ON THE THEORY OF APPROXIMATE REASONING

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1 - INTRODUCTION

Zadeh (1979) introduced the theory of approximate reasoning. This theory, based upon the use of fuzzy sets, provides a powerful framework for reasoning in the face of uncertain information. Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables. In this tutorial, we provide a formal framework for the reasoning system we shall call Approximate Reasoning (AR). We then show how this system provides a mechanism for modeling and making inferences from imprecise functional relationships. This mechanism forms the basis of the fuzzy systems modeling technique used in many of the applications of fuzzy logic control.

In Yager *et al.* (1987) one can find a collection of Zadeh's pioneering papers on this topic. Dubois and Prade (1991; 1991a) provide a comprehensive overview of this topic. A large number of researchers have made important contributions to this area, Dubois *et al.*(1993) provide, in one source, a collection of some of the seminal papers in this discipline as well as an annotated guide to the literature. Yager & Filev (1994) provide a introductory text on fuzzy modeling and control.

2 - PRIMARY ELEMENTS OF AN AR SYSTEM

The primary elements of an AR system are a collection of variables

$$V_1, \cdots V_n$$

and an associated collection of sets $X_1, \cdots, X_n$ where X_k is called the base set (or universe of discourse or domain) of V_k. Essentially X_k provides the set of allowable values for the variable V_k.

A **joint variable** is any collection or tuple of one or more distinct variables. Examples of this are

$$V_1, (V_1, V_2), (V_1, V_5, V_3).$$

A **proposition** in this system is a statement of the form

$$V \text{ is } M.$$

In the above V is a joint variable and M is a fuzzy subset of the cartesian product of the base sets of the atomic variables which make up the joint variable V. We shall call the cartesian product, X, of the base sets of the variables in V the universe of V. Thus M is a fuzzy relationship on X. A statement or proposition which involves only one variable shall be called a **canonical statement.**

Example: If V_1 is a variable corresponding to an objects age and V_2 is a variable corresponding to an objects weight then the joint variable (V_1, V_2) would have as its base set pairs consisting of an age and weight.

We now define a concept called the **possibility** of a proposition. This concept plays a fundamental role in describing inconsistencies and conflicts in our knowledge.

Def.: Given A proposition *V is A* we can define

$$\text{Poss [V is A]} = \text{Max}_x[A(x)]$$

A proposition V is A is called normal (or consistent) if Poss [V is A] = 1, that is if there exists at least one element x in the base set of V such that A(x) = 1. A proposition is called inconsistent (conflicting or a contradiction) if Poss[V is A] = 0, that is A is the null set.

Def: A proposition V is A is called a **tautology** if A(x) = 1 for all x in the domain of V. (That is A is the domain of V).

The manipulation of knowledge, propositions, in AR is based upon two fundamental operations, **conjoin** and **containment**. We first introduce the operation of conjoin which is essentially a generalized conjunction/cartesian product operation. This operation will play a central role in combining propositions.

Def: Assume V_A and V_B are two joint variables on the bases X and Y respectively.

Let

$$V_A \text{ is } M \text{ and } V_B \text{ is } N$$

be two propositions. Their **conjoin** (or conjunction) denoted

$$V_A \text{ is } M \times V_B \text{ is } N$$

is the proposition

$$V \text{ is } P$$

where V is a joint variable consisting of the union of the atomic variables making up V_A and V_B and P is a fuzzy subset of the domain of V, such that for each z in the domain of V

$$P(z) = M(x) \wedge N(y) \quad [\wedge = \text{Min}].$$

In the above x is the element in X which agrees[1] with z on the domains they have in common and similarly y is the element in Y which agrees with z on the domain they have in common.

Example: Assume that V_1, V_2, and V_3 are atomic variables with base sets X, Y and Z respectively. Assume we have two propositions

$$V_a = (V_1, V_2) = \left\{\frac{.7}{(x_1,y_1)}, \frac{.9}{(x_1,y_2)}, \frac{1}{(x_2,y_1)}, \frac{.2}{(x_2,y_2)}\right\}$$

$$V_b = (V_2, V_3) = \left\{\frac{.8}{y_1,z_1}, \frac{1}{y_1,z_2}, \frac{.4}{y_2,z_1}, \frac{1}{y_2,z_2}\right\}$$

the conjunction of these two propositions is

$$V_c = (V_1, V_2, V_3) =$$

$$\left\{\frac{.7}{x_1,y_1,z_1}, \frac{.7}{x_1,y_1,z_2}, \frac{.4}{x_1,y_2,z_1}, \frac{.9}{x_1,y_2,z_2}, \frac{.8}{x_2,y_1,z_1}, \frac{1}{x_2,y_1,z_2}, \frac{.2}{x_2,y_2,z_1}, \frac{.2}{x_2,y_2,z_2}\right\}$$

In the case when the two variables conjoined are the same, this operation reduces to the usual fuzzy set intersection operation,

$$V_A \text{ is } M \cap V_A \text{ is } N = V_A \text{ is } M \cap N.$$

In this case when V_A and V_B are disjoint, have no common variables, this operation gives us the classical cartesian product

$$V_A \text{ is } M \cap V_A \text{ is } N = (V_A, V_B) \text{ is } M \times N.$$

In the case when V_A and V_B have at least one common variable, this operator acts like a join operator.

