PRESCRIPTIVE APPROACHES TO FUZZY CONTROL: YET-TO-BE-REDISCOVERED OLD JEWELS?

Janusz Kacprzyk

Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01–447 Warsaw, Poland
E-mail: kacprzyk@ibspan.waw.pl

ABSTRACT - Models of multistage optimal control under fuzzy constraints and goals are presented, taking – as a point of departure – first, Bellman and Zadeh's (1970) seminal paper, and, second, later extensions presented in Kacprzyk's (1983c) book. Then, an account of most relevant recent developments and newer perspectives is provided. The cases of a deterministic, stochastic and fuzzy systems under control are dealt with, and optimal control over a fixed and specified, implicitly specified, fuzzy, and infinite horizon is discussed. The use of fuzzy logic with linguistic quantifiers is considered. New models based on neural networks are mentioned. The array of models presented – which make possible the optimization of control strategy to be followed, hence being of a clearly prescriptive character – should lead to a new "generation" of fuzzy control involving some optimal control type elements that are not provided by conventional fuzzy control.

1 INTRODUCTION

In the recent years we witness an unprecedented interest in fuzzy control, both among scholars and researchers, and practitioners. This interest is amplified by successful applications (implementations) which range from specialized areas of controlling technological processes (e.g., cement kilns, subway trains, cranes, etc.) to everyday products as, e.g., washing machines, refrigerators, photographic and video cameras, etc. While the former may attract a narrow community of specialists, the later are evidently visible to mass media and general public. Needless to say that although both the above areas of implementation are relevant, the latter has probably played a more important role in creating an appropriate atmosphere that had finally led to the fuzzy boom we had been facing around the globe, principally in Japan.

The applications mentioned above are examples of (simple but working) regulation type control. That is, the rules encode a control strategy known by the human operator from experience. It is usually really good but its "goodness" (optimality) is not explicitly considered. We just assume that an experienced operator knows how to well control the process. This is clearly a *descriptive approach.*

It is unfortunately not widely known that there also exists an earlier approach to fuzzy control, of a *prescriptive* character, that appeared in the late 1960s in the works on fuzzy dynamic programming (Bellman and Zadeh, 1970), was then further developed by Fung and Fu (1977), Kacprzyk (1977–93e), Komolov et al. (1979), etc., and presented in detail in Kacprzyk's (1983c) book.

A large part of this paper is based on the author's book (Kacprzyk, 1983c) which is an extensive account of a wide spectrum of prescriptive fuzzy control models. Unfortunately, it appeared in a bad time, too early and long before the recent interest in, and a good toward fuzzy control. However, in our opinion, it does contain many useful results that may lead to a "new generation" of fuzzy control algorithms incorporating some optimal control type mechanisms; first indications of an interest in such models may be noticed in, e.g., the works of leading research and scholarly institutions exemplified by the LIFE Laboratory in Yokohama (cf. Hoyo, Terano and Masui, 1993). Second, we will discuss newer developments as, e.g., Francelin and Gomide's (1993a,b, 1994) neural network based approach, and Kacprzyk's (1993a-e) interpolative reasoning based approach that can help overcome an inherent low numerical efficiency of those prescriptive models.

Kacprzyk's (1983c) book concerns the following problem class. Let $U = \{u_1, \ldots, u_n\}$ be a set of *controls* (inputs), $X = \{s_1, \ldots, s_m\}$ be a set of *states* (outputs), the *system under control* (deterministic for now) be governed by x_{t+1}

$= f(x_t, u_t)$, where x_{t+1}, x_t are states at time (control stage) t and $t+1$, and u_t is a control at t, $t = 0, 1, \ldots, N-1$; N is the *termination time* (planning horizon). Let $\mu_{C^t}(u_t)$ be a *fuzzy constraint* imposed on u_t, $\mu_{G^{t+1}}(x_{t+1})$ be a *fuzzy goal* imposed on x_{t+1}, and $\mu_D(. \mid .)$ be Bellman and Zadeh's (1970) *fuzzy decision.* We seek an *optimal sequence of controls* $u_0^*, \ldots, u_{N-1}^*$ such that

$$\begin{aligned}\mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ = \max_{u_0,\ldots,u_{N-1}} \mu_D(u_0, \ldots, u_{N-1} \mid x_0) = \\ = \max_{u_0,\ldots,u_{N-1}} \bigwedge_{t=0}^{N-1} (\mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1}))\end{aligned} \quad (1)$$

Problem (1) leads to various problem classes which may be classified, e.g., due to (Kacprzyk, 1983c):

- type of the termination time: (a) fixed and specified in advance, (b) implicitly given (by entering a termination set of states), (c) fuzzy, and (d) infinite;
- type of the system under control: (a) deterministic, (b) stochastic, and (c) fuzzy.

and these case will be discussed below. The fuzzy decision in (1), in which "$\wedge$" is used is called the min-type fuzzy decision, and will be extensively used here.

2 CONTROL WITH A FIXED AND SPECIFIED TERMINATION TIME

This is the basic case, being a point of departure for virtually all next considerations, and will be discussed in more detail. The cases of the deterministic, stochastic and fuzzy system under control, various solution methods, and some extensions will be considered.

2.1 Deterministic system under control

The deterministic system under control is given by the state transition equation

$$x_{t+1} = \mathrm{f}(x_t, u_t); t = 0, 1, \ldots \quad (2)$$

where: $x_t, x_{t+1} \in X = \{s_1, \ldots, s_n\}$ is the *state* (output) at control stages t and $t+1$, respectively, and $u_t \in U = \{c_1, \ldots, c_m\}$ is the *control* (input) at t. The initial state is $x_0 \in X$, and the (finite) termination time N is fixed and specified in advance.

At each t, $u_t \in U$ is subjected to a *fuzzy constraint* $\mu_{C^t}(u_t)$, and on the final state $x_N \in X$ a *fuzzy goal* $\mu_{G^N}(x_N)$ is imposed (fuzzy goals at the subsequent t 's may also be assumed, and the reasoning remains valid).

We require the controls to best fulfill the fuzzy constraints, and the states to best satisfy the fuzzy goals, hence the fuzzy decision (performance function) is

$$\begin{aligned}\mu_D(u_0, \ldots, u_{N-1} \mid x_0) = \\ = \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)\end{aligned} \quad (3)$$

where x_N is obtained from x_0 via (2).

The problem is to find an *optimal sequence of controls*, $u_0^*, \ldots, u_{N-1}^*$ such that

$$\begin{aligned}\mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ = \max_{u_0,\ldots,u_{N-1}} \mu_D(u_0, \ldots, u_{N-1} \mid x_0) = \\ = \max_{u_0\wedge\ldots,u_{N-1}} (\mu_{C^0}(u_0) \wedge \ldots \\ \ldots \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N))\end{aligned} \quad (4)$$

and "$\wedge$", i.e. the minimum, may be replaced by another suitable operation (cf. Kacprzyk, 1983c).

The two traditional solution techniques are:

- dynamic programming, and
- branch-and-bound

which will be discusses below. Moreover, we will also present a novel Francelin and Gomide's (1993a,b, 1994) attempt to use a special neural network.

