
AUTOMATIC MODELLING OF LUMPED PARAMETERS DYNAMIC SYSTEMS

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ABSTRACT: This paper describes the framework and the approach that are utilized in the implementation of a software for automated mathematical modelling of dynamic systems. Application examples illustrate the main features and possibilities of this software. The formalism utilizes a system linear graph representation associated to a systematic procedure in order to obtain a state-space model of lumped-parameters systems. This mathematical modelling method is automated using modern computational symbolic processing capabilities.

RESUMO: Este trabalho descreve a base formal e a abordagem utilizada para a implementação de um ambiente computacional para modelagem simbólica de sistemas dinâmicos. Os recursos implementados são ilustrados através da resolução de problemas-exemplos. O formalismo se aplica a sistemas a parâmetros concentrados e utiliza a representação tipo *grafo de sistemas* associada a um procedimento sistemático para obter um modelo matemático do sistema no espaço de estados. A automatização do processo de modelagem explora os recursos computacionais modernos de processamento simbólico.

1 INTRODUCTION

Mathematical modelling can become an essential part in the process of analyzing a dynamic system, as a mean to make predictions, to optimize performance or to perform qualitative evaluations about the system. In addition, models expressed in symbolic form allow the derivation of alternative models appropriate to the objectives of the analysis; for example, modelling for purpose of control system design.

There are several computer programs available to support dynamic systems analysis and simulations studies; for example, Felez et al., 1990, which utilizes a bond graph approach; Andrews, 1971, which utilizes a vector network approach. However, computer programs designed to derive mathematical models, i.e. models in symbolic form, are scarce.

The *system graph* representation associated with a *systematic procedure* for obtaining state-space equations provides an interesting alternative to automate the process of mathematical modeling of lumped parameters systems, making use of the available modern computational symbolic processing

resources. Automated modelling provides greater reliability on the derived models, reduces the time and effort to obtain them and in general, results in more efficient analysis studies.

In the next two sections we present a theoretical-computational formalism to derive a symbolic state-space model for lumped-parameters dynamic systems. A section dedicated to examples is provided to highlight the main features of the developed software.

2 SYSTEM REPRESENTATION AND THEORETICAL ASPECTS

2.1 Models Comprised In This Analysis

In this paper, we treat lumped-parameter dynamic systems and we consider two classes of systems: linear time-variant systems having two-terminal RLC (resistance-inductance-capacitance) generalized elements, mutual inductances and four-terminal generalized transformer elements; and non-linear time-invariant systems, having two-terminal RLC generalized elements. In both cases the systems may contain independent generalized voltage and current sources (external inputs). A large family of plants is considered, especially those constituted by the interconnection of mechanical, electrical, electro-mechanical, hydraulic and electro-hydraulic components.

Seeking generalization and consistency, it was necessary to impose some hypotheses; in the linear case, they were kept to a minimum in order not to burden the user or to restrain unnecessarily the practical applications of the developed software. In the nonlinear case, they are listed in the section 2.4.2.

2.2 System Graph Representation

In the System Graph (or Linear Graph) representation (Shearer *et alii*, 1969), a set of connected branches, representing the lumped elements of the system, forms a graph describing the real system or its physical model. The generalization of the Kirchhoff laws to electrical circuits can be utilized to impose the graph the constraints to flows and loops (continuity and compatibility conditions). The dynamic equations of the system, i.e. its mathematical model, can be obtained by adding to the constraint equations set, the equations which describe the system elements. The *mobility analogy* (Firestone, 1933) among electrical circuits and the other types of systems,

mechanical, thermal and/or fluid, can be utilized in order to deal with all of them in a generalized way.

So, the problem is to exploit the system graph approach for representing generalized lumped-parameters dynamic systems and a systematic procedure for defining state variables and relate them, in order to implement an integrated software that automates the modelling process for obtaining the state-space equations in a symbolic form as well as its numerical simulation.

By making use of quantities like the *incidence matrix* of the system graph, its *normal tree*, *fundamental cutsets* and *loops*, defined in the section 2.3, it is possible to deduce a general state-space model in terms of special matrices representing the elements of the system and the way they are connected (Kuh and Roher, 1965; Adade F^o, 1992; Gonçalves, 1995). Several algorithms were developed (Gonçalves, 1995) to extract information from the system graph and to implement the systematic procedure for modelling (as commented in the section 3).

2.3 Systematic Procedure For Modelling In The State-Space

A systematic procedure, as described in the sequel, allows us to derive relationships among the variables, needed to automate the process of obtaining state-space models from the system graph representation:

a - define a *normal tree*. This tree is built from the oriented graph of the system, adopting the following sequence for choosing the tree branches: all “across” variable sources, the maximum number of passive elements type-A (generalized capacitors), type-D (generalized dissipators) and a minimum number of type-T (generalized inductors) passive elements; in the end of this process, the elements not included in the normal tree will be included in the complementary graph, called co-tree. Every branch not in the tree is called a *link*.

b - Choose as state variables:

b₁ - the “across” variables of the type-A (generalized capacitors) elements inserted in the normal tree (v_C) and the “through” variables of the type-T (generalized inductors) in the co-tree (i_L); or, alternatively,

b₂ - the integrated “through” variables of the type-A elements inserted in the normal tree (q_C) and the integrated “across” variables of the type-T elements in the co-tree (ϕ_L).

c - Write each (across and/or through) variable of all type-D and type-A elements in the co-tree, and of the type-T elements inserted in the normal tree, as well as the generalized transformers energy ports as functions of the state variables, applying the *loop law* to the fundamental loops and the *node law* to the fundamental cutsets, formed by those branches. A *fundamental loop* is a unique loop formed by every link and some tree branches; a *fundamental cutset* is a unique cutset formed by every tree branch with some links.

d - Apply *node law* to the fundamental cutsets or the *loop law* to the fundamental loops, of every branch that contributes with a state variable.

It should be pointed out that the application of this systematic procedure allows to describe the system utilizing a minimum

set of state variables, since steps (a) and (b) guarantee a non-redundant choice of these variables. Also, from the computational point of view, steps (c) and (d) are represented in the calculation of the matrices presented in the next section.

This systematic procedure is suggested in some texts like (Kuh and Roher, 1965; Wellstead, 1979; Chen, 1984).