As a result of the above observations we shallfind it convenient to use $\times$ and $\cap$ interchangeably to denote the conjoin.

We next define a special conjoin operator. This operation plays a role making propositions which are not necessarily about the same variable be about the same joint variable.

Def: Assume V_A and V_B are two joint variables such that V_B contains all the variables that are in V_A, denoted $V_A \subset V_B$. The **cylindrical extension** of the proposition

$$V_A \text{ is } M$$

to V_B is the proposition

$$V_B \text{ is } M^o$$

defined by

$$V_B \textit{ is } M^o = V_A \textit{ is } M \times V_1 \textit{ is } X_1 \times V_2 \textit{ is } X_2, \ldots \times V_q \textit{ is } X_q,$$

where V_1, V_2, ... V_q are the atomic variables that are in V_B but not in V_A and the X_i's are the respective base sets of these variables.

Thus we see that the cylindrical extension is the conjoin of a proposition with a collection of tautologies.

The cylindrical extension of V_A is M to V_B is M^o can be expressed in terms of membership functions of M and M^o as

$$M^o(y) = M(x)$$

where x is the element in the base set of V_A which agrees with y for the portion of V_B that is V_A.

Example: Assume we have the proposition

$$V_a = (V_1, V_2) = \left\{\frac{.7}{(x_1,y_1)}, \frac{.9}{(x_1,y_2)}, \frac{1}{(x_2,y_1)}, \frac{.2}{(x_2,y_2)}\right\}$$

its cylindrical extension to the variable $V_c = (V_1, V_2, V_3)$ is the set

$$\left\{\frac{.7}{x_1,y_1,z_1}, \frac{.7}{x_1,y_1,z_2}, \frac{.9}{x_1,y_2,z_1}, \frac{.9}{x_1,y_2,z_2}, \frac{1}{x_2,y_1,z_1}, \frac{1}{x_2,y_1,z_2}, \frac{.2}{x_2,y_2,z_1}, \frac{.2}{x_2,y_2,z_2}\right\}$$

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1 When $V_A = (V_1\ V_2\ V_3\ V_4)$ and $V_B = (V_2\ V_3\ V_5)$ then $V = (V_1\ V_2\ V_3\ V_4\ V_5)$. In this situation for the value z = (a, b, c, d, e) the x in agreement is x = (a, b, c, d) and the y in agreement is y = (b, c, e).

We now introduce the second basic operation in the AR system, that of **containment**. This operation will play a significant role in the determination of valid inferences from knowledge bases.

Def: Assume V_A is M and V_B is N are two propositions, we say that V_A is M **contains** V_B is N, denoted V_B is N $\subset$ V_A is M, if

$$M^o(z) \geq N^o(z) \qquad \text{for all z,}$$

where M^o and N^o are the cylindrical extensions of M and N to the base set of the variable V, which is the union of the atomic variables in both V_A and V_B.

The following relates containment and conjunction. Assume that P_1, P_2, and P_3 are three propositions. If $P_3 = P_1 \times P_2$ then

$$P_3 \subset P_1 \text{ and } P_3 \subset P_2.$$

The following observation will subsequently prove to be a useful result which will account for the monotonicity of the AR system.

Theorem: For any three propositions P_1, P_2, and P_3 such that $P_1 \supset P_3$ and $P_2 \supset P_3$ then

$$P_1 \times P_2 \supset P_3.$$

We introduce the concept of the equivalence of two propositions.

Def: Two propositions P_1 and P_2 are **equivalent** if

$$P_1 \subset P_2 \text{ and } P_2 \subset P_1.$$

We denote this as $P_1 = P_2$.

Effectively we see that two propositions V_A *is M* and V_B *is N* are equivalent if their cylindrical extensions to the joint variable V consisting of the union of the variables in V_A and V_B are the same. If V_A and V_B are the same then they are equivalent if N = M, N(x) = M(x) for all x.

Theorem: Assume that P_1 and P_2 are two equivalent propositions, then for any proposition P_3

$$P_3 \subset P_1 \textit{ iff } P_3 \subset P_2$$

$$\textit{and } P_3 \supset P_1 \textit{ iff } P_3 \supset P_2$$

Similarly

$$P_3 \cap P_1 = P_3 \cap P_2$$

In this section we have introduced the basic syntax and operations of the approximate reasoning system. We have also discussed a number of properties associated with these operations which will of later use.

3 - SEMANTICS OF THE AR SYSTEM

In this section we shall introduce a semantics associated with the AR system. When V_i is an atomic variable the specification of X_i as its universe of discourse is meant to indicate that X_i is the set of all the possible values that the variable V_i can assume. A proposition

$$V_i \textit{ is } A_i$$

is meant to indicate that the value of variable V_i lies in the set A_i.

