RELATING TWO STATE FEEDBACK $\mathcal{H}_\infty$ CONTROL APPROACHES*

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ABSTRACT - In this paper, two different approaches to the $\mathcal{H}_\infty$ control by state feedback are related and discussed. The first approach is based on the solvability of an algebraic Riccati equation and the second one explores the convexity of the set defined by a matrix inequality. New insights are provided in order to establish the relationship between the set of stabilizing gains obtained from the two different approaches. The results are illustrated through two examples.

Keywords: $\mathcal{H}_\infty$ Control, State Feedback, Algebraic Riccati Equation, Convex Analysis.

Relacionando Duas Abordagens em Controle $\mathcal{H}_\infty$ por Realimentação de Estado

RESUMO - Neste trabalho, são estudadas e relacionadas duas abordagens para o problema de controle $\mathcal{H}_\infty$ por realimentação de estado. A primeira baseia-se na solvabilidade de uma equação algébrica do tipo Riccati e a segunda explora a convexidade do conjunto definido por uma inequação matricial. A teoria é desenvolvida de forma a evidenciar a relação entre os conjuntos de ganhos estabilizantes gerados pelas duas abordagens acima. Dois exemplos numéricos ilustram os resultados teóricos.

Palavras Chave: Controle $\mathcal{H}_\infty$, Realimentação de Estado, Equações Algébricas de Riccati, Análise Convexa.

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1 - Introduction

This paper is concerned with the problem of $\mathcal{H}_\infty$ control by state feedback. The $\mathcal{H}_\infty$ suboptimal problem, that is, the problem of asymptotic stabilization with a prescribed γ disturbance attenuation level, was first solved by Petersen (Petersen, 1987*b*), who demonstrated to be sufficient to solve a single Algebraic Riccati Equation (ARE) in order to provide a stabilizing feedback gain. Similar ARE-based formulations are described in (Zhou and Khargonekar, 1988), (Khargonekar *et al.*, 1988). It is also worthwhile to recall the seminal paper by Doyle et al. (Doyle *et al.*, 1989). Recently, suboptimal parametrizations in an LMI (Linear Matrix Inequality) framework have been presented in (Gahinet, 1994), (Iwasaki and Skelton, 1994), (Zhou *et al.*, 1995), (Boyd *et al.*, 1994).

The $\mathcal{H}_\infty$ optimal control problem, however, seems more difficult to characterize, since the disturbance attenuation must be reduced to its minimum value γ^* and, as pointed out in (Scherer, 1990), high gain feedback may be necessary to approach the optimal value.

Other works dealing with the $\mathcal{H}_\infty$ control by state feedback are, for instance, (Peres *et al.*, 1993), (Peres *et al.*, 1994), (Scherer, 1989), (Scherer, 1990), (Scherer, 1994). Basically, the control gain is obtained from a positive matrix which is either an element of a convex set or the solution of an algebraic Riccati equation.

The aim of this paper is to highlight the relationship between the sets of stabilizing control gains obtained from these two different approaches. It is shown that the maximal solutions of the Riccati equations associated with the $\mathcal{H}_\infty$ control define the boundary of the convex set which generates the control gain in the convex approach. An-

al., 1994), does not augment the order of the system representation.

1.1 Notation

The notation used in this paper is fairly standard. $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, with $\mathbb{C}$ partitioned as $\mathbb{C}^- \bigcup \mathbb{C}^o \bigcup \mathbb{C}^+$, where $\mathbb{C}^- = \{s \in \mathbb{C} \mid Re(s) < 0\}$, $\mathbb{C}^o = \{s \in \mathbb{C} \mid Re(s) = 0\}$ and $\mathbb{C}^+ = \{s \in \mathbb{C} \mid Re(s) > 0\}$. The space $\mathbb{R}^n$ is equipped with the Euclidean Norm "$\| \|$", $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$, $m, n \in \mathbb{N}$, with the Spectral Norm, i.e., $\|X\| = \sigma_{max}(X)$, where $\sigma_{max}(X)$ is the maximum singular value of the matrix X. Y' stands for the transpose of Y. The boldface characters $\mathbf{I}$ and $\mathbf{0}$ denote, respectively, the identity and the null matrices of convenient sizes. $\mathcal{S}_n$ is the Banach space of real symmetric matrices. $\mathcal{S}_n^+$ is the set of positive definite symmetric matrices, $\mathcal{S}_n^o$ of positive semidefinite and singular symmetric matrices and $\mathcal{S}_n^-$ of nonpositive semidefinite symmetric matrices. For $X, Y \in \mathcal{S}_n$, $X \geq Y$ (respectively, $X > Y$) means $X - Y \geq \mathbf{0}$ (respectively $X - Y > \mathbf{0}$), i.e., $X - Y$ is positive semidefinite (respectively $X - Y$ is positive definite).

Further, $\mathcal{L}_2[0;\infty)$ is the Lebesgue space of square-integrable functions. $H : \mathbb{C} \rightarrow \mathbb{C}^{m\times n}$ is an element of the Lebesgue space $\mathcal{L}_\infty$ if, and only if, H is bounded, i.e., $\|H(jw)\| = \sigma_{max}(H(jw)) \leq k < \infty, \forall\ w \in \mathbb{R}$, except on a set of zero measure. Define in $\mathcal{L}_\infty$ the norm $\|H\|_{\mathcal{L}} = \inf\ \{k \mid \|H(jw)\| \leq k\ \ a.e.\}$ named the essential supreme of H and denoted by $\|H\|_{\mathcal{L}} = \text{ess}\ \sup_{w\in\mathbb{R}}\ \|H(jw)\|$. With that norm, $\mathcal{L}_\infty$ is a Banach space. $F : \mathbb{C} \rightarrow \mathbb{C}^{m\times n}$ is an element of Hardy space $\mathcal{H}_\infty$ if, and only if, F is analytic into $\mathbb{C}^+$ and $\|F\|_\infty \triangleq \sup_{s\in\mathbb{C}^+}\{\|F(s)\|\} = \sup_{s\in\mathbb{C}^+}\{\sigma_{max}(F(s))\} < \infty$. $\|F\|_\infty$ is named $\mathcal{H}$ Infinity Norm (confusing itself with the denomination of the space, "$\mathcal{H}_\infty$-norm").