2.1.1 Solution by dynamic programming

The application of dynamic programming for the solution of (4) was proposed in Bellman and Zadeh (1970). For clarity, it is better to begin the presentation of this approach by slightly rewriting (4) as to find $u_0^*, \ldots, u_{N-1}^*$ such that

$$\begin{aligned}\mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ = \max_{u_0,\ldots,u_{N-1}} (\mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \\ \wedge \mu_{G^N}(\mathrm{f}(x_{N-1}, u_{N-1})))\end{aligned} \quad (5)$$

It is easy to notice that the structure of (5) makes the application of dynamic programming possible as the two last terms, $\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(\mathrm{f}(x_{N-1}, u_{N-1}))$ depend only on u_{N-1} and not on the previous controls, etc. The maximization over $u_0, \ldots, u_{N-1}$ in (5) can be therefore divided into two phases: (1) the maximization over $u_0, \ldots, u_{N-2}$, and (2) the maximization over u_{N-1}, and one can continue this procedure for $u_{N-2}, u_{N-3}, \ldots$.

This backward iteration leads to the following set of dynamic programming recurrence equations

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \\ \quad = \max_{u_{N-i}}(\mu_{C^{N-i}}(u_{N-i}) \wedge \\ \quad \wedge \mu_{G^{N-i+1}}(x_{N-i+1})) \\ x_{N-i+1} = \mathrm{f}(x_{N-i}, u_{N-i}); i = 0, 1, \ldots, N \end{cases} \quad (6)$$

where $\mu_{G^{N-i}}(x_{N-i})$ is regarded as a fuzzy goal at $t = N-i$ induced by the fuzzy goal at $t = N-i+1$.

The $u_0^*, \ldots, u_{N-1}^*$ sought is given by the successive maximizing values of u_{N-i} in (6), u_{N-i}^*, obtained as a function of x_{N-i}, so we obtain an *optimal (control) policy*

$$\mathrm{a}_{N-i}^* : X \longrightarrow U; i = 1, \ldots, N \quad (7)$$

such that

$$u^*_{N-i} = a^*_{N-i}(x_{N-i}); i = 1, \ldots, N \quad (8)$$

2.1.2 Solution by branch-and-bound

The use of a branch-and-bound technique for solving (4) was proposed by Kacprzyk (1978b). It is simple and efficient, and will be used here extensively.

The branch-and-bound procedure starts from x_0. We apply u_0 and proceed to x_1. Next, we apply u_1 and proceed to x_2, etc. Finally, being in x_{N-1}, we apply u_{N-1} and attain x_N. This is clearly equivalent to traversing a decision tree (states associated with nodes, and controls with edges). The procedure is basically an implicit enumeration scheme whose essence may be summarized as

If we currently arrive at some node (state), to which node (out of those traversed so far) should we most rationally add next edges (controls)? Or, at a specified moment of the decision tree traversal, what is the best continuation of the search process?

Let us first denote

$$\begin{cases} v_0 = \mu_{C^0}(u_0) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ v_k = \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^k}(u_k) = \\ \quad = v_{k-1} \wedge \mu_{C^k}(u_k) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ v_{N-1} = \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) = \\ \quad = v_{N-2} \wedge \mu_{C^{N-1}}(u_{N-1}) \\ v_N = \mu_{C^0}(u_0) \wedge \ldots \\ \quad \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N) = \\ \quad = v_{N-1} \wedge \mu_{G^N}(x_N) = \\ \quad = \mu_D(u_0, \ldots, u_{N-1} \mid x_0) \end{cases} \quad (9)$$

Now, due to "$\wedge$" (i.e. the minimum), if we consider $u_0, \ldots, u_k$, then for each $k < w \leq N-1$

$$v_k \geq v_w = v_k \wedge \mu_{C^{k+1}}(u_{k+1}) \wedge \ldots \wedge \mu_{C^w}(u_w) \quad (10)$$

because by "adding" (via "$\wedge$") to v_k any further terms we cannot increase v_w. And, in particular

$$v_k \geq v_N = \mu_D(u_0, \ldots, u_{N-1} \mid x_0) \quad (11)$$

Suppose now that we are at the k-th control stage, and have traversed so far some nodes and edges (from x_0 to x_k). Now we have to most rationally add the edges (apply controls). Evidently, since we seek a trajectory corresponding to the maximal value of $\mu_D(u_0, \ldots, u_{N-1} \mid x_0)$, the most rational choice (currently!) is to choose the most promising node, corresponding to the greatest v_i, $i = 1, \ldots, k$. The other nodes cannot lead (at that particular moment!) to any optimal solution since they cannot obviously yield any higher value of v_i if we add next edges (apply next controls).

This property of the min-type fuzzy decision makes it possible to devise a branch-and-bound algorithm with the branching through the consecutive controls, and the bounding via v_k's, $k = 0, \ldots, N$.

2.1.3 Remarks on the solution by using a neural network

In this section we will briefly sketch the idea of a special type of a neural network to solve problem (4) which was proposed by Francelin and Gomide (1993a,b; 1994).

The approach is based on a special *fuzzy dynamic programming neural network* (FDPNN) that is composed of the so-called m-neurons and M-neurons (cf. Rocha, 1993).

Basically, for the FDPNN, the m-neuron is the one whose output is given by the minimum of its input signals and its bias signal, and the M-neuron is roughly the one whose output signal is given by the maximum of its input signals. In the FDPNN the layers are set alternatively of the ones containing the m-neurons and M-neurons, starting with the layer of the latter ones. The network is based on the so-called *transmitters*, *receptors*, and *controllers* (cf. Rocha, 1993) that encode the fuzzy constraints, goals and decision of (4). Then, the connections between the neurons belonging to the consecutive layers are determined by a special procedure, Finally, it is shown that the results obtained are equivalent to those by using the dynamic programming recurrence equations (6).

This concludes of remarks on this approach because, first, it is very specific and slightly beyond the scope of our optimization based presentation. One should however remember that, first, this is very original and novel approach, one of a very few that have recently appeared, and second, though it is probably not efficient enough now, it should be a viable alternative in a near future when neurofuzzy computers will be available at a reasonable cost. However, even now, an interesting application to a power system planning is reported (Francelin and Gomide, 1993b).

2.1.4 Fuzzy linguistic quantifiers in problem formulation

So far an optimal sequence of controls was sought best satisfying fuzzy constraints and goals at *all* the control stages. This may often be too strict, and a "milder" one as, e.g., *most*, *almost all*, ... may be more appropriate. Such terms, *fuzzy linguistic quantifiers*, may be handled by fuzzy logic using the two basic calculi of linguistically quantified propositions due to Zadeh (1983) and Yager (1983); the former is simpler but often less "adequate". The use of fuzzy linguistic quantifiers in the control setting was proposed by Kacprzyk (1983b) and advanced in Kacprzyk and Iwański (1987).

The source problem (4) may be written as to find an u^*_0,

$\ldots, u^*_{N-1}$ such that

$$\begin{aligned}\mu_D(u^*_0,\ldots,u^*_{N-1} \mid x_0, all) = \\ = \max_{u_0,\ldots,u_{N-1}} \mu_D(u_0,\ldots,u_{N-1} \mid x_0, all) = \\ = \max_{u_0,\ldots,u_{N-1}} (\bigwedge_{t=0}^{N-1} \mid all)(\mu_{C^t}(u_t) \wedge \\ \wedge \mu_{G^{t+1}}(x_{t+1}))\end{aligned} \quad (12)$$

with "*all*" indicating that the satisfaction of C^t's and G^{t+1}'s at *all* the control stages is required.