In the more general setting when V is a joint variable consisting of q atomic variables, V_i, i = 1, ··· q then the statement

$$\text{V is M}$$

indicates that the relationship M contains the collection of tuples among which the actual value of the joint variable V resides. In particular if $V = (V_1, V_2, V_3)$ and M = {(a, b, c), (g, b, f)} then there are only two possible solutions:

$$V_1 = a,\ V_2 = b,\ V_3 = c$$

or

$$V_1 = g,\ V_2 = b,\ V_3 = f.$$

Following Zadeh's notation we can call M(z) the possibility that V assumes the value z.

Example: Consider the two variables V_1 (*city of residence of person*) and V_2 (*age of person*).

If we have the proposition

$$(V_1, V_2) = \left\{\frac{1}{(N.Y.,32)}, \frac{1}{(Boston,29)}, \frac{1}{(N.Y.,29)}, \frac{1}{(L.A.,25)}\right\}$$

This information indicates that the person either lives in N.Y. and is 32 years old or lives in Boston and is 29 years old or lives in N.Y. and is 29 years old or lives in L.A. and is 25 years old.

Under these semantics, we see that the operation of conjoin is indicating the set of possible values on which both propositions agree. The nature of this operation essentially reduces the possible solutions.

With these semantics, we see that for a proposition

$$\text{V is A}$$

the smaller the set A is the more the informative it is with respect to the actual value of V, there are less possible values and our uncertainty is reduced. However this statement is not absolute in the sense that if A becomes to small, subnormal, then we start loosing information, our knowledge base becomes inconsistent.

Zadeh (1979) introduces a number of translation rules which allow us to represent some common linguistic statements in terms of propositions in our language. In the following we describe some of the these translation rules.

Negation of a proposition:

$$\textbf{not } V \text{ is } A \rightarrow V \text{ is } B$$

where B is the negation of A, ie

$$B(x) = \bar{A}(x) = 1 \cdot A(x) =$$

Anding of propositions:

The statement

$$V_A \text{ is } A \textbf{ and } V_B \text{ is } B$$

is defined via our conjunction/conjoin operator as

$$V \text{ is } A \times B$$

where V is the union of the atomic variables in V_a and V_b.

Oring of propositions:

The statement

$$V_A \text{ is } A \textbf{ or } V_B \text{ is } B$$

is defined by V is $A^o \cup B^o$ where V is the union of V_A and V_B, A^o and B^o are the cylindrical extensions of A and B respectively and $\cup$ is the union in the sense that

$$F = A^o \cup B^o$$

is defined by

$$F(z) = \text{Max } (A^o(z), B^o(z)) = A^o(z) \vee B^o(z).$$

Ply operation: The ply or implication operation

$$\text{if } V_A \text{ is } A \text{ then } V_B \text{ is } B$$

translates to the proposition

$$V \text{ is } E$$

where V is the union of V_A and V_B and E is a subset of the base set of V. The set E can be defined in a number of different ways. The two that are used most are the following

I. $E = \bar{A} \cup B$

II. $E(z) = \text{Min } [1, 1 - A^o(z) + B^o(z)]$

This last statement can also be written as

$$E(z) = \text{Min } [1, 1 - A(x) + B(y)]$$

where x and y are the elements in the base set of V_A and V_B that agree with z.

4 - DEDUCTION IN AR

In this section we shall introduce the fundamental inference mechanism used in the theory of approximate reasoning.

Inference Rule-1(The entailment principle): From a proposition

$$V_A \text{ is } A$$

we can deduce (infer) the proposition

$$V_B \text{ is } B$$

if V_A is $A \subset V_B$ is B.

The rational for this rule is based upon the semantic meaning of the propositions in AR as indicating the possible values for the variable. We recall that the statement *V_A is A* indicates that the value for the variable V_A lies in the set A. Then the above rule is reflecting the obvious fact that if the value of the variable lies in a set then it must also lie in a bigger set.

An example of this rule is that from the fact that *John is in his thirties* we can infer that John is over 21.

The next fundamental inference rule reflects the fact that if we have two propositions the information conveyed by these two propositions looked at together is their conjunction, is the set of values which they both say are possible.

Inference Rule-2: From the two propositions V_A is A and V_B is B we can infer the conjuncted proposition

$$V_A \text{ is } A \times V_B \text{ is } B.$$

The basic reasoning process uses the deduction process described in the following.

Def: A **deduction** from a set of premises $(P_1, \cdots, P_q)$ in AR is a sequence of well-formed formulas $B_1, B_2, \cdots, B_n$, where each B_k is either

(1). A premise

(2). A tautology

(3). For some $i < k$, B_k results from B_i as a result of inference rule-1.

(4). For some $i, j < k$, B_k results from B_i and B_j as a result of inference rule-2.

If from the premises $(P_1, \cdots, P_n)$ there exists a deduction terminating in B, we denote this as

$$(P_1, \cdots, P_q) \vdash B$$

and say B is inferable from $(P_1, \cdots, P_n)$.

The deduction process exhibits a **monotonicity** property described below

(Monotonicity): If $(P_1, \cdots P_n) \vdash B$ then $(P_1, \cdots P_{n+1}) \vdash B$.

