\boldsymbol{\epsilon}^*_t = \boldsymbol{\epsilon}^*_{t|t} + \boldsymbol{\epsilon}_t$, with $\boldsymbol{\epsilon}_t \sim \mathcal{N}_{\mathbb{R}^p}(\mathbf{0}, P_{t|t})$. The updated state on Lie group is a translation of the updated $\boldsymbol{\epsilon}^*_t$ to the predicted state, i.e., $g_t = g_{t|t-1}\exp^{\wedge}_{G}(\boldsymbol{\epsilon}^*_t) = g_{t|t-1}\exp^{\wedge}_{G}\left(\boldsymbol{\epsilon}^*_{t|t} + \boldsymbol{\epsilon}_t\right)$. We can use the Baker-Campbell-Hausdorff and approximate $g_t \approx g_{t|t-1}\exp^{\wedge}_{G}\left(\boldsymbol{\epsilon}^*_{t|t}\right)\exp^{\wedge}_{G}(\boldsymbol{\epsilon}_t)$, so that it is obtained that $g_t \sim \mathcal{N}_G(g_{t|t}, P_{t|t})$ with:

$$g_{t|t} = g_{t|t-1}\exp^{\wedge}_{G}\left(\boldsymbol{\epsilon}^*_{t|t}\right) \tag{51}$$

3 D-LG-UKF for Constant Linear Velocity and Constant Turn Rate Model

This section presentes the chosen model for the targets and for the radar measurements; shows how both the state of the target and the radar measurement can be represented on a Lie group and illustrates the advantage of the Lie group trajectory filtering.

3.1 System and Measurement Model

A target performing a trajectory with constant speed and constant angular rate in a plane can be modeled by the following equations (Bar-Shalom et al., 2001):

$$\begin{aligned}x_{t+1} &= q_{1,t} + x_t \\ &\quad + v_t\left[\cos\theta_t \frac{\sin(\omega_t T)}{\omega_t} - \sin\theta_t \frac{1-\cos(\omega_t T)}{\omega_t}\right] \\ y_{t+1} &= q_{2,t} + y_t \\ &\quad + v_t\left[\cos\theta_t \frac{1-\cos(\omega_t T)}{\omega_t} + \sin\theta_t \frac{\sin(\omega_t T)}{\omega_t}\right] \\ \theta_{t+1} &= q_{3,t} + \theta_t + \omega_t T \\ v_{t+1} &= q_{4,t} + v_t \\ \omega_{t+1} &= q_{5,t} + \omega_t\end{aligned} \tag{52}$$

where x_t, y_t, θ_t, v_t and ω_t are the state parameters, corresponding, respectively to the target x-position, y-position, heading, translational speed, and angular rate. The vector $\mathbf{q}_t = [q_{1,t} \quad q_{2,t} \quad q_{3,t} \quad q_{4,t} \quad q_{5,t}]^T \sim \mathcal{N}_{\mathbb{R}^5}(\mathbf{0}, P_{\mathbf{qq}})$ is white Gaussian noise.

It has been already discussed in (Long et al., 2013) that the pose of a differential robot presents a banana-shaped distribution and that the matrix Lie Special Euclidean group $SE(2)$ provides better means of handling the uncertainties distributions than in Euclidean space. Based on this and assuming that moving targets with dynamics modeled as (52) also presents banana-shaped distribution, it has been chosen the group $SE(2)$ to represent the pose of the target. This assumption will be further verified. The translational speed and angular rate will be expressed in a representation of the $\mathbb{R}^2$ as a matrix Lie group in which matrix product is also the group operation, as in $SE(2)$. The resulting group is a 5-dimensional matrix Lie group obtained by the cartesian product of $SE(2)$ and $\mathbb{R}^2$, i.e., $G = SE(2) \times \mathbb{R}^2$. A group element $g_t \in G$, representing the system state, is written as:

$$g_t = \begin{bmatrix} \begin{matrix} \cos\theta_t & -\sin\theta_t & x_t \\ \sin\theta_t & \cos\theta_t & y_t \\ 0 & 0 & 1 \end{matrix} & 0_{3x3} \\ 0_{3x3} & \begin{matrix} 1 & 0 & v_t \\ 0 & 1 & \omega_t \\ 0 & 0 & 1 \end{matrix} \end{bmatrix} \tag{53}$$

Note that although g_t is a 6×6 matrix, G is a 5-dimensional Lie group. This happens because we employ 3×3 matrices to represent $\mathbb{R}^2$, a 2-dimentional space, on matrix Lie groups. This choice allows us to replace the sum operation between two $\mathbb{R}^2$ vectors by the group operation, the matrix product, and keep the sum properties like commutation as such.