Kacprzyk's (1983b) proposal is to find $u^*_0, \ldots, u^*_{N-1}$ which best satisfies the fuzzy constraints and goals at Q (e.g., most, almost all, much more than 50 %, ...) control stages written as

$$\begin{aligned}\mu_D(u^*_0,\ldots,u^*_{N-1} \mid x_0, Q) = \\ = \max_{u_0,\ldots,u_{N-1}} \mu_D(u_0,\ldots,u_{N-1} \mid x_0, Q) = \\ = \max_{u_0,\ldots,u_{N-1}} (\bigwedge_{t=0}^{N-1} \mid Q)(\mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1}))\end{aligned} \quad (13)$$

Later, Kacprzyk and Iwański (1987) proposed a further generalization, including *discounting*, by seeking an $u^*_0, \ldots, u^*_{N-1}$ to best satisfy fuzzy constraints and goals at Q of the *earlier* (B) (such that $t_1 < t_2 \Rightarrow \mu_B(t_1) \geq \mu_B(t_2)$) controls stages, i.e.,

$$\begin{aligned}\mu_D(u^*_0,\ldots,u^*_{N-1} \mid x_0, Q, B) = \\ = \max_{u_0,\ldots,u_{N-1}} \mu_D(u_0,\ldots,u_{N-1} \mid x_0, Q, B) = \\ = \max_{u_0,\ldots,u_{N-1}} (\bigwedge_{t=0}^{N-1} \mid Q, B)(\mu_{C^t}(u_t) \wedge \\ \wedge \mu_{G^{t+1}}(x_{t+1}))\end{aligned} \quad (14)$$

The solution of these problems is however difficult. First, for (13), using Zadeh's (1983) calculus, we obtain a set of some dynamic programming recurrence equations (cf. Kacprzyk, 1983b). This cannot be unfortunately used for Yager's (1983) calculus, and Kacprzyk's (1983b) implicit enumeration algorithm is to be employed.

The solution of (14), i.e. with discounting, is however difficult, and a branch-and-bound type implicit enumeration algorithm was proposed by Kacprzyk and Iwański (1987). The branching is via the consecutive controls, and a sophisticated bounding technique is used.

Recently, Yager (1988) proposed a concept of an OWA (ordered weighted average) operator which may greatly simplify the use of fuzzy linguistic quantifiers. The solution of the resulting control problems may than presumably be easier. This requires however a further research (cf. Kacprzyk, 1992; Kacprzyk and Yager, 1990).

2.2 Stochastic system under control

This is probably the most difficult and challenging case from the analytical point of view as we face the occurrence of both the fuzziness and randomness.

The stochastic system under control is a Markov chain whose state transitions are governed by a conditional probability function $p(x_{t+1} \mid x_t, u_t)$, $t = 0, 1, \ldots, N-1$, which specifies the probability of attaining $x_{t+1} \in X = \{s_1, \ldots, s_n\}$ from $x_t \in X$ and under $u_t \in U = \{c_1, \ldots, c_m\}$. At each t, u_t is subjected to a fuzzy constraint $\mu_{C^t}(u_t)$, $t = 0, 1, \ldots, N-1$, and on x_N a fuzzy goal $\mu_{G^N}(x_N)$ is imposed.

The value of $\mu_D(u_0, \ldots, n_{N-1} \mid x_0)$ is now a random variable, hence in the problem formulation the expected value should somehow be involved.

The following two basic formulations exist:

- Bellman and Zadeh's (1970) formulation: find an optimal sequence of controls $u^*_0, \ldots, u^*_{N-1}$ maximizing the probability of attainment of the fuzzy goal subject to the fuzzy constraints, i.e.

$$\begin{aligned}\mu_D(u^*_0,\ldots,u^*_{N-1} \mid x_0) = \\ = \max_{u_0,\ldots,u_{N-1}} (\mu_{C^0}(u_0) \wedge \ldots \\ \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mathrm{E}\mu_{G^N}(x_N))\end{aligned} \quad (15)$$

- Kacprzyk and Staniewski's (1980) formulation: find an optimal sequence of controls $u^*_0, \ldots, u^*_{N-1}$ maximizing the expected value of the fuzzy decision, i.e.

$$\begin{aligned}\mu_D(u^*_0,\ldots,u^*_{N-1} \mid x_0) = \\ = \max_{u_0,\ldots,u_{N-1}} \mathrm{E}\mu_D(u_0,\ldots,u_{N-1} \mid x_0) = \\ = \max_{u_0,\ldots,u_{N-1}} \mathrm{E}(\mu_{C^0}(u_0) \wedge \ldots \\ \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N))\end{aligned} \quad (16)$$

These two formulations are not the same since

$$\begin{aligned}\mathrm{E}\mu_D(u_0,\ldots,u_{N-1} \mid x_0) \neq \\ \neq \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mathrm{E}\mu_{G^N}(x_N)\end{aligned} \quad (17)$$

The probability of a fuzzy event in (15) and (16) is meant in Zadeh's (1968) sense, i.e. as a real number in $[0,1]$. The use of the fuzzy probability of a fuzzy event, which is a fuzzy number in $[0,1]$ can be found in Kacprzyk (1984).

2.2.1 Bellman and Zadeh's approach

The fuzzy goal is regarded as a fuzzy event in X, and its conditional probability given x_{N-1} and u_{N-1} is

$$\begin{aligned}\mathrm{E}\mu_{G^N}(x_N) = \mathrm{E}\mu_{G^N}(x_N \mid x_{N-1}, u_{N-1}) = \\ = \sum_{x_N \in X} p(x_N \mid x_{N-1}, u_{N-1}) \cdot \mu_{G^N}(x_N)\end{aligned} \quad (18)$$

The structure of (15) is essentially the same as that of (4) for the deterministic system under control, and we can employ the dynamic programming recurrence equations

$$\begin{cases} \mu_{G^{N-i}}(x_{N-1}) = \\ \quad = \max_{u_{N-i}}(\mu_{C^{N-i}}(u_{N-i}) \wedge \\ \quad \wedge \mathrm{E}\mu_{G^{N-i+1}}(x_{N-i+1})) \\ \mathrm{E}\mu_{G^{N-1+1}}(x_{N-i+1}) = \\ \quad = \sum_{x_{N-i}\in X} p(x_{N-i+1} \mid x_{N-i}, u_{N-i}) \times \\ \quad\quad \times \mu_{G^{N-i+1}}(x_{N-i+1}) \\ i = 1, \ldots, N \end{cases} \tag{19}$$

where $\mu_{G^{N-i}}(x_{N-i})$ may be viewed as a fuzzy goal at $t = N - i$ induced by the fuzzy goal at $t = N - i + 1$.

The successive u^*_{N-i}, $i = 1, 2, \ldots, N$, give the optimal sequence of controls (policies) sought, and they relate here the current optimal control to the current state only.

2.2.2 Kacprzyk and Staniewski's approach

Since $\mathrm{E}\mu_D(u_0, \ldots, u_{N-1} \mid x_0) \neq \mathrm{E}(\mu_{C^0}(u_0) \wedge \mathrm{E}(\mu_{C^1}(u_1) \wedge \ldots \wedge \mathrm{E}\mu_{G^N}(x_N))\ldots)$, then the dynamic programming recurrence equations of type (19) cannot be used.

