MINIMUM FUEL TRAJECTORIES FOR THE LUNAR POLAR ORBITER

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Abstract In this paper the problem of sending a spacecraft from Low Earth Orbit (LEO) to the Moon with minimum fuel consumption is considered. The goal is to find the control to be used to perform the tasks involved in the mission that minimizes the fuel expenditure required. The first part of this paper deals with impulsive maneuvers, where the control consists in the application of two impulsive thrusts that changes instantaneously the velocity of the spacecraft, and obtain a set of values for the fuel expenditure and trip time for several trajectories. The second part considers the use of low-thrust trajectories, where the control consists of chosen the instants to start and to stop the application of a finite thrust, as well as the direction of this thrust at every instant of time. The results showed that large savings in fuel consumption could be obtained by using low thrust trajectories for the Earth-Moon part of the mission. A first analysis for the behavior of the orbit around the Moon, including predictions for the frequency and fuel expenditure for the maneuvers to keep the orbit close to the nominal trajectory, is also included.

1 INTRODUCTION

The establishment of a manned lunar base is certainly one of the next big step of the mankind in its journey to the space. To accomplish this goal, a more detailed study of the Moon, including its mineral resources and physical properties, has to be done. It is in this context that the Lunar Polar Orbiter mission appears. It is constituted by one or two spacecrafts in Moon's polar orbit, to make measurements in the Moon's surface and neighborhood. The data obtained will be used for several important tasks, like: site selection of the lunar base; improvements of trajectory calculation around the Moon, study of possible mineral exploitation, etc.

The objective of this paper is to make a preliminary study of the possible trajectories to be used to go to the Moon. It is assumed that the spacecraft begins its trip in Low Earth Orbit (LEO) of about 200 kms above the surface of the Earth and that the equations given by the Two-Body non-perturbed dynamics are valid for each phase of the mission. This dynamics assumes that the only force acting in the satellite is the gravitational force of the Earth, and that the Earth can be considered a point of mass (Bate, 1973, chapter 1). Two scenarios are studied: in the first one a single spacecraft will orbit the Moon; and in the second one the main spacecraft will have a sub-satellite in a higher orbit. It is assumed that this dynamical system can be studied with the assumptions that the only forces involved are the gravitational ones of the bodies involved and the thrust of the space vehicle. The final orbit around the Moon is assumed to be polar. A more detailed study, including the velocity variation required in each phase and an analysis of the effects of the errors involved in the final orbit around the Moon is also presented for one of the trajectories.

Special attention is given to the difference in fuel consumption obtained by using two options of thrust for the Earth-Moon transfer: Infinite (using Hohmann Transfer, like shown in Bate, 1973, pages 163-166) and Low Thrust (using optimal control theory, like in Biggs, 1979 and Prado and Rios-Neto, 1989). For the low thrust maneuver, the Euler-Lagrange equations are used to generate a set of differential equations that are numerically integrated to obtain the final orbit. The difficulty caused by a lack of initial values for all variables in the same point (Two Point Boundary Value Problem) is treated by making iterations in the initial values of the Lagrange multipliers, using the gradient projection method (Bazaraa and Shetty, 1979).

2 IMPULSIVE MANEUVERS

In this case it is assumed that an infinite thrust acting during a negligible time can change the velocity of the spacecraft instantaneously. It is also assumed that the spacecraft will leave the Earth from a circular parking orbit with an altitude of 200 km. The main goal is to obtain the values of the velocity increment, trip time and mass of fuel required for many trajectories, to select one of them for a more detailed analysis. The two scenarios considered here are (International Space University, 1989):

1) A single mission with the spacecraft in a circular orbit around the Moon with an altitude of 100 km and 90 degrees of inclination;

2) A double mission with the two spacecrafts in different orbits:

- The main spacecraft in a circular orbit around the Moon with an altitude of 100 km and 90 degrees of inclination; and
- A sub-satellite (with no engines to perform maneuvers) in a elliptical orbit around the Moon with semi-major axis of 3000 km, eccentricity of 0.37, argument of periapsis of 0.25 degrees West and inclination of 90 degrees.

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Using the above described "Two-Body model" approximation (Bate, 1973, chapter 1) for the Earth-spacecraft system, it is possible to obtain the trip time for different trajectories, all of them assumed to be elliptical with periapsis of 6570 km. The results are shown in Table 1 and Fig. 1 shows a sketch of the trajectories. The eccentricity is calculated by the equation $e = 1 - \frac{r_p}{a}$ (Bate, 1973, chapter 1), where a is the semi-major axis and r_p is the periapsis distance (6570 km) of the orbit. The trip time is half of the period of the orbit, and it is calculated from $T = \frac{\pi a^{3/2}}{\sqrt{\mu}}$ (Bate, 1973, chapter 1), where μ is the gravitational parameter of the Earth (the product of the mass of the Earth by the universal gravitational constant).