It should be highlighted that one impact of inference rule-1 is that if V_A is M is deductible from some premise so is any proposition V_B is N that is equivalent to it.

<u>**Theorem**</u>: Assume $P_1, \cdots, P_q$ are a collection of premises. Let $P_1^*, P_2^*, \cdots, P_r^*$ and $P_1^+, P_2^+, \cdots, P_t^+$ be two sub-collections of these premises such that

$$\left(P_1^*, P_2^*, \cdots, P_r^*\right) \vdash H$$
$$\left(P_1^+, P_2^+, \cdots, P_t^+\right) \vdash G$$

If

$$(G, H) \vdash P$$

then

$$(P_1, \cdots, P_q) \vdash P$$

The proof follows from the definition of deduction.

The following result, the proof of which can be found in Yager & Filev (1994), delineates all the propositions inferable from a collection of propositions.

<u>**Theorem:**</u> A proposition B is inferable from a collection of premises, $(P_1, \cdots P_q) \vdash B$, iff B contains the conjunction of all premises, ie

$$(P_1 \cap P_2 \cap \cdots \cap P_q) \subset B$$

The implication of this theorem is very important and can be seen to form the essential basis of the reasoning mechanism used in the theory of approximate reasoning. What this theorem says is that if one has collection of premises, $(P_1, \cdots P_q)$, in order to find out if some proposition H is inferable from this knowledge base we can proceed as follows

1. Form the conjunction of all the premises

$$P = P_1 \cap P_2 \cap \cdots P_q$$

2. H is inferable from the premises if

$$P \subset H$$

5 - MINIMAL SOLUTIONS AND PROJECTIONS

In this section we are interested in minimal solutions for a variable. The reason for our interest in these minimal solutions is based on the following. Assume $P_1, \cdots P_q$ are a collection of premises. Let V be a variable, if *V is A* can be inferred from these propositions, then for any B such that $A \subset B$, using Inference Rule-1, *V is B* can also be inferred. Thus we are interested in investigating in this section if there exists for any given variable V some proposition **V is G** inferable from $P_1, \cdots P_q$ such that every proposition of the form **V is H** satisfies the condition $H \subset G$. In this case G would serve as some minimal set associated with V from which we can easily generate all the propositions associated with V. Thus we are interested in finding minimal solution sets for any joint variable. This section shall answer this in the affirmative. In order to investigate this issue, we introduce the following definition of the projection of a proposition.

<u>**Def:**</u> Let V_A *is M* be a proposition and let V_B be some variable. The **projection of V_A *is M* *onto* V_B** denoted, $Proj_{V_B}[VA\ is\ M]$, is the proposition

$$V_B \text{ is } N$$

where

$$N(z) = Max_Q\ [M(x)]$$

In the above Q is the set of all x that agree with z on the variables which V_A and V_B have in common. We also stipulate that by definition if V_A and V_B are disjoint then the Max is 1.

(Under this stipulation if V_A and V_B are disjoint, $proj_{V_B}V$ is $M = V_B$ is X where X is the base set of V_B.)

The next theorem, whose proof can be found in Yager & Filev (1994), shows that projection results in a proposition containing the original proposition. This facts leads to the conclusion that projection is a valid inference mechanism, it is a special case of Inference Rule-1..

<u>**Theorem:**</u> Assume

$$Proj_{V_B}[V\ is\ M] = V_B\ is\ N$$

then

$$V_A \text{ is } M \subset V_B \text{ is } N.$$

An immediate corollary to this theorem is the following.

<u>**Corollary:**</u> Let $P_1, P_2, \cdots, P_q$ be a collection of premises. Let V_A be some joint variable and let the proposition V_A is M be inferable from the premises,

$$(P_1, \cdots P_q) \vdash V_A \text{ is } M,$$

then for any variable V if

$$V \text{ is } N = Proj_V\ [V_A \text{ is } M]$$

it follows that

$$(P_1, \cdots, P_q) \vdash V \text{ is } N.$$

We now indicate that the projection of the conjunction of all the premises in a knowledge base provides the desired minimal sets.

Theorem: Let $P_1, \cdots, P_q$ be a collection of premises and denote $P = (P_1 \times P_2 \times \cdots \times P_q)$. Let V be any joint variable, then

$$(P_1, \cdots, P_q) \vdash V \text{ is } N$$

iff it is the case that

$$V \text{ is } M \subset V \text{ is } N$$

where

$$M = \text{Proj}_V\, P.$$

The fundamental implication of this theorem is that there exists minimal sets associated with each variable which can be used to generate any inference related to that variable. For any joint variable V let M be the projection onto V of the conjunction of all the premises, we shall call the set M the **minimal value** of V. Because of the uniqueness of M we can call it simply the value of V.

At this point we can summarize the AR process of finding the value of a variable V from a collection of premises $(P_1, P_2, \cdots, P_q)$:

1. Form the conjunction of the premises:

$$P = (P_1 \times P_2 \times \ldots \times P_q)$$

2. Find $A = \text{Proj}_V(P)$

3. *V is A*

We next introduce the concept of monotonicity and show that AR is monotonic.