Each function on $\mathcal{H}_\infty$ is associated with a single function on $\mathcal{L}_\infty$, in the sense that $H(jw) = \lim_{\xi\rightarrow 0} F(\xi + jw)$, the mapping $F \mapsto H$ of $\mathcal{H}_\infty$ into $\mathcal{L}_\infty$ is linear, injective and preserves the norm (see (Sz.-Nagy and Foias, 1970)), therefore, $\mathcal{H}_\infty$ can be considered as a closed subspace of $\mathcal{L}_\infty$ and furthermore:

$$\|H\|_\infty = \sup_{s\in\mathbb{C}^+}\{\sigma_{max}(H(s))\} =$$

$$= \text{ess}\ \sup_{w\in\mathbb{R}}\ \{\sigma_{max}(H(jw))\} = \|H\|_{\mathcal{L}}$$

2 - Preliminaries

The continuous-time linear system considered in this paper is described by

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) & = & Cx(t) + Du(t) \end{array} \right. \quad (1)$$

where $x: \mathbb{R} \rightarrow \mathbb{R}^n$ is the state variable, $u: \mathbb{R} \rightarrow \mathbb{R}^m$ the control variable, $w: \mathbb{R} \rightarrow \mathbb{R}^k$, $w \in \mathcal{L}_2[0;\infty)$ the exogenous following regularity assumptions are made (see (Doyle *et al.*, 1989)):

i) (A, B_2) is stabilizable and (A, C) is observable
ii) $D'[C \quad D] = [\mathbf{0} \quad \mathbf{I}]$

Moreover, it is assumed that the entire state is available for control, i.e., the measured output is the state. The control law can be chosen as $u = -Kx$, where the gain K is such that

$$K \in \mathcal{K} \triangleq \{K \in \mathbb{R}^{m\times n} \mid A - B_2 K \text{ is asymptotically stable}\}$$

Defining $A_f \triangleq A - B_2 K$ and $C_f \triangleq C - DK$, the map from the disturbances input w to the controlled output z, i.e., the closed-loop transfer function matrix of the system is given by:

$$H(s) \triangleq C_f(s\mathbf{I} - A_f)^{-1} B_1 \quad (2)$$

The $\mathcal{H}_\infty$-optimization problem can be formalized as: find

$$(\mathbf{P1}) \qquad \inf\ \{\|H\|_\infty \mid K \in \mathcal{K}\}$$

In the parameter space of the elements of $\mathcal{K}$, the $\mathcal{H}_\infty$-norm minimization is very difficult to be addressed. Another formulation of this problem can be established considering the solution of a certain Riccati inequality.

Theorem 1 *Let $\gamma > 0$ be given. With the assumptions made for system (1), there exists $K \in \mathcal{K}$ such that $\|H\|_\infty \leq \gamma$ if, and only if, there exists $W = W' > \mathbf{0}$ such that*

$$AW + WA' + WC'CW + \gamma^{-2} B_1 B_1' - B_2 B_2' \leq \mathbf{0} \quad (3)$$

Furthermore, K can be selected as $K = B_2' W^{-1}$.

Proof: See (Scherer, 1990). ■

This result means that for a fixed $\gamma > 0$, defining the set

$$\mathcal{K}(\gamma) \triangleq \{K \in \mathcal{K} \mid \|H\|_\infty \leq \gamma\} \quad (4)$$

it follows that $\mathcal{K}(\gamma) \neq \emptyset$ whenever the hypothesis of the theorem is ensured, and mainly that the optimization problem **(P1)** is equivalent to the problem: find

$$(\mathbf{P2}) \qquad \gamma^* = \inf\ \{\gamma \mid K \in \mathcal{K}(\gamma)\}$$

This formulation of the problem, together with the result of Theorem 1, suggest iterative approaches for its solution, which consist of characterizing at least one element of the set $\mathcal{K}(\gamma)$, while decreasing the value of the disturbance attenuation γ, i.e., to solve at each step a suboptimal problem.

Define, for $\mu \triangleq \gamma^{-2}$, the following matrix expression

$$\mathcal{R}_\mu(W) \triangleq AW + WA' + WC'CW + \mu B_1 B_1' - B_2 B_2' \quad (5)$$

and the sets:

$$\mathcal{M}_\mu \triangleq \{W \in \mathbb{R}^{n\times n} \mid W = W' \text{ and } \mathcal{R}_\mu(W) \leq \mathbf{0}\} \quad (6)$$

$$\mathcal{N}_\mu \triangleq \{W \in \mathbb{R}^{n\times n} \mid W = W' \text{ and } \mathcal{R}_\mu(W) = \mathbf{0}\} \quad (7)$$

where $\mathcal{N}_\mu \subset \mathcal{M}_\mu$.

Considering the dual system of (1) and since $(-A', C')$ is stabilizable, one of the results in (Ran and Vreugdenhil, 1988) can be rewritten as in the following theorem.

Theorem 2 *Let $\mu \geq 0$ be given. If $\mathcal{M}_\mu \neq \emptyset$, then there exists $W(\mu) \in \mathcal{N}_\mu$ such that:*

$$W(\mu) \geq W, \quad \forall\, W \in \mathcal{M}_\mu$$

In particular, $W(\mu)$ is the maximal symmetric solution of $\mathcal{R}_\mu(W) = \mathbf{0}$ and, moreover, all eigenvalues of

$$A(\mu) \triangleq A + W(\mu)C'C \quad (8)$$

are in $\mathbb{C}^+ \bigcup \mathbb{C}^o$.