Table 1 - Orbital parameters and trip time for different trajectories to the Moon

Orbit	Semi-major axis (km)	Eccentricity	Trip Time in hr. (days)
1	500000	0.986	58.5 (2.43)
2	400000	0.983	61.0 (2.54)
3	300000	0.978	67.0 (2.80)
4	250000	0.974	73.9 (3.08)
5	230000	0.971	77.0 (3.21)
6	220000	0.970	83.2 (3.47)
7	200000	0.967	100.0 (4.17)
8	195485	0.966	119.6 (4.98)

Using the impulsive approximation for the control, it is possible to evaluate the velocity increment necessary to send the spacecraft into Lunar Transfer Orbit (LTO). Four different

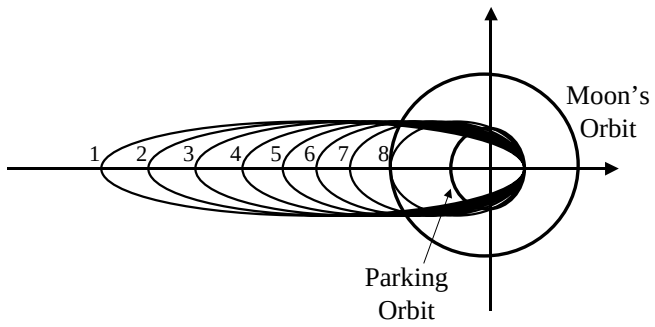

Fig. 1 – Sketch of the orbits showed in Table 1.

orbits were chosen for detailed calculations: 2, 6, 7 and 8. The results are shown in Table 2. The apoapsis distance (r_a) is calculated by the equation $r_a = a(1+e)$ (Bate, 1973, chapter 1) and the velocity increments are calculated by $\Delta V = \sqrt{\frac{\mu r_a}{a r_p}} - \sqrt{\frac{\mu}{r_p}}$ (International Space University, 1989).

Table 2 - Velocity increment for LTO

Orbit	Apoapsis distance (km)	ΔV (km/s)
2	793430	3.18
6	433430	3.14
7	393430	3.13
8	384400	3.11

Then, the patched conic method (Bate, 1973) is used to calculate the maneuver to insert the spacecraft into the Moon's orbit. It divides the trajectories in two parts: a) The first leg neglects the effect of the Moon and the Hohmann method can be used to transfer the spacecraft from its original parking orbit to an orbit that crosses the Moon's path; b) When the spacecraft reaches a position where the Moon's gravity field dominates its motion (sphere of influence of the Moon), the Earth's effects are neglected and the orbit is studied as a Keplerian lunar orbit. Using this approximation, the spacecraft arrives at the sphere of influence of the Moon with hyperbolic excess velocity, that is a velocity so high that does not allow the spacecraft to stay in orbit around the Moon. Then, it is necessary to apply a retrograde impulse to decrease its velocity to achieve an elliptic lunar orbit. Using the basic equations from the Two-Body model described above for the Moon-spacecraft system, it is possible to obtain the velocity of the spacecraft with respect to the Moon (assumed to be in circular orbit around the Earth) and the velocity decrement required. At this point it is necessary to consider the maneuvers in two scenarios, because the velocity decrement depends on how many spacecrafts are in the mission. If there is only one spacecraft, it is possible to assume that with small mid-course corrections it can achieve a hyperbolic arrival at the Moon with the desired periapsis altitude (1840 km) and orbit inclination (90 degrees). Then it is necessary to apply only one impulse, at the periapsis, to obtain the desired circular orbit. The velocity decrements are shown in Table 3. The mathematical expression to obtain the values is: $\Delta V = \sqrt{V_\infty^2 - \frac{2\mu_m}{r}} - \sqrt{\frac{\mu_m}{r}}$, where μ_m is the gravitational parameter of the Moon, V_∞ is the velocity of the spacecraft with respect to the Moon when the approaching starts and r is the radius of the circular final orbit around the Moon.

Table 3 - Velocity decrement to insert one probe into Moon's orbit

Orbit	ΔV (km/s)
2	1.07
6	0.91
7	0.88
8	0.79

If there are two spacecrafts, it will be necessary to use a more complex maneuver, because the sub-satellite has no engine. In this case, the insertion into Moon's orbit will be done with both spacecrafts together, in the orbit desired for the sub-satellite. After the separation of the two spacecrafts, the primary spacecraft will be transferred to its final orbit. Assuming that the insertion is performed at the periapsis of the elliptical orbit of the sub-satellite and that, after separating from the sub-satellite, the main spacecraft will be transferred to its final orbit using a bi-impulsive Hohmann Transfer, the results for the ΔV required can be calculated. They are shown in Table 4, where ΔV_i is the velocity decrement for lunar insertion of both spacecrafts together and ΔV_t is the total ΔV to transfer the main spacecraft to its final orbit. The insertion is performed at the periapsis of the elliptical orbit because this is the point that requires the minimum amount of fuel. The velocity decrement for this phase of the mission is given by: $\Delta V = \sqrt{V_\infty^2 - \frac{2\mu_m}{r_{ps}}} - \sqrt{\frac{2\mu_m}{r_{ps}} - \frac{\mu_m}{2a_s}}$, where r_{ps} is the radius of the periapsis of the orbit of the sub-satellite and a_s is the semi-major axis of this orbit.

The Hohmann transfer is the solution for a bi-impulsive transfer between two circular and coplanar orbits. It was

created by Hohmann (1925). It is the most used result in orbital maneuvers. The transfer is as follows:

a) In the initial orbit a $\Delta V_0 = V_0 \left| \sqrt{\frac{2(r_f/r_0)}{(r_f/r_0)+1}} - 1 \right|$ (where r_0 (r_f) is the radius of the initial (final) orbit and V_0 is the velocity of the spacecraft when in its initial orbit) is applied in the direction of the motion. With this impulse the spacecraft is inserted into an elliptical orbit with periapsis r_0 and apoapsis r_f;

b) The second impulse is applied when the spacecraft is at the apoapsis. The magnitude is $\Delta V_f = V_0 \left| 1 - \sqrt{\frac{2}{(r_f/r_0)+1}} \right| \sqrt{r_0/r_f}$ and it circularizes the orbit.

