A HYBRID CONTROLLER FOR A NONHOLONOMIC SYSTEM*

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Resumo Este artigo descreve um controlador híbrido incorporando lógica simples para o conhecido problema do integrador não-holonômico. Nosso controlador fornece estabilidade no sentido de Lyapunov, e é projetado de maneira a garantir que os sinais de controle mantenham-se uniformemente limitados e tenham no máximo um número finito de descontinuidades em qualquer intervalo finito. Em particular, o fenômeno de oscilações indesejáveis de alta freqüência conhecido como "chattering" não ocorre. Dois controladores alternativos, que empregam apenas informação parcial sobre o estado, também são discutidos.

Abstract This note describes a logic-based hybrid controller for the well-studied nonholonomic integrator. Our controller provides Lyapunov stability and is designed so that the control signals remain uniformly bounded and have at most a finite number of discontinuities in any finite interval. In particular, no "chattering" occurs. Two alternative controllers, which use only partial state information, are also presented.

Keywords: Nonholonomic systems; logic-based control; timed automata.

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1 Statement of the problem

The problem of interest is to regulate to zero the state of the so-called *nonholonomic integrator:*

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv. \end{aligned} \tag{1}$$

Here $u, v \in \mathbb{R}$ are the bounded control inputs. This problem has attracted considerable attention since it was shown that (1) is not smoothly stabilizable (Brockett, 1983). Solutions based on diverse concepts have been proposed in the literature (see for instance the recent survey of (Kolmanovsky and McClamroch, 1995)). Among those solutions is one by Hespanha (1996) which uses a switching control to which ours is similar — the main difference being that the switching scheme in Hespanha (1996) is event-driven and employs a kind of hysteresis to prevent chattering. The present work, in contrast, uses timed automata, closer to the work of Artstein (1995), which discusses a number of schemes employing concepts from automata theory to stabilize linear systems.

In §2 we define what is understood by "hybrid systems" in this paper, and in §3 a two-location hybrid controller which provides Lyapunov stability is described and analyzed. The feedback control signals presented in §4 stabilize the system and depend only on the state z. The bang-bang controller described in §5 achieves stability using measurements of z only. Concluding remarks are made in §6.

2 Hybrid systems

In this section we recall the definition of hybrid system given by Artstein (1995), which will be used in the rest of the paper. A hybrid system consists of:

1. A continuous-time control system $\dot{x} = f(x,u)$, $x \in \mathbb{R}^m$, $u \in U$.

2. The timed automaton triplet (Q, I, M), where $Q = \{q_1, \ldots, q_n\}$ is a finite set of automaton states (called locations), $I = \{\iota_1, \ldots, \iota_m\}$ is the input alphabet, and $M(q,\iota) : Q \times I \to Q$ is the transition map. To each q we associate a time $\Delta T(q)$, the "duration" of state q.

3. The feedback maps $u(x,q) : \mathbb{R}^m \times Q \to U$ and the map $\iota(x) : \mathbb{R}^m \to I$.

A hybrid state is given by a triplet $(x, q, \tau) \in \mathbb{R}^m \times Q \times [0, \infty)$ such that $0 < \tau \leq \Delta T(q)$; x is the plant's state, q the automaton state, and τ the remaining time before the next transition. The hybrid trajectories are determined in the following way. Starting from initial data (x_0, q_0, τ_0), the system evolves for time τ_0 on location q_0, that is, the feedback control is $u(\cdot, q_0)$ and $\dot{\tau} = -1$. After time τ_0 the input is $\iota = \iota(x(\tau_0))$ and the new location is $q_1 = M(q_0, \iota)$, via the transition map. The trajectories then evolve with feedback $u(\cdot, q_1)$ for time $\Delta T(q_1)$, and so on.

Notice that along hybrid trajectories the location $q(t)$ is piecewise constant, the state satisfies the dynamic equation associated to the piecewise continuous control $u(\cdot, q(t))$ and $\tau(t)$ satisfies $\dot{\tau} = -1$ except for the transition points where τ jumps to $\Delta T(q(t))$.