Proof: See (Ran and Vreugdenhil, 1988). ■

This theorem allows to characterize a suboptimal problem, in terms of the algebraic Riccati equation solvability, i.e., $\mathcal{R}_\mu(W) = \mathbf{0}$, for a fixed $\mu \geq 0$ (see Theorem 1).

It is important to remark that, nevertheless, for $0 \leq \nu \leq \mu$, if $W(\mu)$ exists, $\mathcal{R}_\mu(W(\mu)) = \mathbf{0}$ can be rewritten as:

$$AW(\mu) + W(\mu)A' + W(\mu)C'CW(\mu) + \nu B_1 B_1' - (\nu - \mu)B_1 B_1' - B_2 B_2' = \mathbf{0}$$

implying

$$AW(\mu) + W(\mu)A' + W(\mu)C'CW(\mu) + \nu B_1 B_1' - B_2 B_2' = (\nu - \mu)B_1 B_1' \leq \mathbf{0}$$

Thus, $W(\mu) \in \mathcal{M}_\nu$ and by Theorem 2 there exists $W(\nu) \geq W(\mu)$. In other words, the function φ that associates the parameter μ with the corresponding maximal symmetric solution $W(\mu) \in \mathbb{R}^{n\times n}$ of $\mathcal{R}_\mu(W) = \mathbf{0}$ is nonincreasing.

Scherer in (Scherer, 1989), motivated by Theorem 1 and based on the results of Theorem 2, proposes another formulation for problem **(P2)**, described in terms of the maximal solution of the algebraic Riccati equation: find

$$\textbf{(P3)} \qquad \mu^* = \sup\ \{\mu \mid \exists\, W(\mu) \in \mathcal{N}_\mu \text{ and } W(\mu) > \mathbf{0}\}$$

An iterative algorithm for its solution is also provided in (Scherer, 1989).

ing these properties, it is necessary to specify precisely the domain of φ. The following lemma is fundamental to answer this question.

Lemma 1 *Consider μ_1, μ_2 with $\mu_1 \geq \mu_2$ and $W(\mu_1)$ as the maximal solution of $\mathcal{R}_{\mu_1}(W) = \mathbf{0}$. Then, $W(\mu_2)$ is the maximal solution of $\mathcal{R}_{\mu_2}(W) = \mathbf{0}$ if, and only if, $\Delta W = W(\mu_2) - W(\mu_1)$ is the maximal solution of $A(\mu_1)W + WA(\mu_1)' + WC'CW + (\mu_2 - \mu_1)B_1 B_1' = \mathbf{0}$ with $A(\mu)$ defined in (8).*

Proof: Assume $R_{\mu_2}(W(\mu_2)) = \mathbf{0}$. Replacing $W(\mu_2)$ by $\Delta W + W(\mu_1)$ in $R_{\mu_2}(W(\mu_2)) = \mathbf{0}$ and remembering that $A(\mu_1) \triangleq A + W(\mu_1)C'C$,

$$A(\mu_1)\Delta W + \Delta W A(\mu_1)' + \Delta W C'C\Delta W + (\mu_2 - \mu_1)B_1 B_1' = \mathbf{0}$$

implying that ΔW is a solution of the equation proposed, which is maximal from Theorem 2 (since $(-A', C')$ is stabilizable by hypothesis).

Now, assume that ΔW is the solution of the equation proposed. In order to obtain $W(\mu_2)$ as the maximal solution of $R_{\mu_2}(W) = \mathbf{0}$, it suffices to substitute $\Delta W = W(\mu_2) - W(\mu_1)$ and use the fact that $(-A', C')$ is stabilizable. ■

Let μ_{max} be the largest value of μ such that $\mathcal{R}_\mu(W) = \mathbf{0}$ has a real symmetric solution, then $\mu_{max} \leq \infty$. On the other hand, observe that from the regularity assumption *i)*, there exists $W(0) > \mathbf{0}$, the maximal solution of $\mathcal{R}_o(W) = \mathbf{0}$ (standard Riccati equation). Moreover, keeping in mind the definition of μ, i.e., $\mu \triangleq \gamma^{-2}$, and Theorems 1 and 2, it is quite natural to assume the domain of φ as $[0; \mu_{max})$. In fact, φ is well defined since by the previous lemma, $\forall\, \mu \in [0; \mu_{max}], W(\mu)$ exists if, and only if,

$$A(0)\Delta W + \Delta W A'(0) + \Delta W C'C\Delta W + \mu B_1 B_1' = \mathbf{0} \quad (9)$$

has a solution and, in the affirmative case, $W(\mu) = W(0) + \Delta W$. Observe that from the stability of $-A(0)$, equation (9) has a symmetric solution if, and only if, $\|C(s\mathbf{I} + A(0))^{-1}B_1\|_\infty \leq 1/\sqrt{\mu}$ (see (Faibusovich, 1987)).

Based on the stability of $-A(\mu)$, $\mu \in [0; \mu_{max})$, the following result establishes important smoothness properties of φ.

Theorem 3 *Consider $\varphi : [0; \mu_{max}] \rightarrow \mathbb{R}^{n\times n}$ such that*

$$\varphi(\mu) \triangleq W(\mu) \quad (10)$$

where $W(\mu)$ is the maximal symmetric solution of $\mathcal{R}_\mu(W) = \mathbf{0}$. Then the following statements are true:

i) φ is nonincreasing and $\varphi(0) = W(0) > \mathbf{0}$.

ii) For any $0 \leq \mu < \mu_{max}$, $-A(\mu)$ is stable and $\mu_{max} = \mu + \|C(s\mathbf{I} + A(\mu))^{-1}B_1\|_\infty^{-2} \leq \infty$.

iv) If $\mu_{max} < \infty$, then φ is analytic, nonincreasing and concave on $[0;\mu_{max})$. And, moreover, $W(\mu_{max})$ exists and φ is continuous on $[0;\mu_{max}]$.

Proof: See (Scherer, 1990). ■

The next result solves **(P3)** ensuring that $\mu^* = \sup\{\mu \in [0;\mu_{max}] \mid \varphi(\mu) > \mathbf{0}\}$.