Notice first that the probability of attaining the final state x_N from x_0 $u_0, \ldots, u_{N-1}$ is evidently $p(x_1 \mid x_0, u_0) \cdot \ldots \cdot p(x_N \mid x_{N-1}, u_{N-1})$ and

$$\begin{aligned} &\mathrm{E}\mu_D(u_0, \ldots, u_{N-1} \mid x_0) = \\ &= \sum_{(x_0,\ldots,x_N)\in X^{N+1}} \mu_D(u_0, \ldots u_{N-1} \mid x_0) \times \\ &\quad \times p(x_1 \mid x_0, u_0) \cdot \ldots \cdot p(x_N \mid x_{N-1}, u_{N-1}) \end{aligned} \tag{20}$$

We introduce a sequence of functions $\hat{h}_i, h_j$, $i = 0, 1, \ldots, N$, $j = 1, \ldots, N-1$,

$$\begin{cases} \mu_D(u_0, \ldots, u_{N-1} \mid x_0) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \hat{h}_k(x_k, u_0, \ldots, u_{k-1}) = \max_{u_k} h_k(x_k, u_0, \ldots, u_k) \\ h_{k-1}(x_{k-1}, u_0, \ldots, u_{k-1}) = \\ \quad = \sum_{x_k\in X} \hat{h}_k(x_k, u_0, \ldots, u_{k-1}) \times \\ \quad \times\ p(x_k \mid x_{k-1}, u_{k-1}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \hat{h}_0(x_0) = \max_{u_0} h_0(x_0, u_0) \end{cases} \tag{21}$$

meant as that for $u_0, \ldots, u_{k-1}$, and $x_1, \ldots, x_k$, h_{k-1} is the expected value of the fuzzy decision, assuming an optimal continuation of control $u^*_k, \ldots, u^*_{N-1}$. The $\hat{h}_k$ is then roughly the highest attainable expected value of h_k.

Since X and U are finite, then there obviously exist functions w_k such that

$$\begin{aligned} &\hat{h}_k(x_k, u_0, \ldots, u_{k-1}) = \\ &= \ h_k(x_k, u_0, \ldots, u_{k-1}, w_k(x_k, u_0, \ldots, u_{k-1})) \end{aligned} \tag{22}$$

If we introduce now functions g^*_k such that

$$\begin{aligned} &g^*_k(x_0, \ldots, x_k) = \\ &= \ w_k(x_k, g^*_0(x_0), \ldots, g^*_{k-1}(x_0, \ldots, x_{k-1})) \end{aligned} \tag{23}$$

where $g^*_0(x_0) = w_0(x_0)$, then it may be shown (Kacprzyk and Staniewski, 1980) that the g^*_k s are the optimal policies sought, i.e.

$$u^*_k = g^*_k(x_0, \ldots, x_k); k = 0, \ldots, N-1 \tag{24}$$

and $h_0(x_0) = \max_{u_0,\ldots,u_{N-1}} \mathrm{E}\mu_D(u_0, \ldots, u_{N-1} \mid x_0)$.

Notice that the policy relates here the optimal control not only to the current state but also to a "summary" of the past trajectory.

2.3 Fuzzy system under control

Suppose that $\mathcal{U} = \{C_1, \ldots, C_n\}$ is a set of *fuzzy controls* in U, $\mathcal{X} = \{S_1, \ldots, S_m\}$ is a set of *fuzzy states* in X, the fuzzy system under control is governed by the fuzzy state transition equation

$$X_{t+1} = F(X_t, U_t); t = 0, 1, \ldots \tag{25}$$

where $X_t, X_{t+1} \in \mathcal{X}$ are fuzzy states at control stage t and $t+1$, $U_t \in \mathcal{U}$ is fuzzy control at t, $t = 0, 1, \ldots, N-1$.

Let $\mu_{\overline{C}^t}(U_t)$ be a *fuzzy constraint* on U_t, and $\mu_{\overline{G}^{t+1}}(X_{t+1})$ be a *fuzzy goal* on X_{t+1}. Notice that $\overline{G}^{t+1}$ and $\overline{C}^t$ mean that the original fuzzy goal and constraint, G^{t+1} and C^t, are modified to account for the fuzziness of the state and control, e.g., as $\mu_{\overline{G}^N}(X_N) = 1 - \mathrm{d}(X_N, G^N)$ where $\mathrm{d}(.,.)$ is some distance between two fuzzy sets; and similarly for C.

We seek an *optimal sequence of fuzzy controls* $U^*_0, \ldots, U^*_{N-1}$ such that

$$\begin{aligned} &\mu_D(U^*_0, \ldots, U^*_{N-1} \mid X_0) = \\ &= \max_{U_0,\ldots,U_{N-1}} \bigwedge_{t=0}^{N-1} (\mu_{\overline{C}^t}(U_t) \wedge \mu_{\overline{G}^{t+1}}(X_{t+1})) \end{aligned} \tag{26}$$

and this problem can be solved by dynamic programming (Baldwin and Pilsworth, 1982) and branch-and-bound (Kacprzyk, 1979).

In the dynamic programming approach, the set of recurrence equations is

$$\begin{cases} \mu_{\tilde{G}^N}(X_N) = \\ \quad = \max_{x_N}(\mu_{X_N}(x_N \wedge \mu_{G^N}(x_N)) \\ \mu_{\tilde{G}^{N-i}}(X_{N-i}) = \\ \quad = \max_{U_{N-i}}(\max_{u_{N-i}}(\mu_{U_{N-i}}(u_{N-i}) \wedge \\ \quad \wedge \mu_{C^{N-i}}(u_{N-i})) \wedge \mu_{\tilde{G}^{N-i+1}}(X_{N-i+1})) \\ \mu_{X_{N-i+1}}(x_{N-i+1}) = \\ \quad = \max_{x_{N-i}}(\max_{u_{N-i}}(\mu_{U_{N-i}}(u_{N-i}) \wedge \\ \quad \wedge \mu_{X_{N-i+1}}(x_{N-i+1} \mid x_{N-i}, u_{N-i})) \wedge \\ \quad \wedge \mu_{X_{N-i}}(x_{N-i})) \\ i = 1, \ldots, N \end{cases} \tag{27}$$

Though this set of equations can be solved, it is very inefficient as the $\mu_{\tilde{G}^t}(X_t)$'s are to be specified for all X_t's whose number may be huge, theoretically infinite; and similarly $\max_{U_{t-1}}$ is to proceed over a large set of U_{t+1}'s.

A natural approach (Baldwin and Pilsworth, 1982; Kacprzyk and Staniewski, 1982) would be to use some prespecified *reference fuzzy states* and *control*, $\overline{U}_t \in \overline{\mathcal{U}} =$

$\{\overline{C}_1, \ldots, \overline{C}_w\} \subseteq \mathcal{U}$, and $\overline{X}_{t+1} \in \overline{\mathcal{X}} = \{\overline{S}_1, \ldots, \overline{S}_r\} \subseteq \mathcal{X}$, and to express all U_{t-1}'s and X_t's occurring in the solution process by their closest reference counterparts (fuzzy matching!).

