THE EXACT MODEL MATCHING REVISITED

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Abstract: The paper reviews three schemes for the Exact Model Matching (EMM) problem and presents two new schemes, which are logical developments of the third. A comparison is made.

1 INTRODUCTION

The exact model matching (EMM) has been studied in the control theory for a long time. Among the early studies see Wolovich and Falb (1969) and Morse (1973), more recently the problem was addressed by Ichikawa (1997), Yamanaka *et alii* (1997) and Ambrose and Qu (1997), which does not handle properly the EMM problem, but is based on it. Other studies of the EMM can be found in Chen (1984), Devasia *et alii* (1996), Ferreira (1989), Huijberts and Nijmeijer (1990), Kucera (1992), Moog *et alii* (1991), Wolovich (1974), Vardulakis and Karcanias (1985) and Kucera (1991). The concept of implementable matrices, introduced in Chen and Zhang (1985) and studied also in Ferreira (1990), is closely related to EMM. Partial model matching is studied in Kucera *et alii* (1997). Another problem related to the EMM is the so called disturbance decoupling, or zeroing the transfer matrix of a system. The expression "model matching" has been used recently in the literature in the sense of minimizing the H-infinity norm of a transfer matrix.

The EMM is important not only in itself, but because it is a basis for the model reference adaptive control problem.

In this paper we study the EMM problem for linear time-invariant finite dimensional systems. In the next section we study 5 different schemes for the solution of the problem, 3 of which have been handled in the literature. A comparison between the schemes is made in the third section. Proofs of the results are presented in the Appendix.

1.1 Notations and abbreviations

The ring of proper and stable rational functions will be denoted by **S**. It is, in fact, a principal ideal domain (Vidyasagar, 1985). The set of matrices with elements in **S** will be denoted by **S** also, regardless of the dimensions of the matrix. Right coprime will be abbreviated by r.c., left coprime by l.c., such that by s.t., necessary and sufficient by n.a.s.

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2 EMM: FIVE DIFFERENT SCHEMES

The EMM problem is defined as follows: given a plant, find one or more compensators s.t. the controlled output of the (closed-loop) system matches the output of a given model, whatever be the model's input.

More specifically, let $M(s)$ be the (known) transfer matrix of the model, assumed to be proper and stable for all practical purposes, let $P(s)$ be the known transfer matrix of the plant assumed to be proper also. (As we will see, in the second scheme we do not need to assume either that $M(s)$ is proper and stable or that it is known).

Let $z(s)$, $y(s)$ and $u(s)$ denote the controlled output, the measured output and the input of the plant, respectively. Let $P(s)$ be partitioned accordingly, namely, $P(s)=\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix}$ where $P_2(s)$ is assumed strictly proper for convenience in terms of well-posedness of the closed loop. (This restriction is met very often, if not almost always, in practice, but could be dropped easily).

Our goal is to design a system in which the controlled output $z(s)$ matches the output of the model, whatever be the model's input.

Let $w(s)$ be the exogenous signal to the system.

Whatever be the combination of compensators, its output must be $u(s)$, the input to the plant and its inputs must be $w(s)$ and $y(s)$, the measured output of the plant. So, as pointed out by Vidyasagar (1985), the most general equivalent linear compensator must be

$C(s) = [\, C_1(s) \, , - C_2(s)]$, such that

$$u(s) = [\, C_1(s) \, , - C_2(s)] \begin{bmatrix} w(s) \\ y(s) \end{bmatrix}.$$

We have then the following block diagram of the closed loop system:

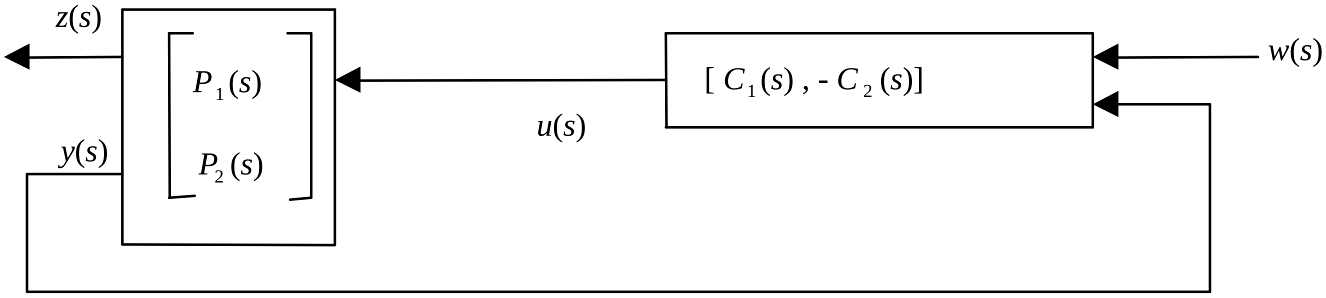


Fig 1

Let $N_1(s)$, $N_2(s)$, $D(s) \in$ **S** be s.t.

$\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} D^{-1}(s)$ - is a right coprime (r.c.) factorization.

Analogously, let $D_c(s)$, $N_{c1}(s)$, $N_{c2}(s) \in$ **S** be s.t.

$[C_1(s), - C_2(s)] = D_c^{-1}(s)\,[N_{c1}, - N_{c2}(s)]$ is a left coprime (l.c.) factorization.

It is well known (see Vidyasagar (1985), among many others) that the closed loop is stable if and only if there exists a unimodular matrix $U(s)$ s.t.

$$D_c(s)\,D(s) + N_{c2}(s)\,N_2(s) = U(s).$$

Redefining, $D_c(s)$ and $N_{c2}(s)$ (and, consequently, $N_{c1}(s)$ also), we have, without loss of generality:

$$D_c(s)\,D(s) + N_{c2}(s)\,N_2(s) = I\,, \qquad (1)$$

where I is the identity matrix, showing, by the way, that $N_2(s)$ and $D(s)$ have to be r.c., while $D_c(s)$ and $N_{c2}(s)$ have to be l.c.

Let us proceed now to the study and comparison of 5 schemes for the EMM.

2.1 First scheme: the "off-line" solution

This is the scheme studied by most authors, actually the EMM is defined by some authors in the strict sense of this scheme, namely, we do not feed either input or output of the model into the closed loop system. Some authors (for example Kucera, 1991) use state-feedback of the plant.

From figure 1 (or the equations which define it) it is easy to obtain (see the Appendix):

$$z(s) = N_1(s)\,N_{c1}(s)\,w(s). \qquad (2)$$

Then, it is clear that EMM is obtained if and only if there exists $N_{cl}(s)$ such that

$$N_1(s)\,N_{c1}(s) = M(s)\,. \qquad (3)$$

Clearly, the EMM is solvable if and only if $N_1(s)$ is a left divisor of $M(s)$.

Notice that $D_c(s)$ and $N_{c2}(s)$ are not involved in the above equation, so the so called "Youla – Kucera parameter" is free.

Notice also that $M(s)$ has to be proper and stable if (3) is to have a solution, an assumption which was made in the Introduction.

Let $N_{c1p}(s)$ be a particular solution of (3). Then it is clear that the set of all solutions of (3) is given by

$$N_{c1}(s) = N_{c1p}(s) + N_{c1h}(s)\,, \qquad (3a)$$

where $N_{c1h}(s) \in$ **S** is any matrix in the (right) kernel of $N_1(s)$.

2.2 Second scheme: Perfect Tracking

This scheme has been addressed recently by Ichikawa (1997), Yamanaka *et alii* (1997) and Devasia *et alii* (1996) and, less recently, by Ferreira (1989).

Here, the model's output is fed into the compensator as its exogenous input, $r(s)$.

With $z(s) = w(s)$, we get immediately from (2):

$$N_1(s)\,N_{c1}(s) = I\,. \qquad (4)$$

From this, EMM problem is solvable if and only if $N_1(s)$ is left unimodular.

In this scheme we do not need to know the model, which could be unstable, improper and even nonlinear, provided of course, in this case, that the model's output is Laplace –transformable.