Theorem 4 *The following statements are true:*

i) The optimal value $\gamma^* = 1/\sqrt{\mu^*}$ is achieved if, and only if, $\varphi(\mu_{max}) > 0$. In this case, $\gamma^* = 1/\sqrt{\mu_{max}}$ and the stabilizing gain is $K^* = B_2'\varphi(\mu_{max})^{-1}$.

ii) If $\varphi(\mu_{max}) \not> \mathbf{0}$, there exists a unique $\mu_s \in [0;\mu_{max}]$, $\mu_s < \infty$, such that $\varphi(\mu_s)$ is positive semidefinite and singular with $\gamma^* = 1/\sqrt{\mu_s}$.

iii) If the optimal value γ^* is not achieved and if (K_ℓ) is a sequence of admissible gains with $\|(C - DK_\ell)(s\mathbf{I} - A + B_2K_\ell)^{-1}B_1\|_\infty \leq \gamma_\ell$ and $\lim_{\ell\to\infty}\gamma_\ell = \gamma^*$, then $\lim_{\ell\to\infty}\|K_\ell\| = \infty$ and the sequence (K_ℓ) is called a high-gain feedback sequence.

Proof: See (Scherer, 1990). ■

Note that, in the case *i)* if $\mu_{max} = \infty$ and $\varphi(\mu_{max}) > \mathbf{0}$, $\varphi(\mu) = \varphi(\mu_{max}), \forall\ \mu \in [0;\mu_{max})$, then from Theorem 1, $\|H\|_\infty < 1/\sqrt{\mu}, \forall\ \mu \in (0;\mu_{max})$ and so $\gamma^* = \mu_{max}^{-2} = 0$. If $\gamma^* = 0$, it follows that $\mu_{max} = \infty$ and $\varphi(\mu) > \mathbf{0}, \forall\ \mu \in [0;\infty)$, then $\varphi(\mu) = \varphi(0), \forall\ \mu \in [0;\infty]$, i.e., φ is constant and then $\varphi(\mu_{max}) > \mathbf{0}$.

In the case *ii)*, either $\mu_{max} < \infty$ with $\varphi(\mu_{max}) \not\geq \mathbf{0}$ or $\mu_{max} = \infty$. The existence of μ_s follows from the continuity of φ, the uniqueness follows from its definition and by the concavity of φ. Indeed, μ_s is the unique value of μ on $[0;\mu_{max})$ such that $\varphi(\mu)$ is positive semidefinite. If $\varphi(\mu_{max})$ is positive semidefinite, but not definite, choose $\mu_s = \mu_{max}$.

Whereas Theorem 1 generically characterizes the state feedback stabilizing gain set $\mathcal{K}(\gamma)$ (4), the smoothness properties of φ stated in Theorem 3 precisely describes $\mathcal{K}(\gamma)$ in the following sense; for a given $\gamma = 1/\sqrt{\nu}, \nu \in [0;\mu_{max}], \mu_{max} < \infty$,

$$\mathcal{K}_I(\gamma) \triangleq \{K \in \mathbb{R}^{m\times n} \mid K = B_2'\varphi(\mu)^{-1}\ ,$$
$$\mu \in [\nu;\mu_{max}], \varphi(\mu) > \mathbf{0}\} \qquad (11)$$

where, from Theorem 4, the actual upper bound for μ is μ_{max} in the case *i)* or μ_s otherwise.

Define $\mu \triangleq \gamma^{-2}$ and the map $\Theta : \mathcal{S}_n \times \mathbb{R} \to \mathbb{R}^{n\times n}$ as

$$\Theta(W,\mu) \triangleq AW + WA' + WC'CW + \mu B_1B_1' - B_2B_2' \qquad (12)$$

and also the sets:

$$\mathcal{G}_\infty \triangleq \{(W,\mu) \in \mathbb{R}^{n\times n} \times \mathbb{R} \mid W = W' > \mathbf{0},$$
$$\mu \geq 0, x'\Theta(W,\mu)x \leq 0,\ \forall x \in \mathbb{R}^n\} \qquad (13)$$

$$\mathcal{G}_\infty(\mu) \triangleq \{W \in \mathbb{R}^{n\times n} \mid W = W' > \mathbf{0},$$
$$x'\Theta(W,\mu)x \leq 0,\ \forall x \in \mathbb{R}^n\} \qquad (14)$$

Note that $(W,\mu) \in \mathcal{G}_\infty$ if, and only if, $W \in \mathcal{G}_\infty(\mu)$ and moreover $\mathcal{G}_\infty(\mu) \subseteq \mathcal{M}_\mu,\ \forall\mu > 0$ ($\mathcal{M}_\mu$ has been defined in (6)). Notice also that

$$\Theta(W,\mu) \triangleq \mathcal{R}_\mu(W) \qquad (15)$$

Theorem 5 *Let $\mu > 0$ be given. For system (1), the following statements are true:*

i) $\mathcal{G}_\infty(\mu)$ defined in (14) is convex.

ii) $\mathcal{G}_\infty(\mu) \neq \emptyset$ if, and only if, (A,B_2) is stabilizable, (A,C) is detectable, $\|H\|_\infty \leq 1/\sqrt{\mu}$ and the stabilizing gain is, in this case, $K = B_2'W^{-1}$.