3 A two-location controller

Consider the system (1), and make the change of variables $(x, y, z) \mapsto (r, \theta, z)$ to cylindrical coordinates:

$$\begin{aligned} x &= r\cos\theta \Rightarrow u = \dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta}, \\ y &= r\sin\theta \Rightarrow v = \dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}. \end{aligned}$$

The equations above, together with the assignment $\theta(0) = 0$ if $r(0) = 0$, suffice to define u and v as a function of r and θ, which is all that is required. It follows that

$$\dot{z} = yu - xv = -r^2\dot{\theta}.$$

The controller we propose is a timed automaton with two locations q_1 and q_2, whose durations are arbitrary positive numbers ΔT_1 and ΔT_2. The control signals in each location are as follows:

Location q_1
$$\begin{aligned} \dot{r} &= -\,\mathrm{sat}\,(ar) \\ \dot{\theta} &= \tfrac{1}{r}\,\mathrm{sat}\,(z/r). \end{aligned}$$

Location q_2
$$\begin{aligned} \dot{r} &= \mathrm{sat}\,|z| \\ \dot{\theta} &= \tfrac{1}{r}\,\mathrm{sat}\,(z). \end{aligned}$$

Notice that the control signals are bounded, as $u^2 + v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \leq 2$ at each location. The constant a satisfies $0 < a < 1$, and the "saturation" function is

$$\mathrm{sat}\,(\xi) = \begin{cases} \xi & \text{if} \quad |\xi| \leq 1 \\ \mathrm{sign}\,\xi & \text{if} \quad |\xi| > 1. \end{cases}$$

The input alphabet contains two symbols, ι_1 and ι_2; if $|z|/r \leq 1$ or if $z = r = 0$ then $\iota(t) = \iota_1$ and if $|z|/r > 1$ or if $r = 0, z \neq 0$ then $\iota(t) = \iota_2$. The transition function satisfies $M(q, \iota_i) = q_i, i = 1, 2$; that is, the information index alone determines the location after a transition. Intuition behind the choice of the hybrid controller is as follows: while in location 1 both z and r are driven to zero (and consequently x and y as well). The location 2 controller moves x and y away from the origin so as to make it subsequently possible to drive z down without employing excessively large control actions.

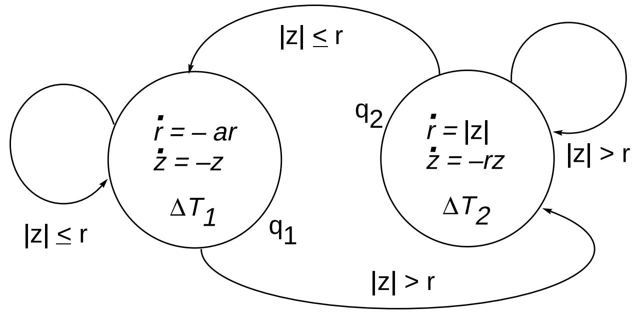

Figure 1 - Transitions diagram (unsaturated controls)

To analyze the closed-loop system, first check that, if the initial condition is such that $\iota(t_0) = \iota_2$, then eventually a transition to location 1 happens. This is because $\dot{r} = \mathrm{sat}\,|z| > 0$ and $\dot{z} = -r\,\mathrm{sat}\,(z)$ has the opposite sign of z at location 2, hence $|z|/r$ is decreasing and will reach the value 1. On the other hand at location 1

$$\frac{d}{dt}\left(\frac{z}{r}\right) = \frac{\dot{z}}{r} - \frac{z\dot{r}}{r^2} = -\frac{z}{r}\left(1 - \frac{\mathrm{sat}\,(ar)}{r}\right)$$

has the opposite sign of z/r, and transition $1 \to 2$ cannot happen. Thus we conclude that the number of discontinuities in the right-hand side of (1) is finite, and the differential equations of the closed-loop system admit a classical solution. In particular, chattering is impossible.