Def: Let $(P_1, \cdots, P_n)$ is be a collection of premises. Assume for some variable V, V is M, is the minimal inferable proposition. Let P_{n+1} be any additional premise and let V is M_2 be the minimal proposition inferable from $(P_1, \cdots P_n, P_{n+1})$. We say that our system is **monotonic in** V if for all P_{n+1}

$$V \text{ is } M_2 \subset V \text{ is } M_1$$

Furthermore we say that our system is **monotonic** if it is monotonic for all V.

One important benefit of monotonicity is that the addition of new information doesn't invalidate any previously inferred conclusions. This characteristic greatly simplifies the use of the reasoning system.

An important characteristic of the AR system is that it is monotonic. This result follows directly from the fact that

$$(P_1 \times P_2 \times \cdots \times P_n) \subset (P_1 \times P_2 \times \cdots \times P_n \times P_{n+1}).$$

From our rules of inference, if two propositions are inferable, then their conjunction is inferable. In particular, if V_1 is A_1 and V_2 is A_2 are two valid propositions where A_1 and A_2 are the minimal values for V_1 and V_2 there conjunction

$$V_1 \text{ is } A_1 \cap V_2 \text{ is } A_2 = (V_1, V_2) \text{ is } E$$

is also a valid deduction. However as the following theorem shows E is not necessarily the minimal value for the joint variable. We shall illustrate this by counter example. Consider an AR system with two variables V_1 and V_2. Assume the knowledge base consists of one proposition

$$(V_1, V_2) \text{ is } A$$

where

$$A = \{(a, 1), (b,2)\}.$$

The above proposition is also the minimal value for the joint variable (V_1, V_2). In addition the minimal value of V_1 is $\{a, b\}$ and that of V_2 is $\{1, 2\}$. However the conjunction of these two propositions is

$$(V_1, V_2) \text{ is } \{(a,b), (a,2), (b,1), (b,2)\}$$

which is not equal to A.

Thus we see that while the conjunction of two minimal statements leads to a valid statement, it may not be minimal with respect to the joint variable. This observation greatly complicates the process of obtaining minimal propositions, it precludes us from simply finding the the minimal values of all the atomic variables and using these to build up the minimal propositions of the joint variables.

While the conjoin of minimal sets doesn't necessarily lead to the minimal sets, the projection of minimal sets onto smaller spaces necessarily does. Assume from a collection of premises that *V is M* is the minimal proposition involving the joint variable V. Let V_A be another joint variable all of whose components lie in V. If

$$V_A \textit{ is } Q = \text{Proj}_{V_A}[V \textit{ is } M]$$

then Q is the minimal set for V_A.

6 - FUNCTIONAL REPRESENTATIONS

In some applications, especially in intelligent control, one is interested in the representation of complex, nonlinear and perhaps ill defined functional relationships in a manner amenable to implementation in the AR environment. In this section, we shall show how the theory of approximate reasoning can be used to provide this kind of representation and how the inference mechanism of AR can allow us to find solutions to these functional relationships.

Assume U and V are two variables taking their values in the sets X and Y respectively. Usually in the control environment these sets are the real line. A function relating these variables is of the form $V = f(U)$. A simple example of a function is $V = 2U^2 + 6U + 9$. As indicated above in many situations the relationship between the variables is not so easily expressed in terms of a simple all encompassing equation.

It is well established that a function of the form $V = f(U)$ can be expressed as a relationship F, a subset of the cartesian space $X \times Y$, consisting of pairs of the form (x, y) such that $(x, y) \in F$ if the pair $V = y$ and $U = x$ is a solution of the above function, $y = f(x)$. Formally we can represent the knowledge contained in this relationship as a proposition

$$(U, V) \text{ is } F.$$

This proposition indicates that any solution for the joint variable (U, V) must be some ordered pair in F.

In the following, we show how the above proposition can be viewed as a disjunction (union) of more basic propositions. Without loss of generality we can denote

$$F = \{a_1, a_2, \ldots, a_n\}$$

where each a_i is a 2-tuple (x_i, y_i) such that $y_i = f(x_i)$. From this we see that F can be viewed as a disjunction of sets

$$F = \bigcup_{i=1}^{n} F_i$$

where each F_i is a singleton set consisting of one of the elements of F, $F_i = \{a_i\}$.

In addition, each set F_i can be viewed as a cartesian product (anding) of X_i and Y_i where

$$X_i = \{x_i\} \text{ and } Y_i = \{y_i\},$$

and

$$F_i = X_i \times Y_i.$$

In this view, each F_i can be described a solution point or proposition of the form

P_i: When U is X_i it is the case that V is Y_i.

This proposition is understood to be interpreted or translated as *the conjoin of the constituents*, U is X_i and V is Y_i. We call these *when* type propositions.

From this point of view, the proposition ***(U, V) is F*** can be seen as the *oring* (disjunction) of a collection of when type propositions:

(U, V) is F = (U, V) is F_1 ***or***(U, V) is F_2 ***or*** (U, V) is F_3, ···***or*** (U, V) is F_n.