The solution of (27), an *optimal control policy*, is represented by "IF $\overline{X}_t = \overline{S}_k$ THEN $\overline{U}_t = \overline{C}_l$", $t = 1, \ldots, N-1$, $\overline{S}_k \in \overline{\mathcal{X}}$, $\overline{C}_l \in \overline{\mathcal{U}}$, which is in turn equated with a fuzzy relation R_t^* in $X \times U$. Thus, for a current X_t (not necessarily reference) the U_t^* sought is determined by the *compositional rule of inference* $U_t^* = X_t \circ R_t^*$ ($\circ$ is, e.g., the max-min composition).

Unfortunately, this dynamic programming approach – in its original version – is evidently not practically efficient. Moreover, for the compositional rule of inference to work properly there should be a high number of "overlapping" reference fuzzy controls and states. However, for computational efficiency this number should be as small as possible! This contradiction will be resolved by assuming a small number of non-overlapping fuzzy states and controls, and then using interpolative reasoning.

The earlier branch-and-bound approach (Kacprzyk 1979) is much simpler and efficient though – by its very nature – it gives optimal controls, not policies. It is virtually the same as for the case of a deterministic system under control presented in Section I.A.2, and will not be presented again.

2.3.1 Interpolative reasoning in the derivation of optimal controls

In the case discussed above there is an inherent conflict between a large number of reference fuzzy controls and states required by the compositional rule of inference to give meaningful results (i.e. a "dense" data set), and a small number of them required for the efficiency of algorithm (i.e. a "sparse" data set).

In the majority of practical problems the efficiency may be vital, and we may be forced to assume the second option, i.e. a small number of non-overlapping reference fuzzy controls and states. Then, we need to use some approach to obtain meaningful results. We will sketch below the idea of Kacprzyk's (1993a-e) approach.

Suppose that we obtain an optimal control policy a_t^* stating that, say, IF $\overline{X}_t = \overline{S}_1$ THEN $\overline{U}_t^* = \overline{C}_{t1}$ ELSE ... ELSE IF $\overline{X}_t = \overline{S}_i$ THEN $\overline{U}_t^* = \overline{C}_{ti}$ ELSE IF $\overline{X}_t = \overline{S}_{i+1}$ THEN $\overline{U}_t^* = \overline{C}_{t(i+1)}$ ELSE ... ELSE IF $\overline{X}_t = \overline{S}_r$ THEN $\overline{U}_t^* = \overline{C}_{tr}$. This optimal policy should now be implemented. Suppose that we wish to determine an optimal fuzzy control U_t^* for a current fuzzy state X_t which is not a reference one. Let X_t be a fuzzy number "between" the two reference fuzzy states $\overline{S}_i$ and $\overline{S}_{i+1}$. We seek therefore an optimal fuzzy control U_t^* corresponding to X_t via the above optimal policy a_t^*. Notice that since X_t is not a reference one, then in general U_t^* will not be a reference fuzzy optimal control either.

The problem is now the determination of U_t^* meant (assuming for simplicity its representation by a triangular fuzzy number) as the determination of the mean value and width. It is reasonable to assume that U_t^* be similar (close) to one of these optimal $\overline{C}_{ti}$ and $\overline{C}_{t(i+1)}$ corresponding to $\overline{S}_i$ and $\overline{S}_{i+1}$. Notice that this has much to do with the so-called gradual rules in approximate reasoning.

The first aspect is the determination of the mean value of the fuzzy optimal control sought. We apply here Kóczy and Hirota's (1992) idea whose essence is

$$\mathrm{d}(\overline{S}_i, X_t)/\mathrm{d}(X_t, \overline{S}_{i+1}) = \mathrm{d}(\overline{C}_i, U_t^*)/\mathrm{d}(U_t^*, \overline{C}_{t(i+1)}) \quad (28)$$

where $d(.,.)$ is a distance between the two fuzzy numbers. That is, (28) means that the relative position of U_t^* with respect to its closest reference counterparts should be the same as that concerning X_t and its reference counterparts.

The second aspect is the width of U_t^*. The reasoning is that the lower the number of reference fuzzy states and controls, i.e. the more sparse the data set, the less precise is the available information. Hence, the fuzzier (of a larger width) U_t^* should be. For instance, we can use a formula as

$$\overline{w}(U_t^*) = \frac{1}{5}(\overline{w}(\overline{S}_i) + \\ + \overline{w}(\overline{S}_{i+1}) + \overline{w}(\overline{X}_t) + \overline{w}(\overline{C}_{ti}) + \overline{w}(\overline{C}_{t(i+1)})) \quad (29)$$

where $\overline{w}(.)$ is a relative width (related to the universe of discourse of the fuzzy states and controls, and the simplest arithmetics mean can be replaced by another formula expressing the above rationale.

Moreover, it may often be expedient to include in (29) some term(s) accounting for the fuzziness of the problem, e.g., in the fuzzy constraints, fuzzy goals, fuzzy system under control, fuzzy termination time, etc. As a first, somehow *ad hoc* attempts, we can use normal degrees of fuzziness of fuzzy sets and calculate the mean fuzziness of fuzzy constraints, fuzzy goals, etc. For the degree of fuzziness of the fuzzy system under control given as IF – THEN rules, we can use Kacprzyk's (1994) approach to the determination of their degree of specificity.

Thus, the relative width of U_t^* defined initially by (29), which involves the fuzziness of the fuzzy states and fuzzy controls only, may be further modified to involve the fuzziness of fuzzy constraints, fuzzy goals and fuzzy system under control.

In general, the approach sketched above works very well, and helps attain more realistic results.

As mentioned, simpler and more efficient than dynamic programming is an earlier branch-and-bound approach by Kacprzyk (1979). An efficient bounding requires a small number of reference fuzzy controls, $\overline{C}_1, \ldots, \overline{C}_w$. As a solution we obtain an optimal control policy $\overline{U}_t^* = a^*(\overline{X}_t)$ which is represented by a fuzzy relation, and for a current (not necessarily reference) fuzzy state the optimal control is determined via the compositional rule of inference. And again, since for numerical efficiency we should have as few as possible nonoverlapping reference fuzzy states and controls, interpolative reasoning can be used.

This concludes our analysis of the basic case with the fixed and specified termination time. Now we will proceed to the next important class with the termination time specified implicitly by entering for the first time some given termination set of states.

3 CONTROL WITH AN IMPLICITLY SPECIFIED TERMINATION TIME

The process terminates now when the state enters for the first time a (specified in advance) termination set of states $W = \{s_{p+1}, s_{p+2}, \ldots, s_n\} \subset X$. This time, $\overline{N} < \infty$, is evidently unknown in advance. The initial state is $x_0 \in X \setminus W = \{s_1, \ldots, s_n\}$. Moreover, if $x_t \in W$, then $f(x_t, u_t) = x_t, \forall u_t$, by assumption.

The fuzzy goal is $\mu_{G^{\overline{N}}}(x_{\overline{N}})$, and $\mu_{G^{\overline{N}}}(x_{\overline{N}}) = 0$ for each $x_{\overline{N}} \in W$. At each t, u_t is subjected to a state-dependent fuzzy constraint $\mu_C(u_t \mid x_t)$.