Proof: For item *i)*, assume there exist matrices $W_1, W_2 \in \mathcal{G}_\infty(\mu)$, $\alpha \in [0;1]$ such that $W = \alpha W_1 + (1-\alpha)W_2$. Obviously $W = W' > \mathbf{0}$, then

$$\Theta(W,\mu) = \alpha(AW_1 + W_1A') + (1-\alpha)(AW_2 + W_2A') +$$
$$+ \mu B_1B_1' - B_2B_2' + \alpha(1-\alpha)W_1C'CW_2 +$$
$$+ \alpha(1-\alpha)W_2C'CW_1 + (1-\alpha)^2W_2C'CW_2 + \alpha^2W_1C'CW_1$$

and

$$AW_1 + W_1A' \leq -W_1C'CW_1 - \mu B_1B_1' + B_2B_2'$$
$$AW_2 + W_2A' \leq -W_2C'CW_2 - \mu B_1B_1' + B_2B_2'$$

implying that

$$\Theta(W,\mu) \leq \alpha(\alpha-1)W_1C'CW_1 - \alpha(\alpha-1)W_1C'CW_2$$
$$- \alpha(\alpha-1)W_2C'CW_1 + \alpha(\alpha-1)W_2C'CW_2$$

which reduces to

$$\Theta(W,\mu) \leq \alpha(\alpha-1)(W_1 - W_2)C'C(W_1 - W_2) \leq \mathbf{0}$$

since $(W_1 - W_2)C'C(W_1 - W_2)$ is positive semidefinite. As a conclusion, $W \in \mathcal{G}_\infty(\mu)$ and $\mathcal{G}_\infty(\mu)$ is a convex set.

For *ii)*, assume that $\mathcal{G}_\infty(\mu) \neq \emptyset$, i.e., there exists $W = W' > \mathbf{0}$ such that

$$AW + WA' + WC'CW + \mu B_1B_1' - B_2B_2' \leq \mathbf{0} \qquad (16)$$

Now, since (A, B_2) is stabilizable and (A, C) is detectable with $\|H\|_\infty \leq 1/\sqrt{\mu}$, from Theorem 1, there exists $W = W' > \mathbf{0}$ such that

$$AW + WA' + WC'CW + \mu B_1 B_1' - B_2 B_2' \leq \mathbf{0} \qquad (17)$$

or, in other words, $\mathcal{G}_\infty(\mu) \neq \emptyset$. ■

This theorem and Theorem 1 establish that the function $\psi : \mathcal{G}_\infty(\mu) \to \mathcal{K}(\gamma)$ is such that

$$\psi(W) \triangleq B_2' W^{-1} \qquad (18)$$

is well defined and is obviously injective; therefore, $\mathcal{K}(\gamma)$ contains a well determined subset of stabilizing gains, which solve the suboptimal problem.

From the definition of the set $\mathcal{G}_\infty$ and from Theorem 5, the $\mathcal{H}_\infty$-optimal control problem, proposed in **(P3)**, can be restated as a convex problem. In other words, find:

$$(\mathbf{P4}) \qquad \mu^* = \sup \ \{\mu \mid (W, \mu) \in \mathcal{G}_\infty\}$$

This formulation deals jointly with the limiting bound of the norm $\mu \triangleq \gamma^{-2}$ and the stabilizing gain K (related to W satisfying (12)). From the results of the above theorem, $\gamma^* = 1/\sqrt{\mu^*}$ and $K^* = B_2' W_*^{-1}$.

Theorem 6 *Problem* **(P4)** *is a convex problem.*

Proof: The convexity of $\mathcal{G}_\infty$ defined in (13) follows immediately from the convexity of $\mathcal{G}_\infty(\mu)$. Since the objective function is linear, the problem is therefore convex. ■

The fact that problem **(P4)** is a convex programming problem, associated to the following results, allows the use of a "External Linearization" method to solve it[1]. Define

$$\mathcal{G} \triangleq \{(W, \mu) \in \mathbb{R}^{n \times n} \times \mathbb{R} \mid (W, \mu) \in \mathcal{G}_\infty, \\ W \geq \mathbf{0} \text{ and } \mu \in [0; \mu_{max}]\} \qquad (19)$$

Theorem 7 *Under the assumptions made concerning system (1), $\mathcal{G}$ defined in (19) is compact.*

Proof: Since $\mathcal{G}$ is closed, it remains to show that it is bounded. This fact, however, is an immediate consequence of the results from Theorems 2.1 and 2.2 in (Ran and Vreugdenhil, 1988). By hypothesis $\mu_{max} < \infty$ and let $(W, \mu) \in \mathcal{G}$ be given, for $\mu \in [0; \mu_{max}]$ fixed, $\varphi(\mu) \geq W$, $\forall (W, \mu) \in \mathcal{G}$ (see Theorem 3), then $\|\varphi(\mu)\| \geq \|W\|$, $\forall\ (W, \mu) \in \mathcal{G}$. Now, let $\mu, \nu \in [0; \mu_{max}]$ be given with $\mu \geq \nu$, $\varphi(\nu) \geq \varphi(\mu)$ and thus $\|\varphi(\nu)\| \geq \|\varphi(\mu)\|$. Therefore $\|\varphi(0)\| \geq \|W\|$, $\forall\ (W, \mu) \in \mathcal{G}$. ■

[1] In fact, other specialized methods could be used as, for instance, interior point methods. The cutting plane technique is employed here in order to highlight the connections between the convex approach described in this section and the iterative one from section III.

$$(\mathbf{P5}) \quad \Big\{ \ \text{subject to} \quad (W, \mu) \in \mathcal{G}$$

Remark 1: Notice that if **(P4)** is feasible, **(P5)** is feasible too, and **(P5)** is a convex problem. Moreover, any $(W, \mu) \in \mathcal{G}$ feasible to **(P5)** is such that, $\varphi(\mu) \geq W$, where $\varphi(\mu)$ is the maximal solution to $\Theta(W, \mu) = \mathbf{0}$ ($\varphi(\mu)$ has been defined in (10) and $\Theta(W, \mu)$ in (12)).