Table 4 - Velocity decrement to insert two probes in Moon's orbit

Orbit	ΔV_i (km/s)	ΔV_t (km/s)
2	0.78	0.29
6	0.63	0.29
7	0.60	0.29
8	0.50	0.29

The total consumption (launch from Earth and insertion around the Moon) for both approaches (ΔV_1 for a single mission and ΔV_2 for a double mission) are summarized in Table 5. To consider errors in the models involved and midcourse corrections, 10% was added to the ΔVs.

Table 5 - Total ΔV for both missions

Orbit	ΔV_1 (km/s)	ΔV_2 (km/s)
2	4.68	4.68
6	4.45	4.47
7	4.41	4.42
8	4.30	4.30

With the total ΔVs obtained, it is possible to calculate the fuel mass required for the mission. Three different values for the specific impulses (I_{sp}) were assumed and 10% was added to include the mass of the hardware required for storage (Burke, 1989). The results are shown in Tables 6 (single mission) and 7 (double mission). In this last case, the difference in mass between the set of the two spacecrafts and the main spacecraft itself was neglected, since the mass of the sub-satellite will be very small. The equation used to convert ΔV into mass of fuel is $m_0 - m_f = m_0 \left(1 - \exp(\frac{-\Delta V}{I_{sp} g_0}) \right)$, where I_{sp} is the specific impulse of the fuel used, g_0 is the acceleration due to Earth's gravity at sea level and m_0 is the initial mass of the spacecraft.

Table 6 - Mass of fuel required for a single mission

Isp(s) \ Orbit	2	6	7	8
240	6.95	6.20	6.07	5.75
290	4.61	4.17	4.09	3.90
340	3.38	3.08	3.03	2.90

Table 7 - Mass of fuel required for a double mission

Isp(s) \ Orbit	2	6	7	8
240	6.95	6.26	6.10	5.75
290	4.61	4.20	4.11	3.90
340	3.38	3.12	3.04	2.90

It is possible to have some alterations in the trajectory for many reasons. If it is necessary to change the orbit during the mission or, if a different parking orbit (or no parking orbit) around the Earth is used, the optimal trajectory may change. However, these results are still valid because these alterations do not affect the ΔV, trip time or mass of fuel consumed by a significant amount. The literature (Cornelisse, 1980) also confirms these values.

The benefits of a short trip time are that it requires less time of tracking and has more safety (because there is some excess in velocity to compensate errors). The price to be paid for these benefits is that it requires more fuel. Considering these facts and the data available it is necessary to find the most economical orbit inside the range of safety, based on previous experience. Following a suggestion from Burke (1989), orbit 6 was considered to be a good choice for a more detailed study.

3 TRAJECTORY ANALYSIS

With one nominal trajectory found, it is necessary to make a more detailed analysis of all the maneuvers involved. This analysis should include ΔV estimation, timing for mid-course corrections and predictions of the effects of the errors in direction and magnitude of the impulses applied in each phase of the mission. The same approach of studying two different scenarios (one or two spacecrafts) will be used.

The first part of the analysis is about the mid-course corrections. After the launch, it is necessary to wait some time to make an accurate orbit determination and then apply the first impulse to correct the trajectory. According to the study done by the Jet Propulsion Laboratory (JPL, 1976), this first maneuver (that is used to correct errors during the launch phase) is expected to require about 40 m/s in ΔV (80 m/s in the worst case) and should be done about 10 hours after launch. The same document estimates that, with this first correction, the errors in the periapsis altitude of the final orbit around the Moon will be between 500 and 800 km, which is unacceptable, due to the high risk of having a collision with the Moon (the nominal periapsis altitude is only 100 km). Then, it is necessary to make one or two more corrections to obtain an acceptable value for this error. In the following calculation it is assumed that a second correction will occur between 20 h and 40 h after launch and the magnitude will be between 5 and 15 m/s, depending on the errors left by the first maneuver. The third maneuver is scheduled to occur about 20 h before the insertion (64 h after launch) and the range in magnitude expected is between 5 and 10 m/s. In spite of the small magnitude involved, this maneuver is very important to avoid a collision with the Moon and to obtain the maximum efficiency of the impulse applied during the orbit insertion.

To have an idea of the errors involved, the errors in periapsis altitude after the third maneuver will be calculated. It is assumed that the maneuver will take place on the incoming asymptote of the lunar approach hyperbola. In this moment, the velocity of the spacecraft relative to the Moon (V_r) is (assuming a 3.47 days trip):

$$V_r = \sqrt{2E} = 1058 \text{ m/s} \quad (1)$$

which corresponds to the following set of orbital parameters:

$$a = -\frac{\mu_m}{2E} = -4377.7 \text{ km} \quad (2)$$

$$e = 1 - \frac{r_p}{a} = 1.42 \quad (3)$$

$$b = -a\left(e^2 - 1\right)^{1/2} = 4415.4 \text{ km} \quad (4)$$

where E is the total energy by mass of the spacecraft (km^2/s^2), a is the semi-major axis of the lunar approach hyperbola (km), μ_m is the gravitational parameter of the Moon (4903 km^3/s^2), e is the eccentricity, r_p is the altitude of periapsis (km), and b is the parameter of impact (km), as shown in Fig. 2.