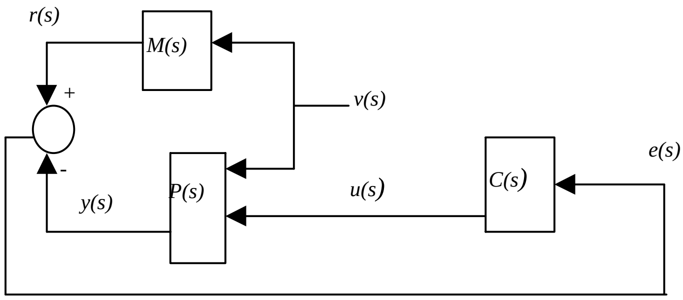
In this scheme we have a "model's output matching" rather than a model matching. But of course, the result is the same, since in the definition of model matching the goal is to match the output of the model, whatever be its input, which certainly happens in this scheme.

As in the previous scheme, the Youla - Kucera parameter is free.

Let $N_{c1p}(s)$ be a particular solution of (4). Then the set of all solutions of (4) has the same form as (3a), N_{c1h}(s) having the same meaning

2.3 Third scheme: Wolovich scheme

The scheme of the next block diagram (fig.2) is proposed only by Wolovich (1974) and in some paper of that author, to the best of our knowledge.



(Figure 2)

$C(s)$ is to be chosen s.t. for any $v(s)$, we have $y(s) = r(s)$. It is clear that to implement this scheme, the plant must have more inputs than the model.

Let $P(s) =: [\,\overline{P}_1 \;,\quad \overline{P}_2\,]$ be an appropriate partition, let

$C(s) =: D_c^{-1}(s)\, N_c(s)$ be a l.c. factorization and let $\overline{P}_2(s) = \overline{N}_2(s)\overline{D}_2^{-1}(s)$, a r.c. factorization

In the Appendix we prove the following necessary and sufficient condition

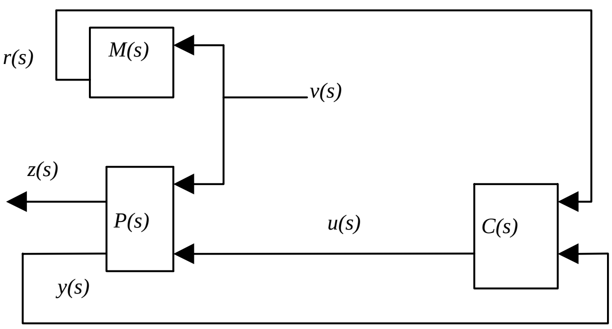
$$(I \;-\; \overline{N}_2(s)\; N_c(s))\; (M(s) \;-\; \overline{P}_1(s)) \;=\; 0\,. \qquad (5)$$

It is clear from the above equality that the left unimodularity of $\overline{N}_2(s)$ is a sufficient (not necessary) condition for the solvability of the problem. Indeed, if this is the case, we choose $N_c(s)$ s.t. $\overline{N}_2(s)\; N_c(s) = I$.

We have no free Youla – Kucera parameter in this case, since $N_c(s)$ is restricted to (5).

2.4 Fourth scheme: generalization of Wolovich scheme

In view of a comparison with the other schemes, let us generalize Wolovich's scheme, assuming that in the plant the controlled output is distinct from the measured output. Besides, in order to improve the controllabitly of the system, we use, like in the other cases, a two-input compensator. This generalization is not studied in the literature, to the best of our knowledge. So we have the following block diagram (fig. 3)



(Figure 3)

Partition $P(s)$ in the obvious way:

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$

Let $P_{22}(s) = N_{22}(s)\; D_{22}(s)^{-1}$, a r.c. factorization, where $D_{22}(s)^{-1}$ incorporates all the unstable poles of $P(s)$ (see Desoer and Gündes (1988) for details).

In the Appendix it is proved that the n.a.s. condition for the solution of the problem is

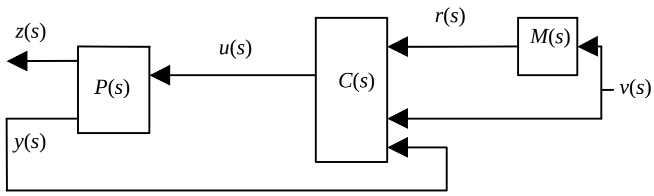
$$(I - N_{12}(s)\, N_{c1}(s))\; M(s) = P_{11}(s) - N_{12}(s)\, N_{c2}(s)\, P_{21}(s), \quad (6)$$

where $N_{12}(s)$ is defined in the proof (Appendix).