An element $X_t \in \mathfrak{g}$ of the Lie algebra associated to the matrix Lie group G is obtained via the logarithm map, which in the case of matrix groups is the matrix logarithm:

$$X_t = \log_G(g_t) = \begin{bmatrix} \begin{matrix} 0 & -\theta_t & p_{x,t} \\ \theta_t & 0 & p_{y,t} \\ 0 & 0 & 0 \end{matrix} & 0_{3x3} \\ 0_{3x3} & \begin{matrix} 0 & 0 & v_t \\ 0 & 0 & \omega_t \\ 0 & 0 & 0 \end{matrix} \end{bmatrix} \tag{54}$$

with

$$\begin{bmatrix} p_{x,t} \\ p_{y,t} \end{bmatrix} = \frac{\theta_t}{2(1-\cos\theta_t)} \begin{bmatrix} \sin\theta_t & 1-\cos\theta_t \\ \cos\theta_t - 1 & \sin\theta_t \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{55}$$

The system model on Lie group is given by $\Omega(g_t)$ so that the state evolves as in (52) when applied in (38). The function $\Omega(g_t)$ captures the dynamics of the system and it can be obtained by $\Omega(g_t) = \log_G^\vee(g_t^{-1} g_{t+1})$, with g_t and g_{t-1}, related by (52) and (53), resulting in:

$$\Omega(g_t) = \begin{bmatrix} v_t T \\ 0 \\ \omega_t T \\ 0 \\ 0 \end{bmatrix} \tag{56}$$

The model in (56) is equivalent to the model in (52) without noise. Although they represent the same dynamics, the model on Lie group is much more compact, which shows that the dynamic model fits better on the Lie group than in Euclidean space. The first two components of $\Omega(g_t)$ can be seen as the components of the body frame; the other componentes are, respectively, the heading, linear speed and angular speed. This means that the system dynamics on Lie group is equivalent to a displacement in the x axis of the body frame, i.e. in the direction of the heading, and a rotation.

For the measurement model, let us consider the measurements of a 2D radar for a single target: azimuth (α_t), range (ρ_t) and radial speed (v_t^ρ):

$$\begin{aligned} \alpha_t &= \operatorname{atan2}(y_t, x_t) \\ \rho_t &= \sqrt{x_t^2 + y_t^2} \\ v_t^\rho &= v_t \cos(\alpha_t - \theta_t) \end{aligned} \tag{57}$$

The radar measurements arrive in polar coordinates and are written in the structure of the Lie group H: $h_t = \exp_H\left([\tilde{\alpha}_t, \tilde{\rho}_t, \tilde{v}_t^\rho]_H^\wedge\right)$. As stated in (Cesic et al., 2016), their uncertainty also resembles banana-shaped contours rather than the eliptical ones. Because of that, the chosen Lie group for the measurements will be constructed as $H = SO(2) \times \mathbb{R}^2$. Thus, the measurement map $\Phi : G \to H$ is given as

$$\Phi(g_t) = \begin{bmatrix} \begin{matrix} \cos\alpha_t & -\sin\alpha_t \\ \sin\alpha_t & \cos\alpha_t \end{matrix} & 0_{2x3} \\ 0_{3x2} & \begin{matrix} 1 & 0 & \rho_t \\ 0 & 1 & v_t^\rho \\ 0 & 0 & 1 \end{matrix} \end{bmatrix} \tag{58}$$

3.2 Stochastic Distribution of System State

When a target moves according to the model in (52), because of the system noise, the final position of the target will not be deterministic. Instead, it is modeled as a joint pdf of the state variables $\mathbf{x}_n = [x_n, y_n, \theta_n, v_n, \omega_n]^T$, where n is the index of the final position.

Let us initialize the system in (52) at $\mathbf{x}_0 = [0, 0, 0, 200, 0]^T$, so that the nominal trajectory is a straight line along the x axis. The Gaussian noise is added only in the linear and angular speed. The system is propagated with $T = 0.01s$ over 800 steps, which is equivalent to $8s$, an usual value for the period of radar measurements. The final state, a $\mathbb{R}^5$ vector, is stored. This procedure is performed a total $N = 10000$ times, resulting in N data points.

The mean and covariance matrix, for the Euclidean Gaussian distribution, can be approximated using the N sample data points obtained.

$$\tilde{\boldsymbol{\mu}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{x}_n \tag{59}$$

$$\tilde{\Sigma} = \frac{1}{N-1}\sum_{n=1}^{N}(\mathbf{x}_n - \tilde{\boldsymbol{\mu}})(\mathbf{x}_n - \tilde{\boldsymbol{\mu}})^T \tag{60}$$

However, this set of data points can also be approximated to a 5-dimentional concentrated Gaussian random variable in the Lie Groups G presented in section 3.1. The mean and covariance for the concentrated Gaussian distribution (Long et al., 2013) can also be obtained by sample approximations:

$$\mu_{k+1} = \mu_k \exp_G\left(\frac{1}{N}\sum_{n=1}^{N}\log_G(\mu_k^{-1}g_n)\right) \tag{61}$$

$$\Sigma = \frac{1}{N-1}\sum_{n=1}^{N}\mathbf{y}_n\mathbf{y}_n^T \tag{62}$$

with $\mathbf{y}_n = \log_G^{\vee}(\mu^{-1}g_n)$ and the mean μ obtained recursively $\mu = \mu_k|_{k\to\infty}$.