What is important to note is that (U, V) is F is made up of a disjunction of each of these basic proposition. Each basic proposition corresponds to a known solution point.

There exists an alternative formulation for F. The view corresponds to an *anding*(conjunction) of a collection of basic propositions. We start again with

$$F = \{a_1, a_2, \cdots, a_n\}$$

where each a_i is a ordered pair, (x_i, y_i), corresponding to a solution. It can be shown that that if we define the subset E_i of $X \times Y$ as

$$E_i = \bar{X}_i \cup Y_i$$

then it is the case that

$$F = \bigcap_{i=1}^{n} E_i$$

This representation provides a disjunctive view of F. In this disjunctive view of F the basic components are the E_i's which have the form above and are semantically equivalent to the proposition

E_i: *If U is X_i then V is Y_i.*

where $E_i = \bar{X}_i \cup Y_i$.

This formulation for E_i is essentially the classical implication operator used in logic. Thus in this view we see that

(U, V) is F = (U, V) is E_1 ***and*** (U, V) is E_2 *and* ··· ***and*** (U, V) is E_n,

which is a conjunction of the primary propositions, each which of is interpreted as a logical implication. This view is more in the spirit of the production rule formulation of expert systems.

In many environments the knowledge available about the relationship F is not of sufficient detail or precision to allow us to specify that when $U = x_i$ it is the case $V = y_i$. In these imprecise environments we may have our information in larger granules.

For example, we may be able to state that

When the sales is between 10-15K it is the case that our profits are between 3-4K.

More formally in these complex environments the propositions necessary to describe F_i involve pairs *U is A_i* and *V is B_i* where A_i and B_i are fuzzy subsets of X and Y which are not necessarily single point sets. Figure 1 and Figure 2 illustrate the two different situations.

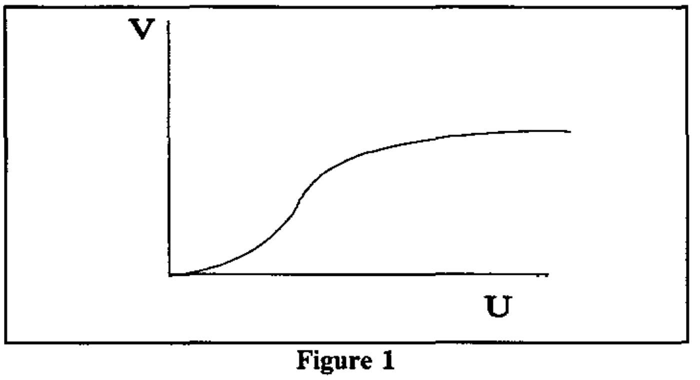

Figure 1

In Figure 1 we know the relationship between the variables U and V precisely whereas in Figure 2 we have the relationship in terms of the rectangles. Each rectangle says that for a U value anywhere in the base of the rectangle, the V will be somewhere in the height of rectangle. Thus, in this case each of the F_i's become a rectangle

$$(U, V) \text{ is } F_i = U \text{ is } A_i \times V \text{ is } B_i.$$

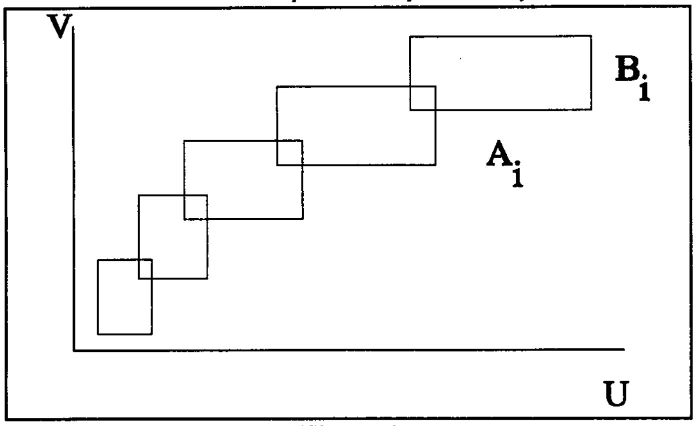

Figure 2

Given the relationship F and the information that U equals x_0, a problem of interest becomes that of finding the value for V. From the mathematical point of view this can be seen as solving the equation. From a logical point of view this can be seen as a generalized form of *modus ponens.* In particular we must use the deductive reasoning mechanism of the theory of approximate reasoning. We recall that the process of finding a solution, making a deduction in the theory of approximate reasoning consists of two steps

1. Combine all propositions via the conjunction operation

2. Project onto the variable of interest

In our problem we have two propositions, the relationship between the two variables U and V and the second proposition is the value of the variable U and we desire to find the value for the variable V. Formally we have

$$P_1: (U, V) \text{ if } F$$
$$P_2: U \text{ is } \{x_0\}.$$

Taking the conjunction of these two datum we get

$$(U, V) \text{ is } G = (U, V) \text{ is } F \times U \text{ is } \{x_0\}.$$

In this situation, we get

$$G = F \cap X_0,$$

where $X_0 = \{x_0\} \times Y = \{(x_0, y_1), (x_0, y_2), \cdots\}$ is the cylindrical extension of $\{x_0\}$. The next step is to take the projection of *(U, V) is G* onto V to obtain the solution for V. Thus if we denote

$$S = \text{Proj}_v[G]$$

then our solution is *V is S.*

The two representations of F, the conjunctive and disjunctive, lead to two formulations for obtaining the solution. The disjunctive view leads to what we shall call the *constructive* solution approach whilst the conjunctive approach leads to a *destructive* approach to solution formulation.