Contrary to the previous cases, this equation is not insightful, no simple sufficient condition is available from it, either. Notice also that no free Youla - Kucera parameter is available, since $N_{c2}(s)$ is restricted to (6).

2.5 Fifth scheme

Feeding the model's input into the plant, as in the previous scheme postulates a larger number of inputs in the plant. So, it does make sense to feed the model's input into the compensator. As we will see, we get a "nicer" n.a.s. condition for the solvability of the problem. The following scheme (fig. 4) has not been studied before in the literature, to the best of our knowledge.



(Figure 4)

The exogenous signal to the two-degree-of freedom compensator is now $\begin{bmatrix} r(s) \\ v(s) \end{bmatrix}$

Partition $C_1(s)$ appropriately, obtaining

$$\begin{aligned} C(s) &= [C_1(s), -C_2(s)] = [C_{11}(s), C_{12}(s), -C_2(s)] \\ &= D_c(s)^{-1}\,[N_{c11}(s), N_{c12}(s), -N_{c2}(s)]\,, \end{aligned}$$

a l.c. factorization.

We obtain the following n.a.s. condition (see the proof in the Appendix):

$$N_1(s)\,(\,N_{c11}(s)\, M(s) + N_{c12}(s)\,) = M(s)\,, \qquad (7)$$

A n.a.s. solvability condition for this equation is that $N_1(s)$ is a left divisor of $M(s)$.

Indeed, necessity is obvious. For sufficiency, let $X(s)$ be a solution of $N_1(s)\, X(s) = M(s)$. Then the equation

$$N_{c11}(s)\, M(s) + N_{c12}(s) = [N_{c11}(s)\,, N_{c12}(s)] \begin{bmatrix} M(s) \\ \mathbf{I} \end{bmatrix} = X(s) \qquad (8)$$

has always a solution for $N_{c11}(s)$ and $N_{c12}(s)$ because $\begin{bmatrix} M(s) \\ \mathbf{I} \end{bmatrix}$ is right unimodular.

Notice also that in this scheme the Youla – Kucera parameter is free.

Let $[\,N_{c11p}(s)\,, N_{c12p}(s)\,]$ be a particular solution of (8). Then it is clear that the general solution of (8) is

$$[\,N_{c11}(s)\,, N_{c12}(s)\,] = [\,N_{c11p}(s)\,, N_{c12p}(s)\,] + Q(s)\,[\,I\,, -M(s)\,]\,,$$

$Q(s)$ being any matrix in **S** .

On the other hand, let $N_h(s) \in$ **S** be any matrix in the (right) kernel of $N_1(s)$ and let

$[\ N_{c11h}(s)\ ,\ N_{c12h}(s)\ ]$ be a particular solution of

$$[\ N_{c11}\ (s)\ ,\ N_{c12}\ (s)\ ] \begin{bmatrix} M(s) \\ I \end{bmatrix} = N_h\ (s)\ .$$

It is clear that the general solution of this equation is

$[\ N_{c11}(s)\ ,\ N_{c12}\ (s)\ ] = [\ N_{c11h}\ (s)\ ,\ N_{c12h}\ (s)\ ] + Q(s)\ [\ I\ ,\ -\ M(s)\ ]$,

$Q(s)$ being any matrix in **S**.

Summing up, the general solution of (7) is

$[\ N_{c11}(s)\ ,\ N_{c12}\ (s)\ ] = [N_{c11p}\ (s) + N_{c11h}\ (s)\ ,\ N_{c12p}\ (s) + N_{c12h}\ (s)] + Q(s)\ [I\ ,\ -\ M(s)]$,

Q(s) being any matrix in **S**, N_{c11h} (s) and N_{c12h} (s) referring to a general N_h (s).

3 A COMPARISON OF THE SCHEMES

Which of the schemes is to be preferred?

3.1 If the model's output is not accessible, we have to pick the first scheme, provided its solvability condition is satisfied.