The level curves of both distributions (marginalized to xy-plane) are ploted in Fig. 1 over the sample points. Note that the sample points present a curved shape, and do not fit very well to the Gaussian distribution in Euclidean space. The modelling of the state as a Lie group element provides a distribution that is a better approximation of the final state of the system, which lead us to believe that a filter evolving the system state as concentrated Gaussian random variables in Lie groups will present an improved performance, when compared to filters that assume a Gaussian distribution in Euclidean space.

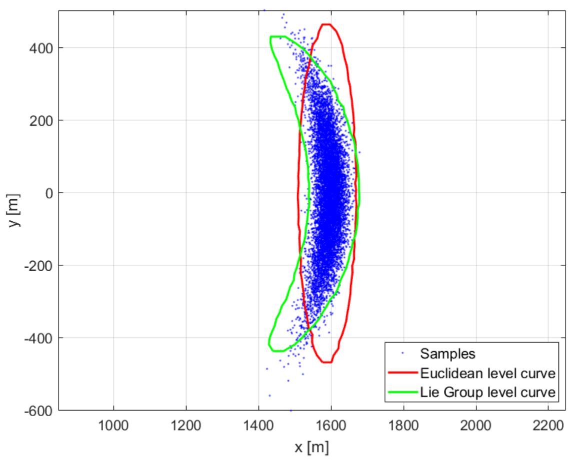

Figure 1: Sample data points and level curves for Gaussian distributions in Euclidean Space and Lie Group.

This result indicates that the banana-shaped distribution is more fitted to data than the Gaussian distribution in Euclidean space. Also, the result motivates the investigation of using the Lie group theory to filter the trajectory of the targets and a better performance is expected.

4 Simulation Results

In order to verify the correct operation of the D-LG-UKF, artificial trajectories are generated, simulating the typical behavior of a target measured by radar systems. It is considered a rectilinear nominal trajectory, passing over the origin, with constant speed. The filter developed in this paper is compared to the conventional D-EKF and to the conventional D-UKF, both implemented to the state space model described in (52) with measurements as vector whose components are given by (57), and the proposed filter is also to the D-LG-EKF, which was previously aplied to this system model in (Magalhaes et al., 2018).

The goal of the simulation is to compare the performance of the filters as the variance of the azimuth's noise increases. In order to do this, the standard deviation of the azimuth noise varies from $0°$ to $10°$. The variances of the measured range and radial velocity are kept constant. Also, a system noise is added, with $0.03\,ms^{-1}$ of standard deviation for the linear speed and $0.2°\,s^{-1}$ of standard deviation for the angular speed. The integration step is $0.01\,s$. The radar is simulated to operate with $7.5\,rpm$ of rotation speed, $5°$ of angular resolution, and $0.1\,s$ between detection attempts. The total duration of the simulation is $800\,s$. The filters are initialized with the same values, the same covariance matrix of the measurement noise is used for the four filters. Although the system noise covariance matrices are not defined the same way for the filters, they are tuned individually for the best achieved performance, so that the results are comparable.

The result is seen in Fig. 2 shows how the root mean square error (RMSE) varies as the standard deviation of the azimuth measurement noise increases. For a given step of standard deviation, the RMSE is calculated individually for each of the 100 trajectories and then it is plotted the mean RMSE and the confidence interval of 95%. It is possible to see that the proposed D-LG-UKF presents the lowest mean for the RMSE among the four filters. Also, it is opssible to see that it is the most stable in the sense that it presents the lowest standard deviation.

5 Conclusion

This paper proposed a construction of the Unscented Kalman Filter to work on Lie groups (D-LG-EKF) and applied it to a system consisting of a moving target with constant linear speed and constant angular rate. It was verified that the presented Lie group representation of the system

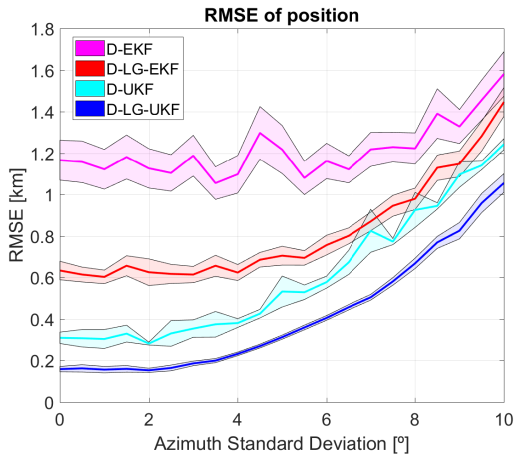

Figure 2: Root Mean Square Error for simulated trajectories varying azimuth standard deviation.

could capture the desired dynamics of the system with precision and simplicity. Also, it was shown how a concentrated Gaussian distribution is a better approximation of the state of a target moving with constant linear speed and constant turn rate model.

The D-LG-UKF proved, via simulation results, to be effective in the contex of radar systems, in which the measurements are performed in polar coordinates and the target pose can be described in the Special Euclidean Group. The filtering on Lie group presents a better performance when compared with the D-EKF, D-LG-EKF, and D-UKF along the entire range of azimuth noise standard deviation. This result is a consequence of the better approximation of the statistics that the Unscented Transform provides and the more proper representation of the system model on the presented Lie group.

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