Assuming errors of 0.5 degrees in pointing, 0.5% in velocity increment, 0.1 m/s resolution, 7 m/s in magnitude for the maneuver, and that the impulse will be applied normal to the velocity (to obtain maximum change in periapsis altitude and to represent the worst case for periapsis error propagation) we have, for the total errors expected in velocity increment ($\delta\Delta V_n$) (JPL, 1976).

$$\delta\Delta V_n = \left[(p\Delta V_n)^2 + R^2 + (\Delta V_n \sin(\delta\alpha))^2\right]^{1/2} = 0.122 \text{ m/s} \quad (5)$$

where p is the proportional error in velocity (0.5%), ΔV_n is the velocity increment to be applied in the maneuver (7 m/s), R is the resolution (0.1 m/s), and $\delta\alpha$ is the precision in pointing (0.5 degree). This error in the normal direction will cause an error in the impact parameter (δb) approximately equal to:

$$\delta b = r_{\Delta V}\sin(\delta B) = 38800\sin(0.122/1058) = 4.47 \text{ km} \quad (6)$$

where δB is the angular error in V_r ($\delta\Delta V_n/V_n$) and $r_{\Delta V}$ is the distance (from the Moon) where the maneuver is performed (38800 km, for 64 h after launch maneuver). This error can be expressed in terms of the error in eccentricity by:

$$\delta e = \frac{b\delta b}{a^2 e} = 0.00073 \quad (7)$$

which means, in terms of error in periapsis altitude (δr_p):

$$\delta r_p = -a\delta e = 3.17 \text{ km} \quad (8)$$

There is still another source of error, because a change in the magnitude of V_r represents an energy error that can be obtained by:

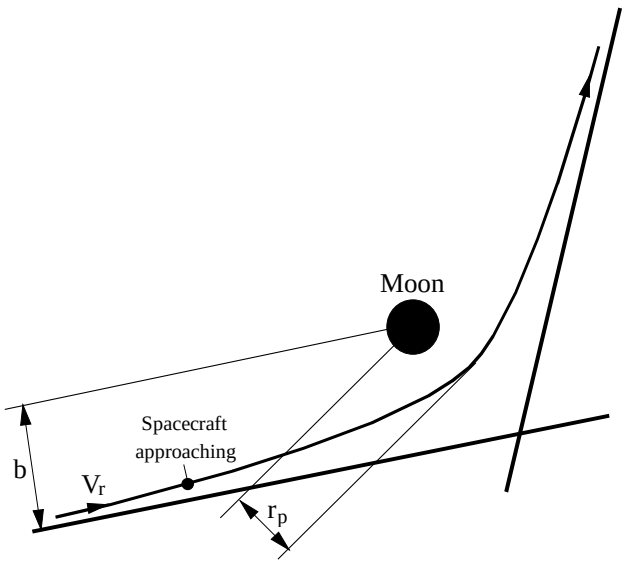

Fig. 2 - Geometry of the Trajectory to the Moon.

$$\delta a = \frac{2a\delta V_r}{V_r} = 0.51 \text{ km} \quad (9)$$

This small component contributes in the variation of the periapsis altitude with:

$$\delta r_p = -\delta a(e-1) = 0.21 \text{ km} \quad (10)$$

Then, the total error is:

$$\delta r_p = \sqrt{3.17^2 + 0.21^2} = 3.18 \text{ km} \quad (11)$$

This value is very good, if it is considered that the nominal altitude of the periapsis is 100 km. It is also interesting to consider, only for safety reasons, the effect of 3σ errors (σ is the standard deviation), since the calculations performed before considered errors of 1σ. Repeating the calculations, the result is an error in periapsis altitude of 9.53 km, which is still very comfortable.

After this, it is necessary to study in more detail the effects of the errors in the orbit insertion impulse. It is assumed that the impulse will occur in the nominal orbit, since errors due to the last maneuver can be corrected before the insertion, if necessary.

According to the assumption that the trip time is 3.47 days, the velocity increment for orbit insertion is about 0.91 km/s. The error in magnitude of the velocity after insertion can be assumed to be proportional to the velocity increment to be applied, and a typical value is about 0.5% (Burke, 1989). This error can be transformed into error in semi-major axis by using the equation:

$$\delta a = \frac{aV_c\delta V}{E} = 10.26 \text{ km} \quad (12)$$

where δV is the error in the velocity after insertion (km/s), V_c is the circular velocity at 100 km (1.63 km/s), and E is the total energy by mass of the spacecraft (km^2/s^2). Then, the maximum error in r_p is 20.52 km (remember that $2a = r_a + r_p$) in case of an overburn. If an underburn occurs, the error will cause an increase of the apoapsis radius, and this error can be corrected very easily with the low thrust available. The pointing error also causes a loss in ΔV proportional to the cosine of the error angle. Assuming 0.5 degree error, we have:

$$\delta\Delta V = (1-\cos(\delta\alpha))\Delta V = 0.035 \text{ m/s} \quad (13)$$

which is very small compared with the 4.56 m/s due to the proportional error assumed, but corroborates with an error of 78.9 m (using equation 12).

The pointing errors are also very important in post-insertion periapsis altitude. For a 0.5 degree pointing error the altitude error is about 8 km (JPL, 1976). If an error of 3σ is considered (1.5 degrees), the new value for the periapsis altitude error is about 24 km.