In this context, from the definition of $\mathcal{G}$, the set $\mathcal{K}(\gamma)$ can be naturally described in the following form; for a given $\gamma = 1/\sqrt{\nu}, \nu \in [0; \mu_{max}]$, $\mu_{max} < \infty$,

$$\mathcal{K}_C(\gamma) \triangleq \{K \in \mathbb{R}^{m \times n} \mid K = \psi(W), \\ (W, \mu) \in \mathcal{G},\ \mu \in [\nu; \mu_{max}]\} \qquad (20)$$

Problem **(P5)** now fulfills the conditions allowing the use of cutting plane methods in order to solve it. Notice that $\mathcal{G}$ is contained in the polytope $P = \{(W, \mu) \mid W = W' \text{ and } \mu \leq \mu_{max}\}$. Considering this polytope as the initial one; the numerical implementation of this method requires the calculation of an $\mathcal{H}_\infty$-norm in order to fix μ_{max} (Theorem 3), but there exist efficient algorithms for this computation in the literature (see, for instance, (Palhares *et al.*, 1997) and references therein). An alternative choice (without $\mathcal{H}_\infty$ norm computations) of μ_{max} is proposed in (Peres *et al.*, 1994), Theorem 3.4.

For conciseness purposes, the algorithm is omitted here. See (Bernussou *et al.*, 1989), (Luenberger, 1984) for details and convergence properties.

The figure 1 illustrates the behavior of the algorithm, assuming (in the jth step) that $(W_j, \mu_j) \notin \mathcal{G}$, thereby the separating hyperplanes L_j and H_j produce the polytope P_{j+1}. If $(W_{j+1}, \mu_{j+1}) \notin \mathcal{G}$, the next polytope P_{j+2} is obtained from the separating hyperplanes H_{j+1} and L_{j+1}, such that $P_{j+2} \subset P_{j+1}$. The procedure extends until the optimal solution is achieved, i.e., $(W, \mu) \in \mathcal{G}$ (whenever such pair exists).

Observe that from Theorem 3, $\varphi(\mu) = W(\mu)$ (where $W(\mu)$ is the maximal solution of $\Theta(W, \mu) \triangleq \mathcal{R}_\mu(W) = \mathbf{0}$) is nonincreasing and concave on $[0; \mu_{max}]$; the graph of φ is depicted by the concave curve in figure 1, which illustrates the case *ii)* discussed in Theorem 4.

Furthermore, by construction, it follows that for $\mathcal{K}_I(\gamma)$ and $\mathcal{K}_C(\gamma)$ defined, respectively, in (11) and (20): $\mathcal{K}_I(\gamma) \subset \mathcal{K}_C(\gamma)$ since $\{(\varphi(\mu), \mu) \mid \mu \in [0; \mu_{max}],\ \varphi(\mu) > \mathbf{0}\}$ is part of the boundary of $\mathcal{G}$.

The occurrence of high-gain feedbacks can be prevented just imposing $W \geq \epsilon\mathbf{I}$, for $\epsilon > 0$ arbitrarily small (Peres *et al.*, 1994), (Scherer, 1989).

Remark 2: Problem **(P5)** could also be handled in an LMI setting by means of the Schur complement applied to $\Theta(W, \mu)$ (see (Boyd *et al.*, 1994), (Palhares and Peres, 1995)).

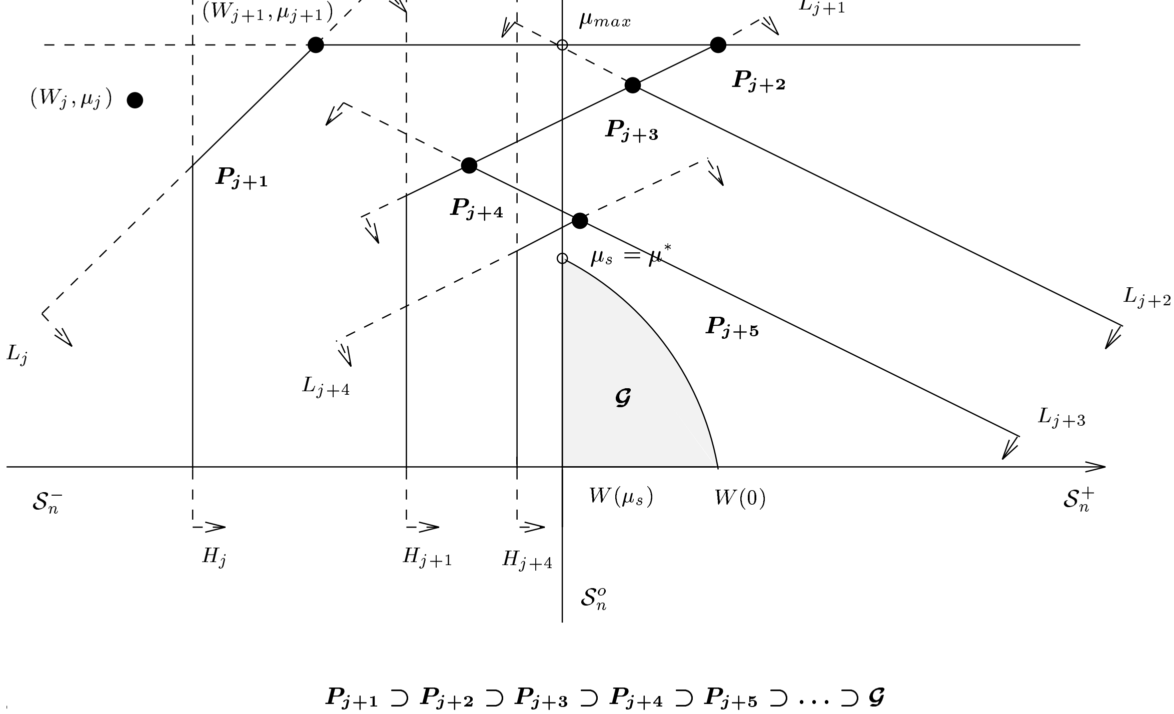

Figure 1 - Illustration of the cutting plane method. P_j is the j*th* polytope with L_j and H_j denoting the separating hyperplanes.