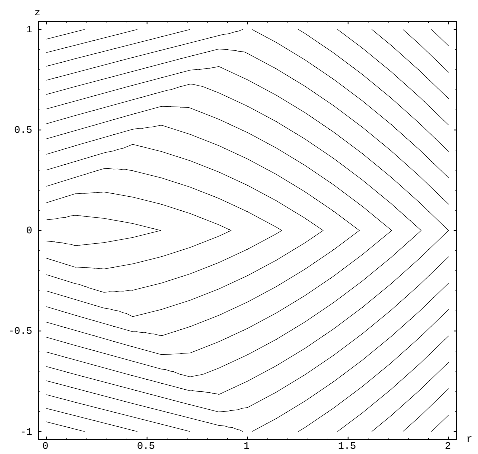

Figure 2 - Level curves of V on the $z \times r$ plane

Now consider the candidate Lyapunov-like function

$$V(x,y,z) = \begin{cases} \frac{1}{2}r^2 + 2|z| & \text{if} \quad |z| \leq r, \\ -r + \frac{1}{2}z^2 + 3|z| & \text{if} \quad |z| > r. \end{cases} \tag{2}$$

with $r = \sqrt{x^2+y^2}$. Call $m = \sqrt{x^2+y^2+z^2}$ the Euclidean norm of the system's state. It is straightforward to verify that

$$\frac{1}{4}m^2 \leq V \leq \frac{1}{2}m^2 + 3m,$$

so V as defined above is positive and proper. V is also continuous, because at points such that $|z| = r$ the two definitions coincide; it is not differentiable however[1]. Let us compute V's rate of change. At location 1, $|z| \leq r$, hence

$$\begin{aligned} \dot{V} &= r\dot{r} + 2\,\mathrm{sign}\,(z)\dot{z} \\ &= -r\,\mathrm{sat}\,(ar) - 2\,\mathrm{sign}\,(z)z < 0; \end{aligned}$$

and at location 2, if $|z| \leq r$,

$$\dot{V} = r\,\mathrm{sat}\,|z| - 2r\,\mathrm{sign}\,(z)\,\mathrm{sat}\,(z) = -r\,\mathrm{sat}\,|z| < 0,$$

while if $|z| > r$,

$$\begin{aligned} \dot{V} &= -\dot{r} + z\dot{z} + 3\,\mathrm{sign}\,(z)\dot{z} \\ &= -\,\mathrm{sat}\,|z| - 2rz\,\mathrm{sat}\,(z) - 3r\,\mathrm{sign}\,(z)\,\mathrm{sat}\,(z) < 0. \end{aligned}$$

One can also certify that V is also decreasing at points where it is not differentiable, namely $|z| = r$ and $z = 0$. Therefore V is decreasing everywhere except for the origin, establishing Lyapunov stability.

The rate of convergence is exponential in the sense that for every initial data there exist $T, C > 0$ such that $m(t) \leq Ce^{-a(t-T)}$ for $t \geq T$. The time T and the constant C are bounded on compact sets. However they cannot be uniformly bounded due to the use of bounded controls.

If one desired to use smooth controllers, one alternative would be to increase the number of locations rather than defining the controls at each location to saturate. More precisely we can construct eight locations splitting each one of q_1, q_2, (in this case it will not be possible to choose all ΔT arbitrarily).

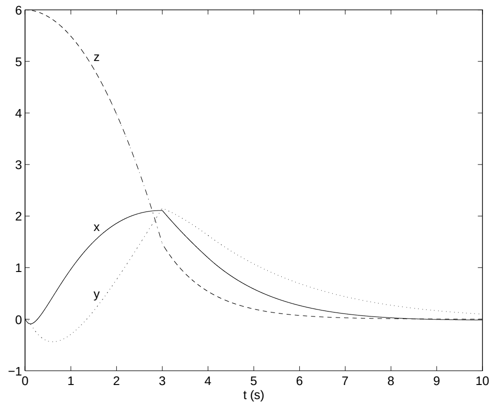

Figure 3 - Simulation results

In a simulation with $z(0) = 2$ and zero initial conditions on x and y, only one transition happens, from location 2 to location 1 at $t = 3$ seconds. Performance of the overall control system could be adjusted by changing the value of the design parameter a, here chosen equal to $1/2$, and of ΔT_1 and ΔT_2, both chosen 1 second.