2.4 System Equations For Automated Modelling

2.4.1 Linear systems

The algebraic manipulation utilizing the systematic procedure to deduce a linear state-space model,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (1)$$

is extensive and can be found in details in Gonçalves (1995). In this sub-section it is presented the final form of the models considering two different sets of state-variables. Consider,

- the vector of independent sources:

$$u(t) = \begin{bmatrix} j^T & e^T \end{bmatrix}^T \quad (2)$$

where:

$j(nra,1) \rightarrow$ vector of “through” variables sources, the k-th element being the algebraic sum of the “through” variables sources belonging to the k-th fundamental cutset.

$e(nb,1) \rightarrow$ vector of “across” variables sources, the i-th element being the algebraic sum of the “across” variables sources belonging to the i-th fundamental loop.

Partitioning the vector of independent sources in accordance to the type of the elements, using the notation for subscripts,

C : capacitors in the normal tree; S : capacitors in the co-tree; L : inductors in the co-tree; Γ : inductances in the normal tree; R : resistors in the co-tree; G : conductances in the normal tree; θ : transformer port in the normal tree; α : transformer port in the co-tree,

we obtain,

$$[j] = \begin{bmatrix} 0 \\ j_C \\ j_G \\ j_\Gamma \end{bmatrix} \quad \text{and} \quad [e] = \begin{bmatrix} e_S \\ e_R \\ e_L \\ 0 \end{bmatrix} \quad (3)$$

The fundamental loops and cutsets equations can be written as:

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & -F \\ F^T & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} e \\ j \end{bmatrix} \quad (4)$$

where the vectors v_2 and v_1 (i_2 and i_1) denote the across (through) variables of the passive elements inserted in the normal tree (cotree), respectively. This matrix F is called the fundamental loop matrix and its transpose is equal to the fundamental cutset matrix. Partitioning F in accordance to the systematic procedure to choose the normal tree (see the Symbol List to a comment about each sub-matrix):

$$F = \begin{bmatrix} 0 & F_{SC} & 0 & 0 \\ F_{R\theta} & F_{RC} & F_{RG} & 0 \\ F_{L\theta} & F_{LC} & F_{LG} & F_{L\Gamma} \\ F_{\alpha\theta} & F_{\alpha C} & F_{\alpha G} & 0 \end{bmatrix} \quad (5)$$

Utilizing the equations describing the generalized RLC and transformers elements, as well as the constraints equations from the oriented system graph, the following matrices can be defined (also see Symbol List):

- the capacitive matrix of the system:

$$C = C_2 + F_{SC}^T C_1 F_{SC} \quad (6)$$

- the inductive matrix of the system:

$$L = L_{11} + F_{L\Gamma} L_{21} + L_{12} F_{L\Gamma}^T + F_{L\Gamma} L_{22} F_{L\Gamma}^T \quad (7)$$

- the resistive matrix of the system:

$$R = R_1 + F_{RG} R_2 F_{RG}^T \quad (8)$$

- the conductive matrix of the system:

$$G = G_2 + F_{RG}^T G_1 F_{RG} \quad (9)$$

- the matrices related to the energy ports of the generalized transformers:

$$T_1 = F_{\alpha C}^T N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{R\theta}^T \quad (10)$$

$$T_2 = F_{\alpha C}^T N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{L\theta}^T \quad (11)$$

$$T_3 = F_{L\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha C} \quad (12)$$

$$T_4 = F_{L\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha G} \quad (13)$$

$$T_5 = F_{R\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha G} R_2 F_{RG}^T - [F_{R\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha G} R_2 F_{\alpha G}^T - F_{RG} R_2 F_{\alpha G}^T] N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{R\theta} \quad (14)$$

$$T_6 = F_{R\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha G} R_2 F_{LG}^T + [F_{R\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha G} R_2 F_{\alpha G}^T - F_{RG} R_2 F_{\alpha G}^T] N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{L\theta} \quad (15)$$

$$T_7 = F_{R\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha C} R_2 \quad (16)$$

$$T_8 = F_{\alpha G}^T N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{R\theta}^T G_1 F_{RG} - [F_{\alpha G}^T N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{R\theta}^T + F_{RG}^T] \quad (17)$$

$$G_1 F_{R\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha G}$$

$$T_9 = F_{\alpha G}^T N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{R\theta}^T G_1 F_{RC} + [F_{\alpha G}^T N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{R\theta}^T + F_{RG}^T] \quad (18)$$

$$G_1 F_{R\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha C}$$

$$T_{10} = F_{\alpha G}^T N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{L\theta}^T \quad (19)$$

$$T_{11} = F_{\alpha G}^T N_{21} (I_{I\theta} - F_{\alpha\theta}^T N_{21})^{-1} F_{R\theta}^T \quad (20)$$

$$T_{12} = F_{R\theta} N_{12} (I_{V\alpha} + F_{\alpha\theta} N_{12})^{-1} F_{\alpha C} \quad (21)$$

In addition, consider,

$$y = (F_{RC}^T + T_1)(R + T_5)^{-1}(T_{12} - F_{RC})$$

$$h = (F_{LC}^T + T_2) + (F_{RC}^T + T_1)(R + T_5)^{-1}(T_6 - F_{RG} R_2 F_{LG}^T) \quad (23)$$

$$k = (T_3 - F_{LC}) + (T_4 - F_{LG})(G + T_8)^{-1}(T_9 - F_{RG}^T G_1 F_{RC}) \quad (24)$$

$$z = (T_4 - F_{LG})(G + T_8)^{-1}(T_{10} + F_{LG}^T) \quad (25)$$

To write the state-space equations in the standard form, the state variables vector, as chosen in the step **b**₁ of the systematic procedure, is redefined as:

$$x = \begin{bmatrix} v_C - C^{-1} F_{SC}^T C_1 e_S \\ i_L + L^{-1} (L_{12} + F_{L\Gamma} L_{22}) j_{\Gamma} \end{bmatrix} \quad (26)$$