First let us consider the disjunctive view. In this case we recall

$$F = \bigcup_i F_i$$

where

$$F_i = A_i \cap B_i.$$

In this case

$$G = F \cap X_0 = (\bigcup_i F_i) \cap X_0 = \bigcup_i (F_i \cap X_0).$$

Denoting $D_i = (F_i \cap X_0)$ we get that

$$G = \bigcup_i D_i.$$

The solution set S is obtained as

$$S = \text{Proj}_v\{G\} = \bigcup_i \text{Proj}_v(D_i).$$

It is seen that

$$D_i = (A_i \cap \{x_0\}) \times (B_i \cap Y) = (A_i \cap \{x_0\}) \times B_i.$$

Let us denote $S_i = \text{Proj}_v\,(D_i)$ then $S = \bigcup_i S_i$.

Recalling that

$$S_i(y) = \text{Proj}_v\,(D_i)(y) = \text{Max}_x[D_i(x, y)]$$

From this we see that

$$S_i(y) = A_i(x_0) \wedge B_i(y).$$

Finally from the above it follows that

$$S = \bigcup_i S_i$$

thus we that $S(y) = \text{Max}_i[S_i(y)]$.

We can view our solution S as being *constructed* by the addition(union) of these S_i's each coming from a rule.

We can very nicely summarize the procedure used to obtain the solution.

Assume we have relationship F between the variables U and V represented by a collection of n propositions of the form

$$P_i: \text{If } U \text{ is } A_i \text{ then } V \text{ is } B_i$$

and we have a value for U

$$U = x_0.$$

The procedure used for obtaining the output value for V is the following:

1. For each rule calculate, $A_i(x_0)$. This called the firing level of the rule.

2. For each rule calculate individual output

$$S_i(y) = A_i(x_0) \wedge B_i(y)$$

3. Calculate the system output as

$$S(y) = Max_i[S_i(y)]$$

As we shall see this procedure for calculating the output of relationship forms the basis of fuzzy systems modeling technique.

Let us now look at the conjunctive formulation of F. In this case,

$$F = \bigcap_i E_i$$

where

$$E_i = \bar{A}_i \cup B_i.$$

Again the system output is obtained as the conjunction of our functional representation and our input value for U,

$$G = F \cap X_0.$$

However in this case our representation of F now is such that

$$G = (\bigcap_i E_i) \cap X_0 = \bigcap_i (E_i \cap X_0).$$

Denoting

$$H_i = E_i \cap X_0$$

we get

$$H_i = (\bar{A}_i \cup B_i) \cap (X_0)$$

$$H_i = \bar{A}_i \cap \{x_0\} \cup B_i \cap \{x_0\}.$$

Furthermore the output is

$$S = Proj_V[G] = \bigcap_i Proj_V[H_i].$$

Denoting $Proj_V[H_i] = S_i$ we get

$$S_i(y) = Max_x[\ \bar{A}_i(x) \wedge \{x_0\} \vee B_i(y) \wedge \{x_0\}] = \bar{A}_i(x_0) \vee B_i(y)$$

Using this approach we can summarize the inference procedure as follows

1. For each rule calculate, $1 - A_i(x_0)$, the negation of the firing level of the rule

2. For each rule calculate individual output

$$S_i(y) = \bar{A}_i(x_0) \vee B_i(y)$$

3. Calculate the system output as $S(y) = Min_i[S_i(y)]$

In this case we see that the solution is obtained by reducing the possible answers, that is every rule which has $A_i(x_0) \neq 0$ eliminates from the solution set all those values for V which don't lie in S_i. We call this the *destructive* method.

If there exists only one A_i such that $A_i(x_0) = 1$ and all other rules have firing level zero then both approaches give the same result. However, if there are multiple A_i's with non-zero membership for x_0, then the two approaches don't necessarily result in the same set S.

To more fully appreciate the distinction between these two approaches we can consider the following. Assume we have a collection of pairs (U is A_i, V is B_i) as shown in Figure 3..

The distinction between the two approaches becomes most apparent when we have places where two rectangles overlap. The conjunctive (destructive approach) takes the intersection of the overlapping pieces while the disjunctive takes the union (constructs) of the overlapping pieces (see Figure 4). We see that in the shaded areas the two approaches disagree.