[1]No surprise here — according to the Artstein-Sontag Theorem (Sontag, 1989), a differentiable control–Lyapunov function leads to smooth controls.

4 Stabilization using z

In this section we describe a hybrid control which depends only on z and stabilizes the nonholonomic integrator. The value of r is measured only at transition times to apply the map ι. (Measurement of z is required to construct a controller which brings the system's state to the origin, whether or not x and y are available.)

Location q_1 $\quad \dot{r} = z, \quad \dot{\theta} = z.$

Location q_2 $\quad \dot{r} = -z^3, \quad \dot{\theta} = z.$

Location q_3 $\quad \dot{r} = -z/2, \quad \dot{\theta} = -z.$

Location q_4 $\quad \dot{r} = -1, \quad \dot{\theta} = 0.$

Location q_5 $\quad \dot{r} = 0, \quad \dot{\theta} = -1.$

The input alphabet contains five symbols ι_i, $i = 1, \ldots, 5$. If $z > r$ then $\iota(t) = \iota_1$, if $r \geq z \geq r/2$ then $\iota(t) = \iota_2$, if $r/2 > z > 0$ and $r \leq 2$ then $\iota(t) = \iota_3$, if $r/2 > z > 0$ and $r > 2$ then $\iota(t) = \iota_4$, if $z \leq 0$ then $\iota(t) = \iota_5$. We assign a time for every location: $\Delta T_1 = \Delta T_2 = \Delta T_4 = \Delta T_5 = 1$, $\Delta T_3 = \frac{1}{2}\ln(4/3)$. The transition function satisfies $M(q, \iota_i) = q_i, i = 1, \ldots, 5$; that is, the information index alone determines the location after a transition.

From location 5 we switch to another location in finite time. Moreover we cannot switch to location 5 from any other location. From location 1 we switch to locations 2, 3 or 4 in finite time. From location 4 we switch to locations 2 or 3 in finite time. Moreover the choice of ΔT_4 is such that we cannot switch from location 4 to location 1. Indeed if we are in location 4 we have $r_0 > 2z_0$, $r_0 > 2$, hence after time 1 we have $r(1) > r_0/2 > z_0 = z(1)$.

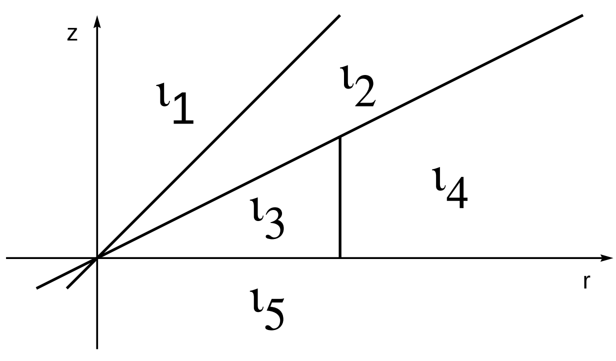

Figure 4 - The input values

From location 2 we can pass to locations 4 and 3, finally from location 3 we can switch to location 2. From location 2 we cannot pass to location 1. Indeed on the boundary between location 1 and 2 (more precisely between the regions associated to the inputs ι_1 and ι_2), we have $r = z$ and using location 2, $\dot{r} = -z^3 = -r^2 z = \dot{z}$, hence we stay on the boundary.