5 - Examples

Example 1 - This example is borrowed from (Petersen, 1987*a*). It concerns the stabilization of the longitudinal short period mode of the F4E fighter aircraft with mach number 0.5 and altitude 5000 ft as operating point. Assume that the entire state vector is available for feedback control. The model is given by

$$A = \begin{bmatrix} -0.9896 & 17.41 & 96.15 \\ 0.2648 & -0.8512 & -11.39 \\ 0 & 0 & -30 \end{bmatrix} \quad , \quad B_2 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Computing μ_{max} from Theorem 3 one gets $\mu_{max} = 4.4446$. By the iterative approach it follows that

$$\varphi(\mu_{max}) = W(\mu_{max}) =$$

$$= \begin{bmatrix} 133.6159 & -4.6459 & -41.0406 \\ -4.6506 & 1.0236 & 1.0084 \\ -41.0397 & 1.0017 & 13.1624 \end{bmatrix} > \mathbf{0}$$

and hence, from Theorem 4, item *i)*, $\mu^* = \mu_{max}$, or $\gamma^* = 1/\sqrt{\mu^*} = 0.4743$, with the optimal control gain

$$K_I = [\,-2.4372 \quad -6.3305 \quad -4.8350\,]$$

The convex approach, started with $\mu_M = 26.1168$ (obtained from Theorem 3.4 in (Peres *et al.*, 1994)), yields as the optimal solution of **(P5)** $\mu^* = 4.4446$, $\gamma^* = 0.4743$,

$$W_* = \begin{bmatrix} 124.2346 & -4.5672 & -38.1367 \\ -4.5672 & 1.0198 & 0.9799 \\ -38.1367 & 0.9799 & 12.2624 \end{bmatrix} > 0$$

with the associated control gain

$$K_C = [\,-2.6297 \quad -6.7909 \quad -5.1895\,]$$

As expected, both methods achieve the minimum value γ^*. Indeed, K_I from the iterative approach yields $\|(C - DK_I)(s\mathbf{I} - A + B_2K_I)^{-1}B_1\|_\infty = 0.4743$ and K_C from the convex approach furnishes $\|(C - DK_C)(s\mathbf{I} - A + B_2K_C)^{-1}B_1\|_\infty = 0.4743$ (see, for instance, (Palhares *et al.*, 1997) for computing the $\mathcal{H}_\infty$ norm).

Observe that, in this case, the iterative approach generates $\mathcal{K}_I(1/\sqrt{\mu^*}) = \{B_2'\varphi(\mu_{max})^{-1}\}$, while the convex approach provides $\mathcal{K}_C(1/\sqrt{\mu^*}) = \{\psi(W) \mid \mathbf{0} < W \leq \varphi(\mu_{max})\}$ (where $\psi(W)$ has been defined in (18)). In particular, the choice of $\epsilon = 10^{-3}$ in the convex approach is such that, $\epsilon\mathbf{I} \leq W \leq \varphi(\mu_{\max})$ (see remark 1). This is illustrated in figure 2.

Notice that $\mathcal{K}_C(1/\sqrt{\mu^*}) \supset \mathcal{K}_I(1/\sqrt{\mu^*})$ and furthermore, from Theorem 3, $\varphi(\mu)$ is nonincreasing and concave over $[0; \mu_{max}]$ (the graph of φ is denoted by the concave curve in figure 2).

Example 2 - Consider the following linear time invariant system described also in (Petersen, 1987*a*):

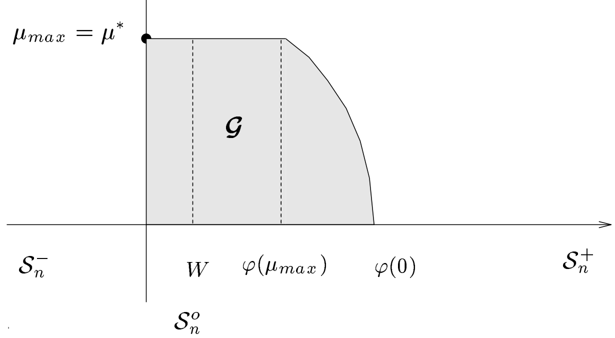

Figure 2 - μ^* and its associated positive definite matrices $\mathbf{0} < W \leq \varphi(\mu_{max})$.

$$A = \begin{bmatrix} -1.7020 & 50.7200 & 263.5000 \\ 0.2201 & -1.4180 & -31.9900 \\ 0 & 0 & -30.0000 \end{bmatrix} \;,\; B_2 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;,\; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \;,\; D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Carrying out the computation of μ_{max} and the associated $\varphi(\mu_{max})$ in Theorem 3, one gets $\mu_{max} = 2.6368$ and

$$\varphi(\mu_{max}) = \begin{bmatrix} 173.2376 & -11.6157 & -35.7981 \\ -11.6157 & -8.3706 & 4.2507 \\ -35.7981 & 4.2507 & 7.4652 \end{bmatrix} \ngeq \mathbf{0}$$

In this case, from Theorem 4, a high gain feedback sequence is necessary to approach the optimal level $\gamma^* = 1/\sqrt{\mu^*} = 0.7772$. Using the algorithm generated by the iterative approach (see (Scherer, 1989)), the optimal level μ^* can be approximated with an error given by $|\mu_{\ell+1} - \mu_\ell|/|\mu_\ell| \leq \delta$, $\delta > 0$, where $\lim_{\ell\to\infty} \mu_\ell = \mu^* = \mu_s$.

On the other hand, the convex approach (problem **(P5)**) can deal with this singularity by simply imposing $W \geq \epsilon\mathbf{I}$, $\epsilon > 0$ arbitrarily small, in the definition of set $\mathcal{G}$ (19). An illustration of this problem is depicted in figure 1.

Table 1 shows how the choice of $\epsilon > 0$ (convex approach) and $\delta > 0$ (iterative approach) affect the disturbance attenuation levels γ_I and γ_C, with the respective gain norms.