In this case, the matrices A(t) e B(t) of the state-space model are given by:

$$A(t) = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} y - \dot{C} & h \\ k & z - \dot{L} \end{bmatrix} \quad (27)$$

$$B(t) = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} I & B_1 & B_{11} & B_2 & B_3 & 0 \\ 0 & B_4 & B_5 & B_{12} & B_6 & I \end{bmatrix} \quad (28)$$

where,

$$B_1 = (F_{RC}^T + T_1)(R + T_5)^{-1}(T_7 - F_{RG} R_2) \quad (29)$$

$$B_{11} = -h L^{-1} (L_{12} + F_{L\Gamma} L_{22}) \quad (30)$$

$$B_2 = y C^{-1} F_{SC}^T C_1 \quad (31)$$

$$B_3 = (F_{RC}^T + T_1)(R + T_5)^{-1} \quad (32)$$

$$B_4 = (T_4 - F_{LG})(G + T_8)^{-1} \quad (33)$$

$$B_5 = -z L^{-1} (L_{12} + F_{L\Gamma} L_{22}) \quad (34)$$

$$B_{12} = kC^{-1}F_{SC}^T C_1 \quad (35)$$

$$B_6 = (T_4 - F_{LG})(G + T_8)^{-1}(F_{RG}^T + T_{11})G_1 \quad (36)$$

For the same reason, that is, to write the state-space equations in the standard representation (without explicit derivatives of the input), the vector of state variables, as chosen in the step **b**₂ of the systematic procedure, is redefined as:

$$x(t) = \begin{bmatrix} q_C - C_2 C^{-1} F_{SC}^T C_1 e_S \\ \phi_L + L_{11} L^{-1} (L_{12} + F_{LG} L_{22}) j_\Gamma \end{bmatrix} \quad (37)$$

In this case, the matrices A(t) e B(t) of the state-space model are given by:

$$A(t) = \begin{bmatrix} C^{-1}C_2 & 0 \\ 0 & L^{-1}L_{11} \end{bmatrix} \begin{bmatrix} yC_2^{-1} - \frac{d}{dt}(CC_2^{-1}) & hL_{11}^{-1} \\ kC_2^{-1} & zL_{11} - \frac{d}{dt}(LL_{11}^{-1}) \end{bmatrix} \quad (38)$$

$$B(t) = \begin{bmatrix} C^{-1}C_2 & 0 \\ 0 & L^{-1}L_{11} \end{bmatrix} \begin{bmatrix} I & B_1 & B_{11} & B_2 & B_3 & 0 \\ 0 & B_4 & B_5 & B_{12} & B_6 & I \end{bmatrix} \quad (39)$$

2.4.2 Nonlinear systems

This section presents the formulation of a dynamic model to systems which have non-linear RLC generalized elements. Utilizing matrices conveniently chosen, this formulation allows an automatization of the modelling process through the use of symbolic manipulation software.

Only nonlinear generalized elements whose characteristics can be represented by parametric equations are considered. Conditions to the existence of a state-space model are presented in terms of the topological constraints and the characteristics of the elements, allowing, by inspection of the system graph, to check the existence of the required model. This section is based on (Brayton and Moser, 1964; Chua and Roher, 1965; Desoer and Katzenelson, 1965; Stern, 1966).

Lets define a parametric equation as a subset C in the Euclidean plane E²,

$$C = \{(w, z) \in E^2 : w = w(\xi), z = z(\xi), \xi \in [a, b]\} \quad (40)$$

such that w e z are continuous functions of the parameter ξ , being differentiable within the interval of interest.

With this definition it is possible to characterize the nonlinear RLC generalized elements as follows.

The constitutive relationship to a nonlinear type-A element can be represented by a curve in the q-v plane, being q the integrated "through" variable and v the "across" variable of the element, which equation is:

$$f_C(v, q) = 0 \quad (41)$$

If equation (41) can be solved for v as a function of q, $v = V(q)$, the element is said to be integrated "through" variable controlled; on the other hand, if his equation can be solved for q as a function of v, $q = Q(v)$, the element is said to be "across" variable controlled.

Similarly, the characteristics of a nonlinear type-T element can be represented by a curve in the ϕ -i plane, being ϕ the integrated "across" variable and i the "through" variable, which equation is:

$$f_L(\phi, i) = 0 \quad (42)$$

If equation (42) can be solved for i as a function of ϕ , $i = I(\phi)$, the element is said to be integrated "across" variable controlled; on the contrary, if this equation can be solved for ϕ as a function of i, $\phi = \Phi(i)$, the element is said to be "through" variable controlled.

The constitutive relationship of a type-D element can be represented by a curve in the v-i plane, the equation being:

$$f_R(v, i) = 0 \quad (43)$$

If equation (43) can be written as $v = V(i)$, the element is said to be "through" variable controlled; on the contrary, if $i = I(v)$, the element is said to be "across" variable controlled.

Equations (41)-(43) make possible to deduce a dynamic model in the sate-space,

$$\dot{x} = f(x, u) \quad (44)$$

In addition, it is assumed the following hypotheses:

H₁ - the system is time-invariant;

H₂ - there are no two-ports (four terminals) elements;

H₃ - the non-linear generalized RLC elements can be described by parametric equations;

H₄ - there are no fundamental loops formed by type-D elements;

H₅ - all type-D elements in the co-tree are "across" variables controlled;

H₆ - all type-D elements in the normal tree are "through" variables controlled;

H₇ - the type-A elements are "across" variables controlled;

H₈ - the type-T elements are "through" variables controlled.

Hypotheses **H₄-H₆** allow us to express, in a direct way, the variables of type-D elements in terms of the adopted state variables; otherwise, we should define inverse functions for the characteristics of the elements (Desoer and Katzenelson, 1965). Hypotheses **H₇-H₈** allow us to derive, in a direct way, a dynamic model in terms of the "across" variables of the type-A elements in the normal tree and "through" variables of type-T elements in the co-tree.

In accordance to the hypotheses **H₁-H₈** the equations for the generalized RLC elements can be written using explicit functions,

$$\begin{bmatrix} i_S \\ i_C \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v_S \\ v_C \end{bmatrix} \quad (45)$$

for the generalized capacitors;

$$\begin{aligned} v_G &= R_2 \circ (F_{LG}^T i_L + j_G) \\ i_R &= G_1 \circ (-F_{RC} v_C + e_R) \end{aligned} \quad (46)$$

for the generalized resistors; and

$$\begin{bmatrix} v_L \\ v_\Gamma \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_L \\ i_\Gamma \end{bmatrix} \quad (47)$$

for the generalized inductors.