Notice that after the first switching from location 3 to location 2 we cannot reach location 4 anymore. Hence the only possibility of chattering is between locations 2 and 4, and between locations 2 and 3. But in both cases r is always uniformly decreasing. Therefore after a certain time we stay in locations 2 and 3, r tends to zero and from the condition $r \geq z > 0$ we obtain that z goes to zero as well. The choice of times prohibits switching from location 3 to location 1. To prove this observe that if we start from r_0, z_0 and we use location 3 then:

$$z(t) \leq e^{r_0^2 t} z_0, \qquad r(t) \geq r_0 - \frac{e^{r_0^2 t} z_0}{2}.$$

To obtain $z(\Delta T_3) \leq r(\Delta T_3)$ it is sufficient to check that:

$$e^{r_0^2 \Delta T_3} z_0 \leq r_0 - \frac{e^{r_0^2 \Delta T_3} z_0}{2}$$

or

$$\frac{3}{2} e^{r_0^2 \Delta T_3} \leq \frac{r_0}{z_0},$$

and this is ensured by $r_0/z_0 \geq 2$ and the definition of ΔT_3.

5 Stabilization via bang-bang controls

In this section we describe bang-bang stabilization via a particularly simple hybrid control. Starting from point (x_0, y_0, z_0) we describe two different possible bang-bang trajectories called respectively LT (long trip) and ST (short trip), which can be viewed as two different locations (there will be four locations overall, running clockwise or counter-clockwise). The LT (clockwise) trajectory corresponds to the control $u = 0, v = -1$ for a time length $2|y_0|$; control $u = -1, v = 0$ for a time length $2|x_0|$; control $u = 0, v = 1$ for time $2|y_0|$ and finally control $u = 1, v = 0$ for time $2|x_0|$. Hence the total time for LT is $4(|x_0| + |y_0|)$. The ST (clockwise) trajectory corresponds to control $u = 0, v = -1$ for time $3|y_0|/2$ and then control $u = -1, v = 0$ for time $3|x_0|/2$. Hence the total time for ST is $\frac{3}{2}(|x_0| + |y_0|)$. The LT and ST counterclockwise are defined similarly.

Now if $z_0 \geq 8|x_0 y_0| > 0$ then we apply LT clockwise; if $8|x_0 y_0| > z_0 \geq 0$ then we apply ST clockwise. For the case $z_0 < 0$, we do the same with z_0 replaced by $-z_0$ and clockwise replaced by counterclockwise. In this way the control is defined everywhere except if $x_0 = 0$ or $y_0 = 0$. But in this case we can apply any nonzero constant controls u, v for a short time.

To check convergence let us treat the case $z_0 \geq 0$, the other case being similar. We first apply LT until $z < 8|x_0 y_0|$. Notice that after every application of LT we have $z = z_0 - 8|x_0 y_0|$. Then we apply ST.

Hence assume that $0 \leq z_0 < 8|x_0 y_0|$. We have two cases: 1) $z_0 > (9/4)|x_0 y_0|$, 2) $z_0 \leq |x_0 y_0|$. Let us call x_1, y_1 and z_1 the coordinates of the point reached after applying