In both approaches, $\mathcal{K}_I(1/\sqrt{\mu_s}) = \emptyset$ and $\mathcal{K}_C(1/\sqrt{\mu_s}) = \emptyset$ (where $\mathcal{K}_I$ and $\mathcal{K}_C$ have been defined in (11) and (20), respectively). Note that μ_s always exists in the sense of Theorem 4, but not a suitable state feedback gain.

6 - Conclusion

The relationship between two approaches for the $\mathcal{H}_\infty$ control by state feedback has been discussed in this paper. In both cases, the stabilizing feedback gain is obtained from a positive definite matrix which can be either the maximal the set described by the convex approach contains as its boundary the solutions of the iterative one. Particularly, for precisely known systems, the iterative approach seems to be more attractive numerically, since it involves only Riccati equations computation. On the other hand, for systems with uncertain paramenters in the model, only the convex approach can be used in a quadratic stabilization sense (for references see, for instance, (Boyd *et al.*, 1994), (Peres *et al.*, 1993), (Palhares and Peres, 1995)).

REFERENCES

Bernussou, J., Geromel, J. C. and Peres, P. L. D., (1989). "A linear programming oriented procedure for quadratic stabilization of uncertain systems", *Systems & Control Letters* Vol. 13, pp. 65–72.

Boyd, S. P., El Ghaoui, L., Feron, E. and Balakrishnan, V., (1994). *Linear Matrix Inequalities in Systems and Control Theory*, SIAM Studies in Applied Mathematics, USA.

Doyle, J. C., Glover, K., Khargonekar, P. P. and Francis, B. A., (1989). "State-space solution to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems", *IEEE Transactions on Automatic Control*, Vol. 34, pp.831–847.

Faibusovich, L. E., (1987). "Matrix Riccati inequality: existence of solutions", *Systems & Control Letters*, Vol. 9, pp. 59–64.

Gahinet, P., (1994). "A new parametrization of $\mathcal{H}_\infty$ suboptimal controllers", *International Journol of Control*, Vol. 59, pp. 1031–1051.

Iwasaki, T. and Skelton, R. E., (1994). "All controllers for the general $\mathcal{H}_\infty$ control problem: LMI existence conditions and state space formulas", *Automatica*, Vol. 30, pp.1307–1317.

Khargonekar, P. P., Petersen, I. R. and Rotea, M. A., (1988). "$\mathcal{H}_\infty$-optimal control with state-feedback", *IEEE Transactions on Automatic Control*, Vol. 33, pp. 786–788.

Luenberger, D. G., (1984). *Linear and Nonlinear Programming*, Addison-Wesley Publishing Company, USA.

Palhares, R. M. and Peres, P. L. D., (1995). "Controle $\mathcal{H}_\infty$ por realimentação de estado: um procedimento convexo de otimização via LMIs", *Anais do XVIII Congresso Nacional de Matemática Aplicada e Computacional*, Vol. 1, pp. 210–214, Curitiba, Brasil.

$10^{[illegible]}$	0.7796	$2.3739 \times 10^{[illegible]}$	0.8133	125.6799
10^{-3}	0.7774	2.3533×10^4	0.7806	1.2837×10^3
10^{-4}	0.7772	2.3306×10^5	0.7775	1.2869×10^4
10^{-5}	0.7772	2.3078×10^6	0.7772	6.4489×10^4
10^{-6}	0.7772	2.2853×10^7	0.7772	1.3316×10^6

Table 1 - The disturbance attenuation γ and the gain-norm varying with $\epsilon = \delta$.

Palhares, R. M., Takahashi, R. H. C. and Peres, P. L. D., (1997). "$\mathcal{H}_\infty$ and $\mathcal{H}_2$ Guaranteed costs computation for uncertain linear systems", *International Journal of Systems Science*, Vol. 28, pp. 183–188.

Peres, P. L. D., Geromel, J. C. and Souza, S. R., (1993). "$\mathcal{H}_\infty$ guaranteed cost control for uncertain continuous-time linear systems", *Systems & Control Letters*, Vol. 20, pp. 413–418.

Peres, P. L. D., Geromel, J. C. and Souza, S. R., (1994). "Optimal $\mathcal{H}_\infty$ state feedback control for continuous-time linear systems", *Journal of Optimization Theory and Applications*, Vol. 82, pp. 343–359.

Petersen, I. R., (1987*a*). "A procedure for simultaneously stabilizing a collection of single input linear systems using non-linear state feedback control", *Automatica*, Vol. 23, pp. 33–40.

Petersen, I. R., (1987*b*). "Disturbance attenuation and $\mathcal{H}_\infty$ optimization: a design method based on the algebraic Riccati equation", *IEEE Transactions on Automatic Control*, Vol. 32, pp. 427–429.

Ran, A. C. M. and Vreugdenhil, R., (1988). "Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems", *Linear Algebra and Its Applications*, Vol. 99, pp. 63–83.

Scherer, C., (1989). "$\mathcal{H}_\infty$-control by state-feedback: an iterative algorithm and characterization of high-gain occurrence", *Systems & Control Letters*, Vol. 12, pp. 383–391.

Scherer, C., (1990). "$\mathcal{H}_\infty$-control by state-feedback and fast algorithms for the computation of optimal $\mathcal{H}_\infty$-norms", *IEEE Transactions on Automatic Control*, Vol. 35, pp. 1090–1099.

Scherer, C., (1994). "The state-feedback $\mathcal{H}_\infty$-problem at optimality", *Automatica*, Vol. 30, pp. 293–305.

Sz.-Nagy, B. and Foias, C., (1970). *Harmonic Analysis of Operator on Hilbert Space*, North-Holland Publishing Company.

Zhou, K. and Khargonekar, P. P., (1988). "An algebraic Riccati equation approach to $\mathcal{H}_\infty$ optimization", *Systems & Control Letters*, Vol. 11, pp. 85–91.

Zhou, K., Khargonekar, P. P., Stoustrup, J. and Niemann, H. H., (1995). "Robust performance of systems with structured uncertainties in state space", *Automatica*, Vol. 31, pp. 249–255.