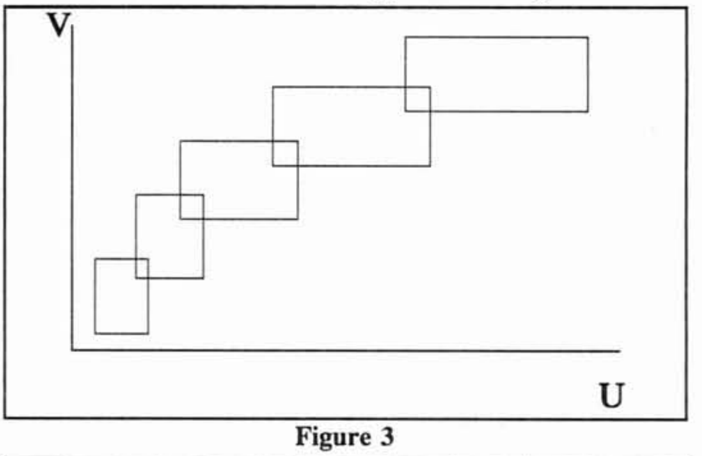

Figure 3

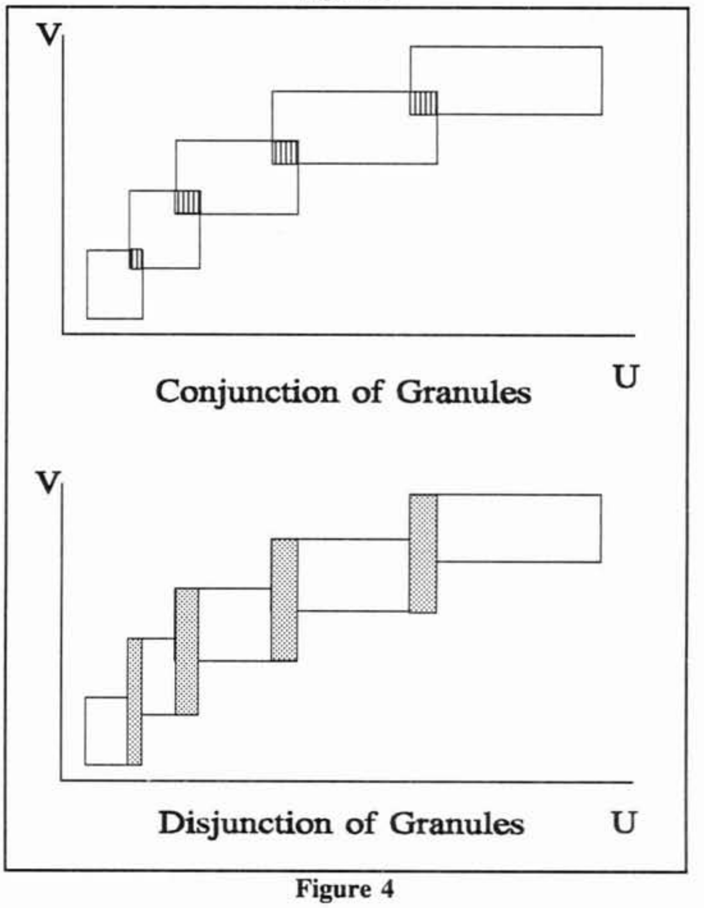

Figure 4

The constructive method implies an increase in uncertainty at places where rules overlap. The destructive method implies a decrease in uncertainty at this overlap of rules.

There are two cases where one can clearly see the distinction between these two approaches. To more clearly illustrate these cases in the following we shall assume all subsets are crisp. The first case is that of conflict. Starting with the two pairs:

$$(U \text{ is } A_1, V \text{ is } B_1)$$

$$(U \text{ is } A_2, V \text{ is } B_2),$$

assume that

$$A_1 \cap A_2 \neq \Phi$$

$$B_1 \cap B_2 = \Phi$$

and that $x_0 \in A_1 \cap A_2$. The above condition says that for $U = x_0$ the rules have conflicting (incompatible) values for V.

Under the above assumption the disjunctive method indicates that

$$V \text{ is } (B_1 \cup B_2).$$

Thus, if for $U = x_0$ we have two solutions for V, V is **a** and V is **b**, then the disjunctive method says V is either a or b. This approach resolves conflict by taking the union of the conflicting values.

Using the conjunctive method we obtain the result that V is Φ. That is, it reports a conflict. Thus it appears that the disjunctive tries to ameliorate conflict by extending the possible solutions to include the conflicting value.

Consider next the case of more specific knowledge. Again assume we have two pairs as above. However, in this case assume

$$A_2 \subset A_1 \text{ and } B_2 \subset B_1.$$

This case corresponds to the situation where have more specific knowledge. Assume $x_0 \in A_2$, consequently both granules are satisfied for $U = x_0$. Using the disjunctive approach we get

$$S = B_1 \cup B_2,$$

however since $B_2 \subset B_1$, then

$$S = B_1.$$

Using the conjunctive approach we get

$$S = B_1 \cap B_2$$

and the condition $B_1 \subset B_2$ here implies

$$S = B_2.$$

Thus we see that the conjunctive method reports the more specific knowledge. Thus, if $A_1 = [30—40]$, $B_1 = [50—60]$, $A_2 = 36$ and $B_2 = 57$ then if $A = 36$ the conjunctive approach reports $V = 57$, while the disjunctive method reports that V lies in the interval [50—60]. We that in this case the conjunctive has provided a more informative answer.

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