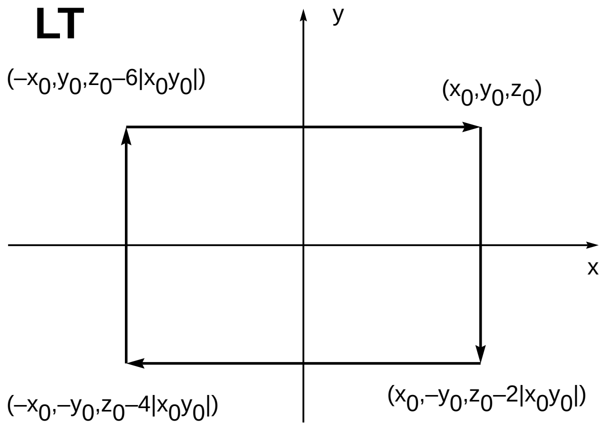

Figure 5 - The clockwise long trip

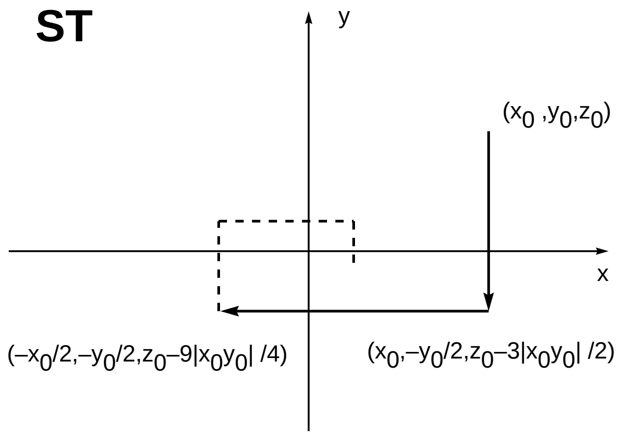

Figure 6 - The clockwise short trip

ST so that $z_1 = z_0 - (9/4)|x_0 y_0|$. We have $|x_1| = |x_0|/2$, $|y_1| = |x_0|/2$. Moreover, in the first case, $0 < z_1 < z_0 - (9/4)|x_0 y_0| < 6|x_0 y_0| < 24|x_1 y_1|$, while in the second case $|z_1| < (9/4)|x_0 y_0| < 9|x_1 y_1|$. In any case, after applying ST we possibly apply LT again for at most two times. Indeed if z_2 is the time after (possibly) two LT's, we have:

$$|z_1| < 24|x_1 y_1| \Rightarrow |z_2| < 8|x_1 y_1|.$$

Then we apply again ST.

Therefore, if we call (x_i, y_i, z_i) the position after the ith application of ST and ΔT_i the time between the ith and the $i + 1$st application, we obtain

$$|x_i| = \frac{1}{2^i}|x_0|, \qquad |y_i| = \frac{1}{2^i}|y_0|,$$

$$\begin{aligned} \Delta T_i &\leq \left(4 + 4 + \frac{3}{2}\right)(|x_{i-1}| + |y_{i-1}|) \\ &< 10\,(|x_{i-1}| + |y_{i-1}|) = \frac{10}{2^i}(|x_0| + |y_0|). \end{aligned}$$

Therefore

$$\sum_{i=1}^{+\infty} \Delta T_i < 10(|x_0| + |y_0|) \sum_{i=1}^{+\infty} \frac{1}{2^i} = 10(|x_0| + |y_0|),$$

hence in finite time x and y tend to zero. Moreover from the above estimates $z_i < 24|x_{i-1} y_{i-1}|$, thus z tends to zero as well.

Notice that if we apply the constant control $u = 0$, $v = 1$ for an interval of duration 1 we can measure the x coordinate knowing the z coordinate. In the same way we can measure the y coordinate using the control $u = 1$, $v = 0$. Using bang-bang controls we can stabilize the system observing only the z coordinate. Indeed we can initially apply these controls to measure x and y, and then use the strategy above, which permits us to keep track of these coordinates. Indeed the LT leaves the coordinates x and y unchanged, while ST divides both x and y by -2.

Through this scheme we have obtained convergence to the origin in finite time, but every control is chattering as the trajectories approach the origin. Moreover it is easy to check Lyapunov stability. Several similar schemes could be devised, changing the durations and the controls in order to avoid chattering, speed up or slow down convergence rates, lower the control effort, etc.

6 Concluding Remarks

The literature registers diverse approaches to the control problem discussed in the present paper. In our opinion the techniques exemplified here provide a flexible tool to the design of nonlinear controllers. Because there is a clear intuition behind the hybrid controllers presented, it is not difficult to modify them in order to satisfy given engineering requirements, without discarding the stability analysis. Whether hybrid controllers can be designed to reach better performance in comparison with the sundry competing schemes available in the literature (Kolmanovsky and McClamroch, 1995) remains to be seen.

The hybrid systems in §3 and §4 can be seen as discontinuous controls as well (modulo the timing of the transitions), but the hybrid formulation results more intuitive for implementation. The control proposed in §5 is genuinely hybrid in the sense that it cannot be described as a discontinuous feedback. It is a hybrid system in the sense explained in §2 if the duration ΔT of locations is allowed to depend on the state at each transition time. This new definition of hybrid system is more powerful but may be more complicated to implement.

The use of hybrid controllers for stabilization of higher-dimensional nonholonomic systems in Chaplygin form is currently under investigation.

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