The submatrices in the equations (45) and (47) are determined by:

$$C_1 = \frac{\partial}{\partial v_C} [Q_S \circ (-F_{SC} v_C + e_S)] \quad (48)$$

$$C_2 = \frac{\partial}{\partial v_C} [Q_C(v_C)] \quad (49)$$

$$L_{11} = \left[\frac{\partial}{\partial i_L} \Phi_{11}(i_L, i_\Gamma) \right] \circ (F_{L\Gamma}^T i_L + j_\Gamma) \quad (50)$$

$$L_{12} = \left[\frac{\partial}{\partial i_\Gamma} \Phi_{11}(i_L, i_\Gamma) \right] \circ (F_{L\Gamma}^T i_L + j_\Gamma) \quad (51)$$

$$L_{21} = \left[\frac{\partial}{\partial i_L} \Phi_{22}(i_L, i_\Gamma) \right] \circ (F_{L\Gamma}^T i_L + j_\Gamma) \quad (52)$$

$$L_{22} = \left[\frac{\partial}{\partial i_\Gamma} \Phi_{22}(i_L, i_\Gamma) \right] \circ (F_{L\Gamma}^T i_L + j_\Gamma) \quad (53)$$

The symbol (\circ) in these equations denotes function of function.

In case hypothesis **H₄** is not verified (this condition is analysed by the implemented software, which provides a warning message in case of its violation), we can introduce to the original system graph, a linear capacitor in parallel to the type-D elements of the co-tree that are forming the respective fundamental loops; a linear capacitor inserted in the system graph allows to simulate a residual parasitic capacitance, which is a characteristic of physical systems. Alternatively, we can insert linear inductors in series to type-D elements in the normal tree, that are forming the respective fundamental loops; a linear inductor inserted this way allow us to simulate parasitic inductance, which is a characteristic of physical systems. It is assumed no mutual coupling for the linear inductances inserted in the original system graph.

The algebraic manipulation to derive state-space models is similar to the linear case and can be found in details in Gonçalves (1995). The results are presented in the sequel for two different sets of state variables.

For the first set, $v_C \in i_L$, the dynamic model is given by:

$$C \dot{v}_C = F_{RC}^T G_1 \circ (-F_{RC} v_C + e_R) + j_C + F_{LC}^T i_L + F_{SC}^T C_1 \dot{e}_S \quad (54)$$

$$L \dot{i}_L = -F_{LC} v_C - F_{LG} R_2 \circ (F_{LG}^T i_L + j_G) - (L_{12} + F_{L\Gamma} L_{22}) \dot{j}_\Gamma + e_L \quad (55)$$

where,

$$C = F_{SC}^T C_1 F_{SC} + C_2 \quad (56)$$

$$L = L_{11} + F_{L\Gamma} L_{21} + L_{12} F_{L\Gamma}^T + F_{L\Gamma} L_{22} F_{L\Gamma}^T \quad (57)$$

To derive a dynamic model for the second set of state variables, $q_C \in \phi_L$, let's consider hypotheses **H₉-H₁₂** in substitution of hypotheses **H₇-H₈** as below:

H₉ - the type-A elements in the normal tree are of controlled "through" variables;

H₁₀ - the type-T elements in the co-tree are of controlled integrated "across" variables;

H₁₁ - the type-A elements in the co-tree are of controlled "across" variables;

H₁₂ - the type-T elements in the normal tree are of controlled "through" variables;

Utilizing hypotheses **H₉-H₁₂**, we obtain the following dynamic model:

$$CC_2 \dot{q}_C = F_{RC}^T G_1 \circ [-F_{RC} V_C(q_C) + e_R] + F_{LC}^T I_L(\phi_L) + j_C + F_{SC}^T C_1 \dot{e}_S \quad (58)$$

$$LL_1 \dot{\phi}_L = -F_{LC} V_C(q_C) - F_{LG} R_2 \circ [F_{LG}^T I_L(\phi_L) + j_G] - F_{L\Gamma} L_{22} \dot{j}_\Gamma - e_L \quad (59)$$

where,

$$C_1 = \frac{\partial}{\partial v_C} \{Q_S \circ [-F_{SC} V_C(q_C) + e_S]\} \quad (60)$$

$$C_2 = \frac{d}{dq_C} [V_C(q_C)] \quad (61)$$

$$L_{11} = \frac{d}{d\phi_L} [I_L(\phi_L)] \quad (62)$$

$$L_{22} = \frac{\partial}{\partial \phi_L} \{ \Phi_{22} \circ [F_{L\Gamma}^T I_L(\phi_L) + j_\Gamma] \} \quad (63)$$

$$C = F_{SC}^T C_1 F_{SC} + C_2^{-1} \quad (64)$$

$$L = L_{11}^{-1} + F_{L\Gamma} L_{22} F_{L\Gamma}^T \quad (65)$$

3 ASPECTS OF THE COMPUTER IMPLEMENTATION

The automatic modelling software (called MASD, an acronym for "Modelagem Automática de Sistemas Dinâmicos") is constituted by Turbo Pascal routines which implement the user interface program (problem definition) and the system graph characterization, as well as a program written in Mathematica to manipulate and obtain the state-space symbolic form equations. MASD main features are illustrated in this paper through application examples.

A flow diagram of MASD is shown in figure 1. In this diagram, *system definition* refers to the user input of data describing the system graph topology (number of nodes, number of branches, initial and final node numbers for each branch element), the type of elements of the system and the simulation parameters. This numerical data input is done in a

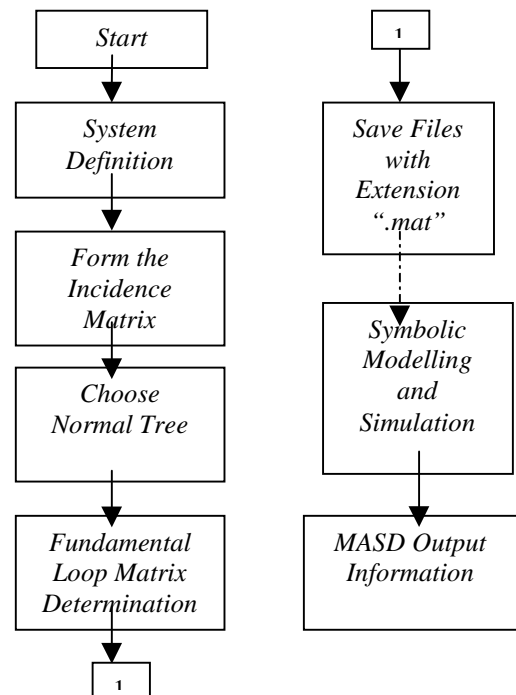


Figure 1 – MASD Program Flow Diagram

questions and answers session, controlled by the user interface routine. The dialog boxes for data input are presented in the Appendix. Concluded the data input, the information about the graph topology are then reorganized by MASD in a matrix form through an *incidence matrix*, which is the base for the systematic classification of tree branches and links, in the process of defining a normal tree and its respective co-tree. Tree search algorithms are utilized in this process as well as to obtain the *fundamental loop matrix* (Gonçalves, 1995).