3.2 If the model's output is accessible, the second scheme is clearly the best solution, provided its solvability solution is satisfied, since the model need not be known. This scheme is extremely "robust", as far the model is concerned.

3.3 If the solvability condition of the second scheme is not satisfied, but that of the first scheme is, we might as well pick the fifth scheme which has the same solvability condition, provided that both input and output of the model are accessible. Notice however that the first and the fifth schemes have different sensitivities with respect to the difference between the real model and the nominal one, as shown next.

Let $M(s)$ be the nominal model and $M^*(s)$ the real one. Omitting the argument (s), we have:

In the first scheme, the output of the real model is

$z^* = M^*\ v$, while the output of the closed loop system based on the nominal model is

$z = N_1\ N_{c1}\ v = M\ v$.

So the difference is

$$z^* - z = (M^* - M)\ v =: \Delta_M\ v \tag{9}$$

On the other hand, in the fifth scheme, the output of the closed loop system with the real model is

$z = N_1\ (\ N_{c11}\ M^* + N_{c12}\ )\ v = N_1(\ N_{c11}\ M + N_{c12}\ )\ v + N_1\ N_{c11}\ \Delta_M\ v =$

$(M^* - \Delta_M\ + N_1\ N_{c11}\ \Delta_M\ )\ v = M^*\ v + (\ N_1\ N_{c11}\ - I\ )\ \Delta_M\ v$.

$$\therefore\ z^* - z = (\ I - N_1\ N_{c11}\ )\ \Delta_M\ v \tag{10}$$

Comparing this with (9), it is clear that if there exists N_{c11} s.t.

$\| I - N_1 N_{c11} \|_\infty < 1$,

the fifth scheme would be employed with advantage over the first, as far sensitivity with respect to Δ_M is concerned. This result is not surprising, since in the fifth scheme the model is imbedded into the loop. It might be added that the norm of $I - N_1\ N_{c11}$ might not be smaller than 1 in the whole range of frequencies, and still be much less than one in the band of interest, in which case choosing the fifth scheme would make sense. An example is given next.

Example:

Let a plant be s.t. $N_2(s)$ and $D(s)$ are r.c. and $N_1(s) = (s - 10)(s + 10)^{-1}$, and let the

model be $M(s) = (s - 10)(s + 20)^{-1}$. It is clear that the solvability condition of the second scheme is not satisfied, but that of the first and fifth scheme is.

$N_{c2}(s)$ and $D_c(s)$ are to be chosen s.t. (1) is satisfied. Using the first scheme, (3) gives the solution

$N_{c1}(s) = (s + 10)(s + 20)^{-1}$.

Using the fifth scheme, one possible solution of (7) is

$N_{c11}(s) = (s - 12)(s + 12)^{-1}$,

$N_{c12}(s) = 44s\ (\ (s + 12)(s + 20)\ )^{-1}$. Hence,

$1 - N_1(jw)\ N_{c11}(jw) = j44w\ (120 - w^2\ + j22w)^{-1}$, whose module is

$44w\ (w^4\ + 224w^2\ + 14400)^{-1/2}$, which is equal to 1 if $w \cong 2.9$.

So, if the band of interest is pretty below $w = 2.9$, it would make sense to use the fifth scheme. If this were not the case, the most commonly used scheme, the first, should be picked up

3.4 If the conditions for the solvability of the first (and fifth) and second scheme are not satisfied, we should check the fourth scheme, as shown in the following

Example:

The model to be followed is $M(s) = (s - 2)(s + 1)^{-1}$. The plant is

$$P(s) = \begin{bmatrix} (s^2 + 2s - 5)(s+1)^{-2} & (s-1)(s+1)^{-1} \\ 1 & (s+2)^{-1} \end{bmatrix}$$

Since the plant is stable, we have

$N_1(s) = P_1(s) = [(s^2\ + 2s - 5)(s + 1)^{-2}\ ,\ (s - 1)(s + 1)^{-1}\ ]$,

which is neither left unimodular nor a left divisor of $M(s)$, so there are no solutions to (4), (3) or (7): the problem is not solvable through the first, second or fifth schemes. Let us try the fourth:

It is clear that $P_{ij}(s) = N_{ij}(s)$; i , j = 1, 2 , are the four elements of $P(s)$ above. It is straightforward to verify that equation (6) is satisfied with $N_{c1} = N_{c2} = 1$, the EMM problem being solved with $C = [1 , 1]$.