Another important source of error is the mass of the spacecraft, which is not very well known due to the uncertainty in the mass of fuel consumed in the mid-course maneuvers. To transform this mass error to a velocity increment error ($\delta\Delta V$) it is necessary to use the equation:

$$\delta\Delta V = \frac{g_0 I_{sp}\delta m_0}{m_0} \quad (14)$$

where g_0 is the gravity acceleration on the Earth's surface (9.8 m/s^2), I_{sp} is the specific impulse of the combination fuel/motor used (290 s), δm_0 is the error in mass (kg), and m_0 is the total mass of the spacecraft.

Assuming a total error of 50 m/s in the velocity increment for mid-course corrections, the corresponding $\delta\Delta V$ is about 2.84 m/s. This error in ΔV implies an error of 6.4 km in semi-major axis and 12.8 km in the periapsis altitude (in case of overburn). In case of a 3σ error, this result changes from 12.8 km to 38.4 km. Then, considering all errors described, the total error is (for 1σ case):

$$\delta r_p = \sqrt{20.52^2 + 0.0789^2 + 8^2 + 12.8^2} = 25.47 \text{ km} \quad (15)$$

For the 3σ case, the equation (15) gives the value 76.4 km. This means that a collision with the Moon is very improbable, even if an underburn technique (a technique of designing the motor to apply an impulse slightly smaller than the nominal value, and corrects the final orbit later) is not used. In the double mission (two spacecrafts), the insertion will be done in a higher altitude, which means that there is no necessity for studying the errors involved to avoid a collision with the Moon.

4 LOW-THRUST TRAJECTORY

This study was done with the objective of comparing the differences in fuel consumption for an Earth-Moon transfer with the use of a engine of low thrust.

The spacecraft is supposed to be in a planar Keplerian motion controlled only by the thrust, whenever it is active. A Keplerian motion is a trajectory obtained when only the gravitational force of a point of mass is included in the dynamical system. It can be circular, elliptic, parabolic or hyperbolic. This thrust is assumed to have the following characteristics:

i. Fixed magnitude;
ii. Constant Ejection Velocity of the gases eliminated by the engine;
iii. Free angular motions;
iv. Operation in on-off mode.

The solution is given in terms of the time-histories of the thrust (pitch angle), fuel consumed and duration of the propelled phase. The time-history of the thrust is a plot that shows the direction of the thrust in every instant that it is on. It is the control of the satellite.

4.1 Formulation of the Optimal Control Problem

This is a typical optimal control problem, and it is formulated as follows.

Objective Function to be minimized: $J = m_0 - m_f = X_4$, the difference between the initial and final mass of the spacecraft, that represents the fuel consumed.

This objective function has to be minimized with respect to the control u(.), that is the time to start and to stop the engine and the pitch angle of the thrust at every instant of time (α: $[t_0,t_f] \rightarrow R$, $\alpha \in C^1$ in $[t_0,t_f]$), since the magnitude of the thrust is assumed to be constant and the maneuver is planar ($\beta = 0$ in the equations below).

This system is subject to the following equations of motion:

$$dX_1/ds = f_1 = SiX_1F_1 \quad (16)$$
$$dX_2/ds = f_2 = Si\{[(Ga+1)\cos(s)+X_2]F_1+\nu F_2\sin(s)\} \quad (17)$$
$$dX_3/ds = f_3 = Si\{[(Ga+1)\sin(s)+X_3]F_1 -\nu F_2\cos(s)\} \quad (18)$$
$$dX_4/ds = f_4 = Si\nu F(1-X_4)/(X_1W) \quad (19)$$
$$dX_5/ds = f_5 = Si\nu(1-X_4)m_0/X_1 \quad (20)$$
$$dX_6/ds = f_6 = - SiF_3[X_7\cos(s)+X_8\sin(s)]/2 \quad (21)$$
$$dX_7/ds = f_7 = SiF_3[X_6\cos(s)-X_9\sin(s)]/2 \quad (22)$$
$$dX_8/ds = f_8 = SiF_3[X_9\cos(s)+X_6\sin(s)]/2 \quad (23)$$
$$dX_9/ds = f_9 = SiF_3[X_7\sin(s)-X_8\cos(s)]/2 \quad (24)$$

where:

$$Ga = 1 + X_2\cos(s) + X_3\sin(s) \quad (25)$$
$$Si = (\mu X_1^4)/[Ga^3m_0(1-X_4)] \quad (26)$$
$$F_1 = F\cos(\alpha)\cos(\beta) \quad (27)$$
$$F_2 = F\sin(\alpha)\cos(\beta) \quad (28)$$
$$F_3 = F\sin(\beta) \quad (29)$$

and F is the magnitude of the thrust, W is the velocity of the gases when leaving the engine, ν is the true anomaly of the spacecraft.

In those equations the state was transformed from the Keplerian elements (a = semi-major axis, e = eccentricity, i = inclination, Ω = argument of the ascending node, ω = argument of periapsis, ν = true anomaly of the spacecraft), in the variables X_i, to avoid singularities, by the relations:

$$X_1 = [a(1-e^2)/\mu]^{1/2} \quad (30)$$
$$X_2 = e\cos(\omega-\phi) \quad (31)$$
$$X_3 = e\sin(\omega-\phi) \quad (32)$$
$$X_4 = (\text{Fuel consumed})/m_0 \quad (33)$$
$$X_5 = t = \text{time} \quad (34)$$
$$X_6 = \cos(i/2)\cos((\Omega+\phi)/2) \quad (35)$$
$$X_7 = \sin(i/2)\cos((\Omega-\phi)/2) \quad (36)$$
$$X_8 = \sin(i/2)\sin((\Omega-\phi)/2) \quad (37)$$
$$X_9 = \cos(i/2)\sin((\Omega+\phi)/2) \quad (38)$$
$$\phi = \nu + \omega - s. \quad (39)$$

and s is the range angle of the spacecraft.