Once the *fundamental loop matrix* is obtained and adequately partitioned, the next step of the program is to save the required information (those matrices defined in the modelling equations, parameters of the elements, parameters of simulation etc) for the symbolic processing session; such information are saved as files *.mat*. The algebraic manipulation program to deduce the dynamic mathematical model, as well as its simulation, is part of MASD and implemented using the software *Mathematica for Windows®* v2.1. In the flow diagram of figure 1 the dashed line indicates that the symbolic manipulation program must be activated by the user, inside Mathematica. Other algorithms, as commented above, were implemented in Turbo Pascal v5.5.

MASD computational program, for the linear case, provides as output the matrices $A(t)$ e $B(t)$ and the vectors $x(t)$ and $u(t)$; for the non-linear case it provides a model in accordance to eq.(45). In both cases, if required by the user, it provides a numerical simulation of the model.

4 APPLICATION EXAMPLES

4.1 Linear System

This example shows results obtained using MASD for modelling the mechanical system (Wellstead, 1979) represented in the figures 2 and 3.

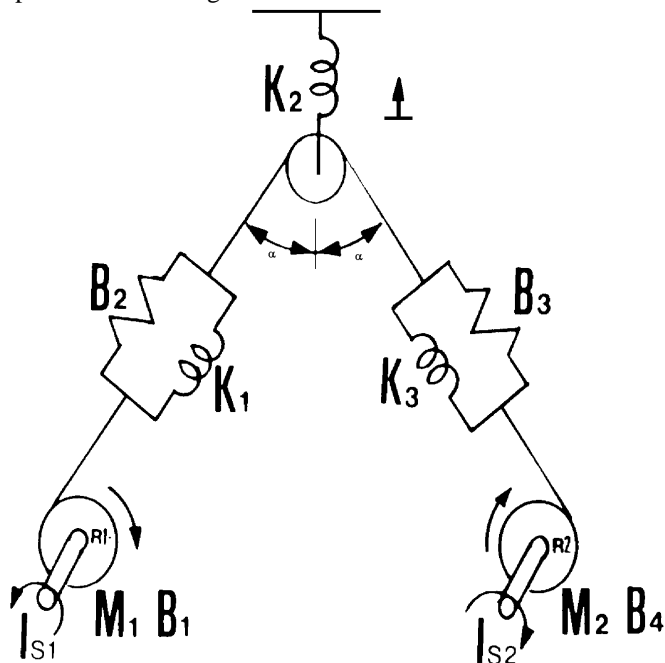


Figure 2 - Mechanical System

Assuming numerical simulation of the system is required, consider the following fictitious values for the lumped elements of this system

$$\begin{aligned} M_1 &= 3 \text{ [kg]} & M_2 &= 3 \text{ [kg]} \\ I_{S1} &= 1 \text{ [Nm]} & I_{S2} &= 2 \text{ [Nm]} \\ B_1 &= 1.9 \text{ [Ns/m]} & B_2 &= 1.2 \text{ [Ns/m]} \\ B_3 &= 1.2 \text{ [Ns/m]} & B_4 &= 1.9 \text{ [Ns/m]} \end{aligned}$$

$$K_1 = 1.8 \text{ [N/m]} \quad K_2 = 1.8 \text{ [N/m]}$$

$$K_3 = 1.5 \text{ [N/m]}$$

$$n_1 = r_a = 0.1 \text{ [m]} \quad n_2 = 2 \cos(\alpha)$$

$$n_3 = r_b = 0.1 \text{ [m]} \quad \alpha = 0.524 \text{ [rad]}$$

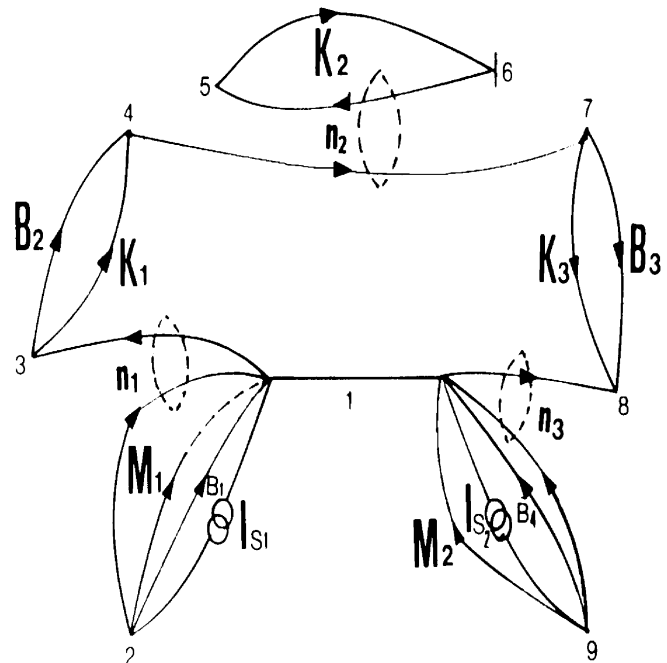


Figure 3 - System Graph

: After data input, which consists in defining the system graph as shown in the Fig.3, in a questions and answers session controlled by the user interface program (see the Appendix), MASD represents internally the graph through an *incidence matrix*; also, utilizes an specific terminology in accordance to each type of element (Cg, Bg, Kg for generalized capacitance, resistance and inductance, respectively; Fe, Fa for the generalized across- and through-variable sources; Te, Ta for the transformer ports inserted in the normal tree and co-tree, respectively), and utilizes a numeric index for each element, depending on the order they are entered by the user. Five files are generated by MASD after data input, with information for the next phase of the symbolic modelling process. For example, MASD provides, in the file *arq1.mat* a description of the elements of the system, in this problem the capacitance matrix (C_2), the resistance matrices (R_1 e R_2) and the inductance matrix (L_{11}); also, the transformation ratio matrix (n_{12}) as well as the external sources vector (F_a). In the file *arq2.mat*, MASD provides the sub-matrices of the fundamental loop matrix F (see eq. (5)):

Arq1.mat
 $C_2 = \{\{Cg1,0\},\{0,Cg2\}\};$
 $R_1 = \{\{1/Bg1,0\},\{0,1/Bg4\}\};$
 $R_2 = \{\{1/Bg2,0\},\{0,1/Bg3\}\};$
 $L_{11} = \{\{1/Kg1,0,0\},\{0,1/Kg2,0\},\{0,0,1/Kg3\}\};$
 $n_{12} = \{\{1/n1,0,0\},\{0,1/n2,0\},\{0,0,1/n3\}\};$
 $F_a = \{\{Fa1,0\},\{0,Fa2\}\};$