We seek $u_0^*, \ldots, u_{\overline{N}-1}^*$ such that

$$\begin{aligned}\mu_D(u_0^*, \ldots, u_{\overline{N}-1}^* \mid x_0) &= \\ &= \max_{u_0, \ldots, u_{\overline{N}-1}} (\mu_C(u_0 \mid x_0) \wedge \ldots \\ &\ldots \wedge \mu_C(u_{\overline{N}-1} \mid x_{\overline{N}-1}) \wedge \mu_{G^{\overline{N}}}(x_{\overline{N}}) \end{aligned} \quad (30)$$

where $x_0, \ldots, x_{\overline{N}-1} \in X \setminus W$, and $x_{\overline{N}} \in W$.

The solution of the problem sketched above may proceed by using the following three basic approaches:

- an iterative approach by Bellman and Zadeh (1970),
- a graph-theoretic approach by Komolov et al. (1979),
- a branch-and-bound approach by Kacprzyk (1978b, 1983c).

We will present the first approach in some detail only since, first, the graph-theoretic one is somehow beyond the scope of this paper, and the branch-and-bound approach is very similar to the one presented in Section I.A.2.

Basically, Bellman and Zadeh's (1970) approach uses some *functional equation* that relates the fuzzy decision from some specified t on to that from $t+1$ on.

This functional equation is

$$\begin{aligned}\mu_D(u_t, u_{t+1}, \ldots, u_{\overline{N}-1} \mid x_t) &= \\ = \mu_C(u_t \mid x_t) &\wedge \\ \wedge \mu_D(u_{t+1}, u_{t+2}, \ldots, u_{\overline{N}-1} \mid x_{t+1})\end{aligned} \quad (31)$$

and it can be solved, i.e. a unique *stationary policy function* $a : X \setminus W \longrightarrow U$ such that $u_t = a(x_t), t = 0, 1, \ldots$ can be found.

The case of a stochastic approach is difficult, and has not been practically solved, while the case of a fuzzy system may be dealt with by using reference fuzzy states and controls as in Sections II.C, and then interpolative reasoning.

4 CONTROL WITH A FUZZY TERMINATION TIME

Denote the set of possible control stages by $S = \{0, 1, \ldots, K-1, K, K+1, \ldots, N\}$; N is a fixed, highest possible termination time. The fuzzy termination time is now given as a fuzzy set $\mu_T : S \longrightarrow [0,1]$ such that $\mu_T(t) \in [0,1]$ is a measure of how preferable $t \in S$ is as the termination time, from 1 for the most preferable to 0 for the least preferable (unacceptable) through all intermediate values.

The process should therefore terminate at some $M \in \operatorname{supp} T = \{t \in S : \mu_T(t) > 0\}$. Moreover, suppose that $\operatorname{supp} T = \{K, K+1, \ldots, N\}$, i.e. that the termination time should occur at some "later" control stages.

As previously, the $\mu_{C^t}(u_t)$'s are imposed on the controls $u_t \in U$ at $t = 0, 1, \ldots, N-1$, and the $\mu_{G^N}(x_N)$ is imposed on $x_N \in X$. The systems under control may be, as in the previous sections, deterministic, stochastic, and fuzzy.

For a deterministic system under control, the fuzzy termination time can be introduced into the control problem formulation (i.e. into the fuzzy decision) in the following two basic ways:

- due to Fung and Fu (1977)

  $$\begin{aligned}\mu_D(u_0, \ldots, u_{M-1} \mid x_0) &= \\ = \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{M-1}}(u_{M-1}) &\wedge \\ \wedge \mu_{G^M}(x_M) \wedge \mu_T(M)\end{aligned} \quad (32)$$

  i.e. the termination time concerns the whole fuzzy decision, and

- due to Kacprzyk (1977, 1978a)

  $$\begin{aligned}\mu_D(u_0, \ldots, u_{M-1} \mid x_0) &= \\ = \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{M-1}}(u_{M-1}) &\wedge \\ \wedge (\mu_T(M) \cdot \mu_{G^M}(x_M))\end{aligned} \quad (33)$$

  i.e. the fuzzy termination time concerns the fuzzy goal only.

and will assume in the sequel the latter one.

For both the above approaches, (32) and (33), the control problem is now formulated as to find an *optimal termination time* M^* and an *optimal sequence of controls* $u_0^*, \ldots, u_{M^*-1}^*$ such that

$$\begin{aligned}\mu_D(u_0^*, \ldots, u_{M^*-1}^* \mid x_0) &= \\ = \max_{M, u_0, \ldots, u_{M-1}} \mu_D(u_0, \ldots, u_{M-1} \mid x_0)\end{aligned} \quad (34)$$

Now, we will consider the cases of the particular systems under control: deterministic, stochastic and fuzzy, and their respective control problems.

4.1 Deterministic system under control

To simplify further notation, we introduce a *modified fuzzy goal* $\mu_{\overline{G}^M}(x_M)$ given as

$$\mu_{\overline{G}^M}(x_M) = \mu_T(M) \cdot \mu_{G^M}(x_M), \forall x_M \in X \qquad (35)$$

Then we seek an M^* and $u_0^*, \ldots, u_{M^*-1}^*$ such that

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{M^*-1} \mid x_0) = \\ &= \max_{M, u_0, \ldots, u_{M-1}} [\mu_{C^0}(u_0) \wedge \ldots \wedge \\ &\qquad \wedge \mu_{C^{M-1}}(u_{M-1}) \wedge \mu_{\overline{G}^M}(x_M) \end{aligned} \qquad (36)$$

The structure of this problem is virtually the same as that of (4) which implies that the same two basic solution techniques can be used:

- dynamic programming, and
- branch-and-bound.

The first approach to the solution of (36) was proposed by Kacprzyk (1977; 1978a, c) and was based on dynamic programming. Stein's (1980) later improvement employed a dynamic programming scheme too.

First, recall that $\operatorname{supp} T = \{M\} = \{t \in S : \mu_T(t) > 0\} = \{K, K+1, \ldots, N-1, N\}$, i.e. K is the earliest possible and N is the latest possible termination time. Hence, $u_0, \ldots, u_{N-1}$ may be partitioned into two parts:

- $u_0, \ldots, u_{K-2}$, i.e. the controls that do not lead to the termination of the control process, and
- $u_{K-1}, \ldots, u_{M-1}$, i.e. the controls that may lead to the termination of the process.

In $u_0^*, \ldots, u_{M^*-1}^*$, its part from $t = K-1$ to the final control stage, $u_{K-1}^*, \ldots, u_{M^*-1}^*$, must evidently be by itself optimal. Hence, (36) may be rewritten as

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{M^*-1}^* \mid x_0) = \\ &= \max_{u_0, \ldots, u_{K-2}} (\mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{K-2}}(u_{K-2}) \wedge \\ &\quad \wedge \max_{M, u_{K-1}, \ldots, u_{M-1}} (\mu_{C^{K-1}}(u_{K-1}) \wedge \ldots \\ &\quad \ldots \wedge \mu_{C^{M-1}}(u_{M-1}) \wedge \mu_{\overline{G}^M}(x_M))) \end{aligned} \qquad (37)$$

Now, if we denote, for $i = 1, \ldots, M-K+1$,

$$\begin{aligned} &\overline{\mu}_{G^{M-i}}(x_{M-i}, M) = \\ &= \max_{u_{M-i}} (\mu_{C^{M-i}}(u_{M-i}) \wedge \overline{\mu}_{G^{M-i+1}}(x_{M-i+1}, M)) \end{aligned} \qquad (38)$$

where $\overline{\mu}_{G^M}(x_M, M) = \mu_{\overline{G}^M}(x_M)$, and perform the dynamic programming backward iterations analogous to, say, (6), then we will obtain

$$\begin{cases} \overline{\mu}_{G^{M-i}}(x_{M-i}, M) = \\ \quad = \max_{u_{M-i}} (\mu_{C^{M-i}}(u_{M-i}) \wedge \\ \quad \wedge \overline{\mu}_{G^{M-i+1}}(x_{M-i+1}, M)) \\ x_{M-i+1} = f(x_{M-i}, u_{M-i}) \\ i = 1, \ldots, M-K+1; M = K, K+1, \ldots, N \end{cases} \qquad (39)$$

By solving this set of recurrence equations we determine $\overline{\mu}_{G^{K-1}}(x_{K-1}, M)$ and its corresponding optimal sequence of controls $u_{K-1}^*, \ldots, u_{M-1}^*$, for all $M \in \{K, K+1, \ldots, N\}$.