The number of state variables defined above is greater than the minimum required to describe the system, which implies that they are not independent and relations between than exist, like: $X_6^2 + X_7^2 + X_8^2 + X_9^2 = 1$. This system is also subject to the constraints in state, because five of the the Keplerian elements of the initial and the final orbit are fixed: a, e, i, ω, Ω. All the parameters (gravitational force field, initial values of the satellite, etc...) are assumed to be known.

The equations for the Lagrange multipliers (p_i) are (adjoint equations):

$$\frac{dp_1}{ds} = -\frac{1}{X_1}\left[4\sum_{j=1}^{9} p_j f_j + p_1 f_1 - p_4 f_4 - p_5 f_5\right] \quad (40)$$

$$\frac{dp_2}{ds} = \frac{\cos(s)}{Ga}\left[3\sum_{j=1}^{9} p_j f_j - p_4 f_4 - p_5 f_5\right] - Si p_2 F_1 - Si\cos^2(s)(p_2 F_1 - p_3 F_2) - Si\cos(s)\sin(s)(p_2 F_2 + p_3 F_1) \quad (41)$$

$$\frac{dp_2}{ds} = \frac{\cos(s)}{Ga}\left[3\sum_{j=1}^{9} p_j f_j - p_4 f_4 - p_5 f_5\right] - Si p_2 F_1 - Si\cos^2(s)(p_2 F_1 - p_3 F_2) - Si\cos(s)\sin(s)(p_2 F_2 + p_3 F_1) \quad (42)$$

$$\frac{dp_4}{ds} = -\frac{1}{m_0(1-X_4)}\left[\sum_{j=1}^{9} p_j f_j - p_4 f_4 - p_5 f_5\right] \quad (43)$$

$$\frac{dp_5}{ds} = 0 \quad (44)$$

$$\frac{dp_6}{ds} = -\frac{SiF_3}{2}[p_7\cos(s) + p_8\sin(s)] \quad (45)$$

$$\frac{dp_7}{ds} = \frac{SiF_3}{2}[p_6\cos(s) - p_9\sin(s)] \quad (46)$$

$$\frac{dp_8}{ds} = \frac{SiF_3}{2}[p_6\sin(s) + p_9\cos(s)] \quad (47)$$

$$\frac{dp_9}{ds} = -\frac{SiF_3}{2}[p_8\cos(s) - p_7\sin(s)] \quad (48)$$

The condition that comes from the Principle of Pontryagin can be written as:

Extremize $\sum_{i=1}^{9} p_i f_i$ with respect to α.

This condition can be converted to give equations (using the values of f_i and making the derivatives with respect to α equal to zero) that are used to give the value of α at every instant. The result is:

$$\sin(\alpha) = q_2/S' \quad (49)$$

$$\cos(\alpha) = q_1/S' \quad (50)$$

where:

$$S' = \pm\sqrt{q_1^2 + q_2^2} \quad (51)$$

$$q_1 = p_1 X_1 + p_2[X_2 + (Ga+1)\cos(s)] + p_3[X_3 + (Ga+1)\sin(s)] \quad (52)$$

$$q_2 = p_2 Ga\sin(s) - p_3 Ga\cos(s) \quad (53)$$

This problem can be solved by using the basic approach used in optimal control theory. First order necessary conditions for a local minimum are used to obtain the Euler-Lagrange equations and the optimal control angles at each range angle, leading to a "Two Point Boundary Value Problem" (TPBVP), where the difficulty is to find the initial values of the Lagrange multipliers (Prado and Rios-Neto, 1989). The range angle (s) is the angle that the line between the spacecraft and the center of the Earth makes with an arbitrary reference direction in space. It replaces the time as the independent variable in this problem. The treatment given here is the hybrid approach of guessing a set of values for the initial Lagrange multipliers, integrating numerically the Euler-Lagrange equations and then search for a new set of values for the initial Lagrange multipliers, based on a nonlinear programming algorithm. With this approach, the problem is reduced to parametric optimization with the initial values of the Lagrange multipliers as variables to be optimized.

The method proposed by Biggs (1979) was used here, the "adjoint-control" transformation is performed to allow us to guess the control angles and its rates at the beginning of thrusting, instead of the initial values of the Lagrange multipliers. It is constituted by a set of equations that can be solved for the Lagrangian multipliers as a function of the control angles and its rates at the beginning of thrusting. With this, it is easier to find a good initial guess, and the convergence is faster.

4.2 Numerical Method

To solve the nonlinear programming problem, the well-known gradient projection method was used (Bazaraa and Shetty, 1979). The algorithm was coded in single precision (48 bits) FORTRAN IV, and the calculations were performed in a VAX/VMS computer.