Arq2.mat
 $F_{re} = \{\{0,0,0\},\{0,0,0\}\};$
 $F_{rc} = \{\{-1,0\},\{0,-1\}\};$
 $F_{rg} = \{\{0,0\},\{0,0\}\};$
 $F_{le} = \{\{0,0,0\},\{0,0,0\},\{0,1,0\}\};$
 $F_{lc} = \{\{0,0\},\{0,0\},\{0,0\}\};$
 $F_{lg} = \{\{-1,0\},\{0,-1\},\{0,0\}\};$

$F_{ae} = \{\{0,0,0\}, \{1,0,-1\}, \{0,0,0\}\};$
 $F_{ac} = \{\{-1,0\}, \{0,0\}, \{0,-1\}\};$
 $F_{ag} = \{\{0,0\}, \{1,1\}, \{0,0\}\};$
 $F_{ic} = \{\{-1,0\}, \{0,1\}\};$
 $F_{ig} = \{\{0,0\}, \{0,0\}\};$

Also, at the end of the data input MASD generates the following information for the user:

TREE BRANCHES:

Element	initial node	final node	magnitude
Te1	3	1	0.1
Te2	5	6	$2\cos[0.524]$
Te3	8	1	0.1
Cg1	2	1	3
Cg2	9	1	3
Bg2	3	4	1.2
Bg3	7	8	1.2

LINKS (CO-TREE BRANCHES):

element	initial node	final node	magnitude
Bg1	2	1	1.9
Bg4	9	1	1.9
Kg1	3	4	1.8
Kg2	7	8	1.8
Kg3	5	6	1.5
Fa1	2	1	1
Fa2	1	9	2
Ta1	1	2	0.1
Ta2	7	4	$2\cos[0.524]$
Ta3	1	9	0.1

Choosing a standardized set of state variables we have that the state vector is composed by the integrated through-variable of the mechanical capacitance Cg_1 and the integrated across-variables in the mechanical inductances kg_1 and kg_2 , respectively:

$$x(t) = \{\{qc[1][t]\}, \{qc[2][t]\}, \{fi[1][t]\}, \{fi[2][t]\}, \{fi[3][t]\}\}$$

The input vector is:

$$u(t) = \{\{-Fa1\}, \{Fa2\}, \{0\}, \{0\}\}$$

At the end of the symbolic modelling process, the ij-elements of the system matrix $A(t)$ were obtained by MASD as:

$$\begin{aligned}
 a[1,1] &= -(Bg1/Cg1), a[1,2] = 0, a[3,1] = 0, a[4,1] = 0, \\
 a[5,1] &= -((Kg3*n1)/n2) \\
 a[2,1] &= 0, a[2,2] = -(Bg4/Cg2), a[3,3] = 0, a[3,4] = 0, \\
 a[3,5] &= (Kg3*n3)/n2 \\
 a[3,1] &= 0, a[3,2] = 0, a[3,3] = -(Kg1/Bg2), a[3,4] = 0, \\
 a[3,5] &= Kg3/(Bg2*n2) \\
 a[4,1] &= 0, a[4,2] = 0, a[4,3] = 0, a[4,4] = -(Kg2/Bg3), \\
 a[4,5] &= Kg3/(Bg3*n2) \\
 a[5,1] &= n1/(Cg1*n2), a[5,2] = -(n3/(Cg2*n2)), \\
 a[5,3] &= -(Kg1/(Bg2*n2)), \\
 a[5,4] &= -(Kg2/(Bg3*n2)), \\
 a[5,5] &= ((Bg2 + Bg3)*Kg3)/(Bg2*Bg3*n2^2)
 \end{aligned}$$

and the ij-elements of the matrix $B(t)$ were obtained as:

$$\begin{aligned}
 b[1,1] &= 1, b[1,2] = 0, b[3,1] = 0, b[4,1] = 0 \\
 b[2,1] &= 0, b[2,2] = 1, b[2,3] = 0, b[2,4] = 0 \\
 b[3,1] &= 0, b[3,2] = 0, b[3,3] = Bg2^{(-1)}, b[3,4] = 0 \\
 b[4,1] &= 0, b[4,2] = 0, b[4,3] = 0, b[4,4] = Bg3^{(-1)} \\
 b[5,1] &= 0, b[5,2] = 0, b[5,3] = 1/(Bg2*n2), \\
 b[5,4] &= 1/(Bg3*n2)
 \end{aligned}$$

4.2 A NONLINEAR SYSTEM

A hydraulic reservoir system is shown in figure 4 and its analogous circuit (Q-i) is shown in figure 5. This system, consisting of a large hilltop reservoir feeding two smaller town reservoirs, is described in Athans *et alii* (1974),

where:

C_r hydraulic capacitance of the main reservoir

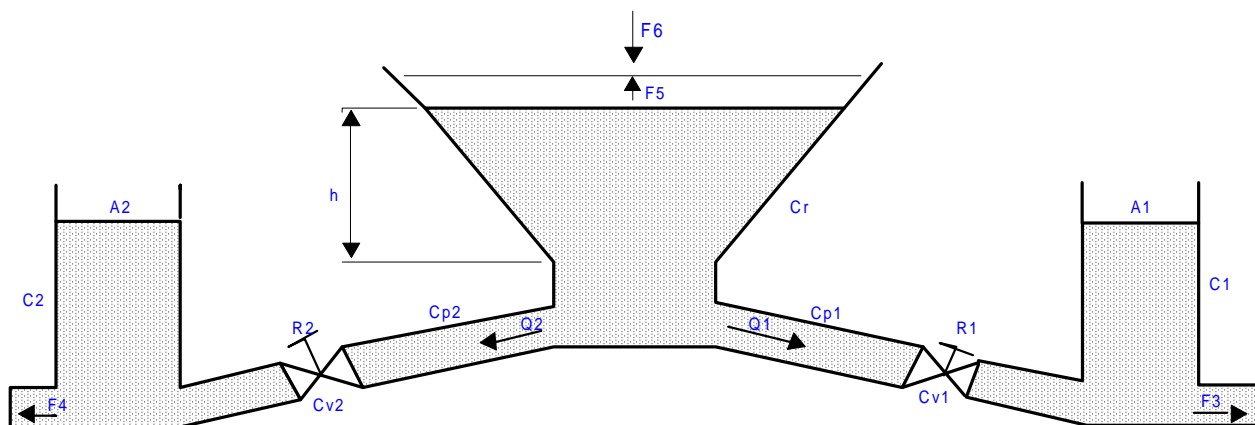


Figure 4 - Hvdraulic

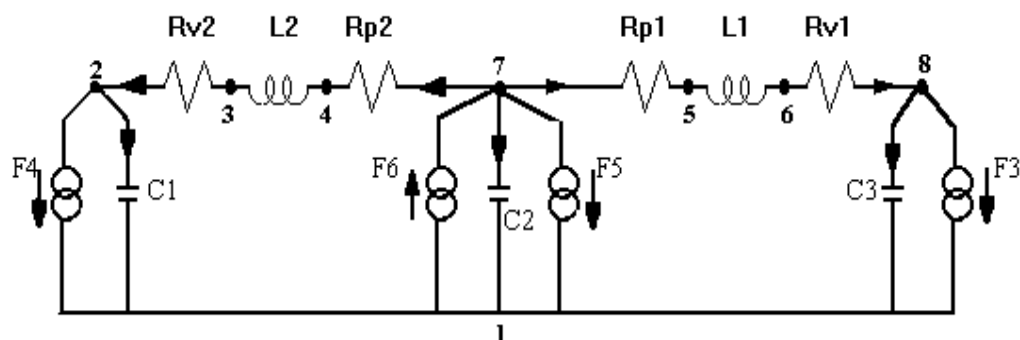


Figure 5 - Analogous Circuit

C_1	hydraulic capacitance of the reservoir 1
C_2	hydraulic capacitance of the reservoir 2
R_{v1}	hydraulic resistance of the retention valve linked between tanks 1 and main
R_{v2}	hydraulic resistance of the retention valve linked between tanks 2 and main
C_{p1}	orifice coefficient of discharge of the pipe 1
C_{p2}	orifice coefficient of discharge of the pipe 2
C_{v1}	orifice coefficient of discharge of the R_1
C_{v2}	orifice coefficient of discharge of the R_2
Q_j	water flow rate into tank [ft ³ /s] j; (j = 1, 2)
A_j	cross-sectional area of the reservoir j [ft ²]; (j = 1, 2)

The mathematical model for the water flow rate in a reservoir system:

$$Q = (C_p) \sqrt{\Delta P_{pipe}} \quad [ft^3/min] \quad \text{flow in pipe}$$

$$Q = (C_v)(r) \sqrt{\Delta P_{valve}} \quad [ft^3/min] \quad \text{flow in valve}$$

$$\Delta \Gamma = (L_f)(Q) \quad [lb/(ft \min)] \quad \text{fluid momentum in the pipe 1 and 2}$$

$$V_L = (C_f)(\Delta P) \quad [ft^3] \quad \text{volume in the reservoir 1 and 2}$$

$$V_L = f(\Delta P) \quad [ft^3] \quad \text{volume in the reservoir principal}$$

$$\frac{d}{dt} V_L = \sum_{j=1}^2 Q_j \quad [ft^3/min] \quad \text{flow in tanks}$$

where:

ΔP	pressure drop [lb/ft ²]
r	valve position ($0 \leq r \leq 1$)
V_L	volume of liquid [ft ³]
L_f	hydraulic inductance [lb/ft ⁴]
C_f	hydraulic capacitance [(ft ⁴ min ²)/lb]

Based on the oriented system graph of this hydraulic system, we can write the following set of equations:

$$F_4 + Q_{C2} - Q_{Rv2} = 0 \quad (66)$$

$$F_6 - F_5 - Q_{CR} - Q_{Rp1} - Q_{Rp2} = 0 \quad (67)$$

$$F_3 + Q_{C1} - Q_{Rv1} = 0 \quad (68)$$

$$Q_{L2} = Q_{Rp2} = Q_{Rv2} \quad (69)$$

$$Q_{L1} = Q_{Rp1} = Q_{Rv1} \quad (70)$$

$$P_{CR} - (P_{L1} + P_{Rv1} + P_{C1} + P_{Rp1}) = 0 \quad (71)$$

$$P_{CR} - (P_{L2} + P_{Rv2} + P_{C2} + P_{Rp2}) = 0 \quad (72)$$

Substituting eq. (69) in the eq. (66) and then using the capacitance constitutive equation, we obtain

$$\frac{d}{dt} P_{C2} = -\frac{1}{C_{f2}} (F_4 - Q_{L2}) \quad (73)$$

Similarly, substituting equations (69) and (70) in eq. (67), we obtain,

$$\frac{d}{dt} P_{CR} = -\frac{1}{C_R} (Q_{L1} + Q_{L2} + F_5 - F_6) \quad (74)$$

Notice that,

$$V_{CR} = f(P_{CR}) \rightarrow \frac{d}{dt} V_{CR} = Q_{CR} = C_R \frac{d}{dt} P_{CR}$$

$$C_R = \frac{\Delta}{\partial P_{CR}} f(P_{CR})$$

In the same way, substituting eq. (70) in eq. (68), we obtain

$$\frac{d}{dt} P_{C1} = -\frac{1}{C_{f1}} (F_3 - Q_{L1}) \quad (75)$$

The constitutive equation to the hydraulic inductance L_1 , is

$$\frac{d}{dt} Q_{L1} = \frac{1}{L_1} P_{L1} \quad (76)$$

Substituting eq. (71) in eq. (76) and utilizing the resistance constitutive equations, we obtain

$$\frac{d}{dt} Q_{L1} = \frac{-\left(\frac{Q_{L1}}{rc_v}\right)^2 - \left(\frac{Q_{L1}}{c_p}\right)^2 - P_{C1} + P_{CR}}{L_1} \quad (77)$$

In a similar way, utilizing eq. (72) and resistance constitutive equations into the hydraulic inductance constitutive equation L_2 , we obtain,

$$\frac{d}{dt} Q_{L2} = \frac{-\left(\frac{Q_{L2}}{rc_v}\right)^2 - \left(\frac{Q_{L2}}{c_p}\right)^2 - P_{C2} + P_{CR}}{L_2} \quad (78)$$

Equations (73)-(75), (77) e (78) can be compared with the model obtained utilizing MASD.