Then, the $\mu_{G^{K-1}}(x_{K-1})$ sought, and the corresponding M^* and $u_{K-1}^*, \ldots, u_{M^*-1}^*$ are determined by solving

$$\mu_{G^{K-1}}(x_{K-1}) = \max_{M \in \{K, K+1, \ldots, N\}} \overline{\mu}_{G^{K-1}}(x_{K-1}, M) \qquad (40)$$

The first part of the optimal sequence of controls, $u_0^*, \ldots, u_{K-2}^*$, is now found by solving

$$\begin{cases} \mu_{K-1-i}(x_{K-1-i}) = \\ \quad = \max_{u_{K-1-i}} (\mu_{C^{K-1-i}}(u_{K-1-i}) \wedge \\ \quad \wedge \mu_{G^{K-i}}(x_{K-i})) \\ x_{K-i} = f(x_{K-1-i}, u_{K-1-i}); i = 1, \ldots, K-1 \end{cases} \qquad (41)$$

where $\mu_{G^{K-1}}(x_{K-1})$ is determined from (40).

Stein's (1980) approach is an improved dynamic programming based scheme for solving (36) that increases the numerical efficiency, in particular for large $\operatorname{supp} T$, i.e. for the case of a large number of possible termination times.

Its idea is based on the following reasoning. If the process under control is at the control stage $(N-i)$, $i = 1, \ldots, N$, then we face two possible options. First, we can immediately stop and attain

$$\mu_{\overline{G}^{N-i}}(x_{N-i}) = \mu_T(N-i) \cdot \mu_{G^{N-i}}(x_{N-i}) \qquad (42)$$

or, second, we can apply control u_{N-i} and obtain

$$\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{\overline{G}^{N-i}}(x_{N-i}) \qquad (43)$$

Evidently, the better [of a higher value of (42) or (43)] alternative should be chosen (at this particular control stage!).

This leads to the following set of recurrence equations

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \mu_{\overline{G}^{N-i}}(x_{N-i}) \vee \\ \quad \vee \max_{u_{N-i}} [\mu_{C^{N-i}}(u_{N-i}) \wedge \\ \quad \wedge \mu_{G^{N-i+1}}(x_{N-i+1})] \\ x_{N-i+1} = f(x_{N-i}, u_{N-i}); i = 1, \ldots, N \end{cases} \qquad (44)$$

The optimal termination time sought, M^*, is determined by such a control stage $N-i$ at which, in the optimal sequence of controls, the terminating control occurs, i.e. when while solving (44) there occurs

$$\begin{aligned} &\mu_{\overline{G}^{N-i}}(x_{N-i}) > \\ &\quad > \max_{u_{N-i}} (\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1})) \end{aligned} \qquad (45)$$

The solution of (44) usually requires less computational effort than the solution of (39)–(41).

The solution by branch-and-bound, proposed by Kacprzyk (1983c) is clearly analogous to the one presented in Section I.A.2, and will not be repeated. It is simple and efficient.

4.2 Stochastic system under control

The case of a stochastic system under control operating in a fuzzy environment, and with a fuzzy termination time is clearly interesting and challenging.

The system is again a Markov chain whose dynamics is described by a conditional probability $p(x_{t+1} \mid x_t, u_t)$; $x_t, x_{t+1} \in X = \{s_1, \ldots, s_n\}$, and $u_t \in U = \{c_1, \ldots, c_m\}$.

At each t, u_t is subjected to $\mu_{C^t}(u_t)$, and $\mu_{G^M}(x_M)$ is imposed on x_M. We also assume that the fuzzy termination time, $\mu_T(t)$, defined as a fuzzy set in $S = \{1, \ldots, K-1, K, K+1, \ldots, N\}$, concerns the fuzzy goal only, i.e. (33). The modified fuzzy goal, to be entered into (33) is therefore again $\mu_{\overline{G}^M}(x_M) = \mu_T(M) \cdot \mu_{G^M}(x_M)$.

We consider the fuzzy goal to be a fuzzy event in Zadeh's (1968) sense, i.e.

$$\begin{aligned} &\mathrm{E}\,\mu_{G^M}(x_M) = \\ &= \mathrm{E}\,\mu_{G^M}(x_M \mid x_{M-1}, u_{M-1}) = \\ &= \sum_{x_M \in X} p(x_M \mid x_{M-1}, u_{M-1}) \cdot \mu_{G^M}(x_M) \end{aligned} \qquad (46)$$

so that

$$\mathrm{E}\,\mu_{\overline{G}^M}(x_M) = \mu_T(M) \cdot \mu_{G^M}(x_M) \qquad (47)$$

The problem is now to find an optimal termination time M^* and an optimal sequence of controls $u_0^*, \ldots, u_{M^*-1}^*$ such that

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{M^*-1}^* \mid x_0) = \\ &= \max_{M, u_0, \ldots, u_{M-1}} (\mu_{C^0}(u_0) \wedge \ldots \\ &\ldots \wedge \mu_{C^{M-1}}(u_{M-1}) \wedge \mathrm{E}\,\mu_{\overline{G}^M}(x_M)) \end{aligned} \qquad (48)$$

This problem was first formulated and solved by Kacprzyk (1978a, c), and then Stein (1980) proposed a modification to improve the numerical efficiency. Both these approaches are based on dynamic programming.

The solution of (48) proposed by Kacprzyk (1978a, c) is analogous to that for the case of a deterministic system (using Bellman and Zadeh's approach to the control of a stochastic system under a fixed and specified termination time shown in Section I.B.1).

First, since $\operatorname{supp} T = \{M\} = \{t \in S : \mu_T(t) > 0\} = \{K, K+1, \ldots, N\}$, then the sequence of controls $u_0, \ldots, u_{M-1}$ can evidently be partitioned into two parts:

- $u_0, \ldots, u_{K-2}$,
- $u_{K-1}, \ldots, u_{M^*-1}$.