5 RESULTS

The low thrust propulsion system was studied only for the Earth-Moon trajectory and not for the lunar insertion phase. This is done because the insertion phase has to be performed in a short time, because the spacecraft has a high velocity with respect to the Moon and it will escape if a control is not used soon. The satellite is supposed to leave the Earth from a circular orbit with semi-major axis of 6570 km and to go to an orbit with eccentricity of 0.97 and semi-major axis of 220000 km, that is the Lunar Transfer Orbit desired (Orbit 6 in Table 1). This transfer is considered to be planar, because this is the less expensive case (in terms of fuel consumed) and it can be obtained with an adequate choice of the launch time. Two values were considered for the mass of the satellite after the low thrust maneuver: 150 and 180 kg. These values are compatible with a final mass of 100 and 120 kg in lunar orbit, respectively. The motor/fuel combination is supposed to have a specific impulse of 3500 s, and a thrust magnitude of 200 and 20 N was simulated. Figs. 3 to 6 show the optimal control (pitch angle) for all combinations studied. Table 8 shows the fuel consumption and the duration of the propelled phase for all cases.

Table 8 - Fuel consumed and duration of the propelled phase for all trajectories simulated

Mission	E-M (kg)	Ins. (kg)	Total (kg)	Time (h)
Impulsive 1	205.78	31.40	237.18	0.00
Impulsive 2	247.69	38.16	285.85	0.00
L. T. 1	22.46	31.40	53.86	10.70
L. T. 2	14.85	31.40	46.25	0.71
L. T. 3	27.12	38.16	65.28	12.92
L. T. 4	18.26	38.16	56.42	0.87

Impulsive 1: Single mission using an impulsive engine with specific impulse of 340 s;

Impulsive 2: Double mission using an impulsive engine with specific impulse of 340 s;

L. T. 1: Single mission using an engine with 20 N and specific impulse of 3500 s, and assuming a mass of 150 kg for the spacecraft after the Low-Thrust maneuver;

L. T. 2: Single mission using an engine with 200 N and specific impulse of 3500 s, and assuming a mass of 150 kg for the spacecraft after the Low-Thrust maneuver;

L. T. 3: Double mission using an engine with 20 N and specific impulse of 3500 s, and assuming a mass of 180 kg for the spacecraft after the Low-Thrust maneuver;

L. T. 4: Double mission using an engine with 200 N and specific impulse of 3500 s, and assuming a mass of 180 kg for the spacecraft after the Low-Thrust maneuver;

E-M: Fuel consumed for the Earth-Moon trajectory;

Ins.: Fuel consumed for the lunar insertion, assuming an engine with specific impulse of 340 s;

Total: Total mass of fuel required (Earth-Moon trajectory + lunar insertion).

Time: Duration of the propelled arc;

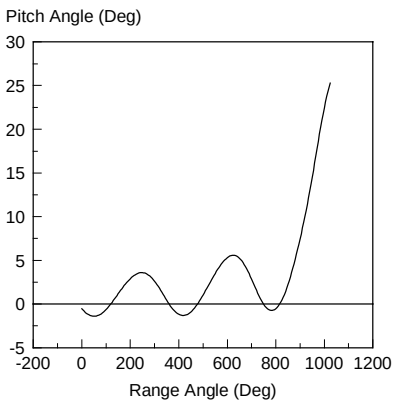

Fig. 3 - Pitch angle (deg) X Range angle (deg) for Low Thrust 1 maneuver.

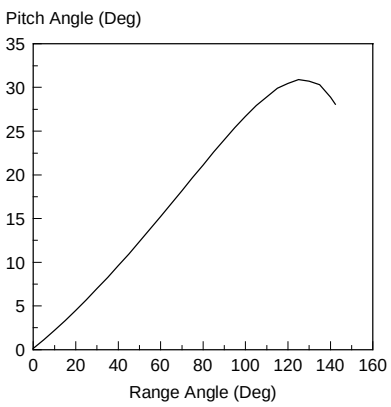

Fig. 4 - Pitch angle (deg) X Range angle (deg) for Low Thrust 2 maneuver.

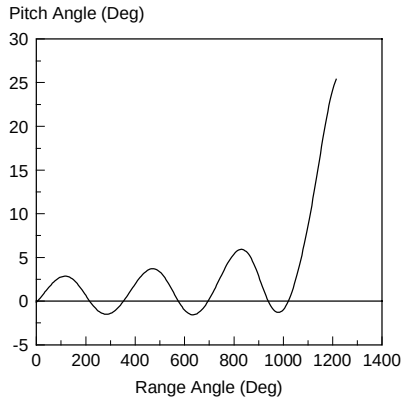

Fig. 5 - Pitch angle (deg) X Range angle (deg) for Low Thrust 3 maneuver.

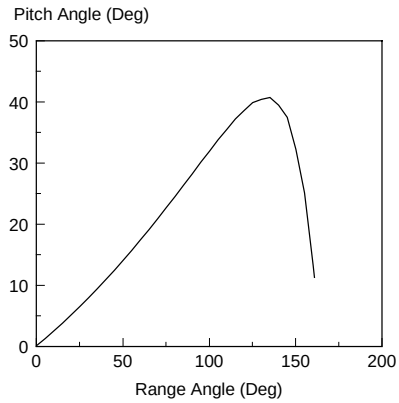

Fig. 6 - Pitch angle (deg) X Range angle (deg) for Low Thrust 4 maneuver.

6 STATION-KEEPING IN LUNAR ORBIT

The purpose of this section is to talk briefly about the effects of the disturbing forces on the final orbit of the spacecraft, and design the maneuvers needed to keep the orbital parameters within an acceptable range of values. It will be assumed that the orbit around the Moon will be circular with a nominal altitude of 100 km.