As commented in the previous example, data input consists in defining the system graph, in a question and answer session controlled by the user interface program (see the Appendix). A constitutive function is required in defining a nonlinear element. It is shown below the information MASD generates:

TREE BRANCHES:

Element	initial node	final node	statevariable
Cg1	8	1	vc[1]
Cg2	5	1	vc[2]
Cg3	2	1	vc[3]
Bg1	7	8	
Bg2	5	6	
Bg3	5	4	
Bg4	3	2	

LINKS (CO-TREE BRANCHES):

Element	initial node	final node	state ariable
Kg1	6	7	il[1]
Kg2	4	3	il[2]
Fa1	8	1	
Fa2	5	1	
Fa3	1	5	
Fa4	2	1	

Choosing a standardized set of state variables, we have that the state vector is composed by the “across” variables (pressures) of the hydraulic capacitance C_{g1} , C_{g2} and C_{g3} and the “through” variables (flows) of the hydraulic inductances k_{g1} and k_{g2} , respectively. A dynamic model for this system was obtained by MASD as:

$$\frac{d}{dt}vc[1][t] = \frac{-Fa1 + il[1][t]}{Cf1}$$

$$\frac{d}{dt}vc[2][t] = \frac{-Fa2 + Fa3 - il[1][t] - il[2][t]}{\frac{\partial(Vcr[vc[2][t]])}{\partial vc[2][t]}}$$

$$\frac{d}{dt}vc[3][t] = \frac{-Fa4 + il[2][t]}{Cf2}$$

$$\frac{d}{dt}il[1][t] = \frac{-\frac{(il[1][t])^2}{Cp^2} - \frac{(il[1][t])^2}{(Cv*r)^2} - vc[1][t] + vc[2][t]}{Lf1}$$

$$\frac{d}{dt}il[2][t] = \frac{-\frac{(il[2][t])^2}{Cp^2} - \frac{(il[2][t])^2}{(Cv*r)^2} - vc[3][t] + vc[2][t]}{Lf2}$$

In these expressions, vc denotes pressure and il denotes flow; Vcr represents the nonlinear function relating volume and pressure in the main reservoir.

5 CONCLUSION

Automated modelling provides greater reliability to the derived models, reduces the time and effort spent to obtain them, provides support to teaching in specific areas, and in general, allows more efficiency in the analysis studies as a whole. This paper describes a framework and the approach that are utilized in the implementation of a software, acronym MASD, for state-space automated symbolic modelling of lumped-parameters dynamic systems. Application examples illustrate the main characteristics and possibilities of this software. The main contribution of this work is the implementation of a dedicated software for mathematical modelling, as described in section 3, and some generalizations in its system theoretical framework which is represented by the equations (4)-(39)/(45)-(65).

Finally, it is worth pointing out that such a kind of a software allows one to approach more complex systems in an efficient and didactic way, also helping in teaching activities and academic investigations.

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APPENDIX

MASD accepts data input through a file or the keyboard. In this option, user interface basically is done through dialog boxes as exemplified in the figures below for the linear case:

CHOOSE THE CLASS OF THE DYNAMIC SYSTEM TO BE MODELLED

[1] - Linear System

[2] - Non Linear System

< Enter your option >:

Figure 6 - To choose the class of the dynamic system

DEFINE THE SYSTEM GRAPH

< Enter the number of nodes > :

< Enter the number of elements > :

Figure 7 - To define the topology of the system graph

CHOOSE A SET OF STATE VARIABLES TO THE SYSTEM

[1] - “Across” Variable of the Generalized Capacitor and “Through” Variable of the Generalized Inductor.

[2] - Integrated “Through” Variable of the Generalized Capacitor and Integrated “Across” Variable of the Generalized Inductor.

Enter your option:

Figure 8 - To choose the state variables

TYPE OF THE SYSTEM\SUB-SYSTEM TO BE MODELLED

[1] - Electrical/Hydraulic/Thermal System

[2] - Mechanical System

< Enter your option >:

Figure 9 - Application of the Mobility Analogy

DEFINE THE ELEMENTS OF THE SYSTEM GRAPH

[1] – Source of “Across” Variable [5] – Source of “Through” Variable

[2] – Generalized Capacitor [6] – Generalized Transformer

[3] – Generalized Resistor [7] – Mutual Inductance

[4] – Generalized Inductor

CHOOSE THE ELEMENT OF THE GRAPH:

ELEMENT :

MAGNITUDE : # INITIAL CONDITION :

INITIAL NODE : # FINAL NODE :

Figure 10 - Menu to define the linear elements

DEFINE THE SIMULATION FINAL TIME

Simulation Final Time ?

Defining simulation final time equal to zero,
the dynamic system will be modelled but not simulated.

Figure 11 - To define the simulation time

SYMBOL LIST

C_1, R_1, L_{11} Co-tree generalized capacitances, resistances and inductances diagonal matrices, respectively.

C_2, G_2, L_{22} Normal tree generalized capacitances, conductances and reciprocal inductances diagonal matrices, respectively.

$L_{12} = L_{21}^T$ Mutual inductances diagonal matrices.

$N_{12} = N_{21}^T$ Transformation ratios diagonal matrices.

F_{SC} Fundamental loop matrix for the capacitors in the co-tree (s) and the capacitors in the normal tree (c).

$F_{S\theta}$ Fundamental loop matrix for capacitors in the co-tree and the transformers ports in the normal tree (θ).

$F_{R\theta}$ Fundamental loop matrix for resistors in the co-tree (R) and the transformers ports in the normal tree.

F_{RG} Fundamental loop matrix for resistors in the co-tree (R) and the conductances in the normal tree ($G = R^{-1}$).

$F_{L\theta}$ Fundamental loop matrix for the inductors in the co-tree (L) and the transformers ports in the normal tree.

F_{LC} Fundamental loop matrix for the inductors in the co-tree (L) and the capacitors in the normal tree.

F_{LG} Fundamental loop matrix for the inductors in the co-tree (L) and the conductances in the normal tree.

$F_{L\Gamma}$ Fundamental loop matrix for the inductors in the co-tree (L) and the reciprocal inductances in the normal tree ($\Gamma = L^{-1}$).

$F_{\alpha\theta}$ Fundamental loop matrix for the transformers ports in the co-tree (α) and the transformers ports in the normal tree.

$F_{\alpha C}$ Fundamental loop matrix for the transformers ports in the co-tree and the capacitors in the normal tree.

$F_{\alpha G}$ Fundamental loop matrix for the transformers ports in the co-tree and the conductances in the normal tree.

$F_{\alpha\Gamma}$ Fundamental loop matrix for the transformers ports in the co-tree and the reciprocal inductances in the normal tree.

I Identity matrix.

n_{ra} Total number of tree branches.

n_b Total number of branches in the graph.