4 CONCLUSION

A review of the EMM problem was made, with the introduction of two new schemes for the solution of it. The solvability condition was presented for each of the schemes and the general solution for three of the schemes was developed. A comparison was made between the schemes, pointing out which should be used in different situations. Comparing two of the schemes (the first and the fifth) the robustness issue was introduced with respect to the difference between the nominal model and the real one. The robustness issue with respect to perturbations of the plant should be an issue in the further development of the research.

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APPENDIX

In what follows we omit the argument (s) of the Laplace transformed functions.

Proof of (3), the "off-line" solution

From figure 1, $u = C_1 w - C_2 y = C_1 w - C_2 P_2 u \therefore (I + D_c^{-1} N_{c2} N_2 D^{-1}) u = D_c^{-1} N_{c1} w \therefore D_c^{-1}(D_c D + N_{c2} N_2) D^{-1} u = D_c^{-1} N_{c1} w \therefore u = D N_{c1} w$ $\therefore z = P_1 u = N_1 N_{c1} w \therefore N_1 N_{c1} = M.$

Proof of (5), Wolovich's scheme

From fig. 2, $u = Cr - Cy = Cr - C(\overline{P}_1 v + \overline{P}_2 u) \therefore$ $(I + C\overline{P}_2)u = Cr - C\overline{P}_1 v$
$\therefore D_c^{-1}(D_c \overline{D}_2 + N_c \overline{N_2})\overline{D_2}^{-1} u = D_c^{-1}(N_c r - N_c \overline{P_1} v)$. In view of the fact that
$D_c \overline{D_2} + N_c \overline{N_2} = I$, we have $u = \overline{D_2} N_c r - \overline{D_2} N_c \overline{P_1} v$; $y = \overline{P_1} v + \overline{P_2} u$
$= \overline{P_1} v + \overline{N_2} N_c r - \overline{N_2} N_c \overline{P_1} v = (I - \overline{N_2} N_c) \overline{P_1} v + \overline{N_2} N_c r.$
But from $r = Mv$ and $y = r$, we have:
$Mv = (I - \overline{N_2} N_c) \overline{P_1} v + \overline{N_2} N_c M v \therefore (I - \overline{N_2} N_c)(M - \overline{P_1}) v = 0 \quad \forall v$. Hence the result.

Proof of (6), generalizing Wolovich's scheme

From the figure 3, we have: $u = C_1 r - C_2 (P_{21} v + P_{22} u)$ $\therefore (I + C_2 P_{22}) u = C_1 r - C_2 P_{21} v.$

Now, from Desoer and Gündes (1988) we have the "canonical" r.c. factorization of $P:\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ Y & P_{22} \end{bmatrix}^{-1}$,
for some $Y \in$ **S**.
Then, we have:
$(I + D_c^{-1} N_{c2} N_{22} D_{22}^{-1}) u = D_c^{-1} N_{c1} r - D_c^{-1} N_{c2} P_{21} v$
$\therefore$ $u = D_{22}(N_{c1} r - N_{c2} P_{21} v)$ $\therefore$ $z = (P_{11} - N_{12} N_{c2} P_{21}) v + N_{12} N_{c1} r$.
But from $r = M v$ and $z = r$, we have
$M v = P_{11} v + N_{12} N_{c1} M v - N_{12} N_{c2} P_{21} v \quad \forall v$,
which is equivalent to
$(I - N_{12} N_{c1}) M = P_{11} - N_{12} N_{c2} P_{21}$.

Proof of (7)

From the figure 4, we have $u = C_{11} r + C_{12} v - C_2 P_2 u$
$\therefore (I + D_c^{-1} N_{c2} N_2 D^{-1}) u = C_{11} r + C_{12} v$
$\therefore u = D N_{c11} r + D N_{c12} v$; $z = P_1 u$;

$r = M v$, $z = r$ $\therefore$ $M v = N_1 N_{c11} M v + N_1 N_{c12} v$ $\forall v$. Hence the result.