The Moon does not have atmospheric drag, and (at an altitude of 100 km) the disturbances caused by the Earth, the Sun and other external sources can be neglected, since the effects of the first three spherical harmonic terms of the gravity field of the Moon (JPL, 1976) are much larger. These assumptions simplify the problem but do not solve it, because there is not enough information available about the gravity field of the Moon to allow an accurate prediction of the behavior of the LPO's orbital parameters. The models currently available are based on the motion of the spacecrafts that were sent to the Moon previously. These models can make accurate predictions only for a spacecraft moving near the positions mapped.

Scientists at JPL compared three different models of the lunar gravity field (Sjogren, 1971; Liu and Lang, 1972; Ferrari, 1975). Their conclusion is that different models give different behaviors and we can only estimate the motion of the lunar polar orbiter (JPL, 1976). Unfortunately, only spacecrafts with low inclination and/or high altitudes were used for gravity mapping, so there are no good models for a low polar orbit.

Using the models available, the main conclusion is that a near circular orbit will become more elliptical, but will retain the value of its semi-major axis. This means that the apoapsis altitude increases by the same amount that the periapsis

decreases (JPL, 1976). With this information, it is necessary to define a limit for the periapsis altitude, where the station-keeping maneuver must be performed to avoid a collision with the Moon. Following the same values used in the JPL document (JPL, 1976) (based in the model developed by Liu and Lang, 1972) the value of 50 km will be used. This choice implies that one maneuver needs to be performed every 70 days, since this is the time required for the periapsis altitude to reach the critical value of 50 km.

With the aforementioned assumptions, the typical station keeping maneuver is to transfer the spacecraft from an elliptical orbit (periapsis altitude of 50 km and apoapsis altitude of 150 km) to a circular orbit with an altitude of 100 km. This can be performed by using the bi-impulse Hohmann Transfer, described previously in this paper. The first impulse is applied at the apoapsis to rise the periapsis to an altitude of 100 km. This first impulse requires a ΔV of 11.20 m/s. The second impulse is retrograde, applied at the periapsis, to decrease the apoapsis altitude to 100 km. This impulse requires a ΔV of 10.90 m/s. So, the total ΔV required is 22.10 m/s per maneuver. Then, since four maneuvers are necessary for one full year of operation, the total ΔV required is 88.40 m/s. If an additional 5% is added to compensate for the non-impulsive characteristic of the thrust, the final result will be about 93.0 m/s per year.

If the mass of the spacecraft is assumed to be 100 kg, the required mass of fuel can be calculated by the equation $m_0 - m_f = m_0\left(1-\exp(\frac{-\Delta V}{I_{sp} g_0})\right)$, and they are shown in Table 9, for three kinds of fuel (including the 5% extra added to the ΔVs).

Table 9 - Mass of fuel required for station keeping

Fuel	I_{sp} (s)	Mass (kg/maneuver)	Mass (kg/year)
Hydrazine	235	1.01	4.04
Bi-prop A	290	0.82	3.28
Bi-prop B	310	0.77	3.08

7 CONCLUSIONS

This paper showed the basic parameters for several trajectories to send, insert and keep a spacecraft in lunar orbit. The numerical results also showed that: with the error parameters used, there is no necessity to use an underburn technique for lunar insertion; the use of low thrust maneuver for the Earth-Moon trajectory can make very large savings in fuel expenditure; a total ΔV of less than 100 m/s, divided into four maneuvers, is enough for station-keeping during one year of operation.

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REFERENCES

Bate, R.R., Mueller, D.D. and White, J.E., 1973, “Fundamentals of Astrodynamics”, New York, Dover Publications, Inc.

Bazaraa, M.S. and Shetty, C.M., 1979, “Nonlinear Programming-Theory and Algorithms”, New York, John Wiley & Sons, Chap. 10.3.

Biggs, M.C.B., 1979, “The Optimization of Spacecraft Orbital Maneuvers. Part II: Using Pontryagin's Maximum Principle”, The Hatfield Polytechnic, Numerical Optimization Center, England.

Burke, J., 1989, Private communication, Strasbourg, France.

Cornelisse, J.W., 1980, “Translunar Trajectory Characteristics and Propulsion Requirements for the Polar Orbiting Lunar Observatory (POLO) Mission”, Internal Estec Working 1250, Noordwijk.

Ferrari, A.J., 1975, “Lunar Gravity: The first Farside Map”, Science, vol. 188, pp. 1297.

Hohmann, W., 1925, “Die erreichbarkeit der himmelskorper,” Oldenbourg, Munique.

International Space University (ISU), 1989, “ARTEMIS, A Program to Identify and Map Lunar Resources”, Final Report of the 1989 summer session, Strasbourg, France.

Jet Propulsion Laboratory (JPL), 1976, “Lunar Polar Orbiter: Dual Spacecraft System Description”, Internal report, Pasadena CA.

Liu, A.S. and Lang, P.A., 1972, “Lunar Gravity Field as Determined by Orbiters”, Space Research XII, pp. 163, Berlin.

Prado, A.F.B.A. and Rios-Neto, A., 1989, “Suboptimal and Hybrid Numerical Solution Schemes for Orbit Transfer Manoeuvres”, Space Dynamics, pp. 749-760. Cépaduès-Éditions, Toulouse, France..

Sjogren, W.L., 1971, “Lunar Gravity Estimate: Independent Confirmation”, Journal of Geophysical Research. Vol. 76, pp. 7021.