In an optimal sequence of controls $u_0^*, \ldots, u_{M^*-1}^*$, its part $u_{K-1}^*, \ldots, u_{M^*-1}^*$ must obviously be itself optimal, so that (48) may be rewritten as

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{M^*-1}^* \mid x_0) = \\ &= \max_{u_0, \ldots, u_{K-2}} (\mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{K-2}}(u_{K-2}) \wedge \\ &\wedge \max_{M, u_{K-1}, \ldots, u_{M-1}} (\mu_{C^{K-1}}(u_{K-1}) \wedge \ldots \\ &\wedge \mu_{C^{M-1}}(u_{M-1}) \wedge \mathrm{E}\,\mu_{\overline{G}^M}(x_M))) \end{aligned} \qquad (49)$$

The structure of (49) is virtually the same as that of, e.g., (15), and makes it possible to employ dynamic programming. Analogously as for (39)–(41), we obtain the following sets of dynamic programming recurrence equations:

- for the control stages $K-1, \ldots, M$

$$\begin{cases} \overline{\mu}_{G^{M-i}}(x_{M-i}, M) = \\ \quad = \max_{u_{M-i}} (\mu_{C^{M-i}}(u_{M-i}) \wedge \\ \quad \wedge \mathrm{E}\,\overline{\mu}_{G^{M-i+1}}(x_{M-i+1}, M)) \\ \mathrm{E}\,\overline{\mu}_{G^{M-i+1}}(x_{M-i+1}, M) = \\ \quad = \sum_{x_{M-i+1} \in X} p(x_{M-i+1} \mid x_{M-i}, u_{M-i}) \times \\ \quad \times \overline{\mu}_{G^{M-i+1}}(x_{M-i+1}, M) \\ i = 1, \ldots, M-K+1; M = K, K+1, \ldots, N \end{cases} \qquad (50)$$

where $\overline{\mu}_{G^M}(x_M, M) = \mu_{\overline{G}^M}(x_M)$, and

$$\begin{aligned} &\mu_{G^{K-1}}(x_{K-1}) = \\ &= \max_{M \in \{K, K+1 \ldots, N\}} \overline{\mu}_{G^{K-1}}(x_{K-1}, M) \end{aligned} \qquad (51)$$

which gives M^* and $u_K^*, u_{K+1}^*, \ldots, u_{M^*-1}^*$;

- for the control stages $0, 1, \ldots, K-2$

$$\begin{cases} \mu_{G^{K-i-1}}(x_{K-i-1}, M) = \\ \quad = \max_{u_{K-i-1}} (\mu_{C^{K-i-1}}(u_{K-i-1}) \wedge \\ \quad \wedge \mathrm{E}\,\overline{\mu}_{G^{K-i}}(x_{K-i})] \\ \mathrm{E}\,\mu_{G^{K-i}}(x_{K-i})) = \\ \quad = \sum_{x_{K-i} \in X} p(x_{K-i} \mid x_{K-i-1}, u_{K-i-1}) \cdot \\ \quad \cdot \mu_{G^{K-i}}(x_{K-i}) \\ i = 1, \ldots, K-1 \end{cases} \qquad (52)$$

which gives $u_0^*, \ldots, u_{K-2}^*$.

An improved dynamic programming based approach for solving (49) was also proposed by Stein (1980), and it increases the numerical efficiency, in particular for large $\operatorname{supp} T$.

Its idea is basically the same as for a deterministic system under control, namely if the process under control is at the stage $(N-i)$, $i = 1, \ldots, N$, then we can either immediately stop and attain

$$\mathrm{E}\,\mu_{\overline{G}^{N-i}}(x_{N-i}) = \mu_T(N-i) \cdot \mathrm{E}\,\mu_{G^{N-i}}(x_{N-i}) \qquad (53)$$

or we can apply u_{N-i} and obtain

$$\mu_{C^{N-i}}(u_{N-i}) \wedge \mathrm{E}\,\mu_{\overline{G}^{N-i}}(x_{N-i}) \qquad (54)$$

and the better alternative should be chosen.

This leads to the following set of recurrence equations

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \\ \quad = \mu_{\overline{G}^{N-i}}(x_{N-i}) \vee \max_{u_{N-i}}(\mu_{C^{N-i}}(u_{N-i}) \wedge \\ \quad \wedge \operatorname{E} \mu_{G^{N-i+1}}(x_{N-i+1})] \\ x_{N-i+1} = f(x_{N-i}, u_{N-i}); i = 1, \ldots, N \end{cases} \tag{55}$$

with M^* determined by such an $N-i$ at which, in the optimal sequence of controls, the terminating control occurs, i.e. when

$$\mu_{\overline{G}^{N-i}}(x_{N-i}) > \\ > \max_{u_{N-i}}(\mu_{C^{N-i}}(u_{N-i}) \wedge \operatorname{E} \mu_{G^{N-i+1}}(x_{N-i+1})) \tag{56}$$

4.3 Remarks on the fuzzy system under control

The case with a fuzzy system under control can be obviously dealt with by combining the idea of a fuzzy termination time presented in this section, and the two approaches to the control of a fuzzy system with a fixed and specified termination time, i.e. those using dynamic programming and branch-and-bound. Moreover, one can use the interpolative reasoning based approach to avoid numerical difficulties.

5 CONTROL WITH AN INFINITE TERMINATION TIME

The finite termination is proper for rather short processes and of a higher variability. In many cases the process is however to proceed over a long time and is low-varying, with the goal just to maintain some (stable) conditions. Then it may be expedient to assume an infinite termination time and use some specific apparatus proposed by Kacprzyk and Staniewski (1982, 1983) (cf. also Kacprzyk, 1983c).

5.1 Deterministic system under control

At each t, u_t is subjected to a *state-dependent fuzzy constraint* $\mu_C(u_t \mid x_t)$. The fuzzy goal $\mu_G(x_t)$ is the same for all t 's. The deterministic system under control is given as usually by a state transition equation $x_{t+1} = f(x_t, u_t); x_t, x_{t+1} \in X, u_t \in U$.

The fuzzy decision is

$$\begin{aligned} & \mu_D(u_0, u_1, \ldots \mid x_0) = \\ & = \mu_C(u_0 \mid x_o) \wedge \mu_G(x_1) \wedge \\ & \quad \wedge \mu_C(u_1 \mid x_1) \wedge \mu_G(x_2) \wedge \ldots = \\ & = \lim_{N \to \infty} \bigwedge_{t=0}^{N} (\mu_C(u_t \mid x_t) \wedge \mu_G(x_{t+1})) \end{aligned} \tag{57}$$

and it is often expedient to introduce a *discount factor* $b > 1$, whose role is basically to reflect the fact that what happens in nearer future matters more; then

$$\begin{aligned} & \mu_D(u_0, u_1, \ldots \mid x_0, b) = \\ & = (\mu_C(u_0 \mid x_0) \wedge \mu_G(x_1)) \wedge \\ & \quad \wedge b(\mu_C(u_1 \mid x_1) \wedge \mu_G(x_2)) \wedge \\ & \quad \wedge b^2(\mu_C(u_2 \mid x_2) \wedge \mu_G(x_3)) \wedge \ldots = \\ & = \lim_{N \to \infty} \bigwedge_{t=0}^{N} b^t(\mu_C(u_t \mid x_t) \wedge \mu_G(x_{t+1})) \end{aligned} \tag{58}$$

The problem is to find $u_0^*, u_1^*, \ldots$ such that

$$\mu_D(u_0^*, u_1^*, \ldots \mid x_0) = \\ = \max_{u_0, u_1, \ldots} \lim_{N \to \infty} \bigwedge_{t=0}^{N} (\mu_C(u_t \mid x_t) \wedge \mu_G(x_{t+1})) \tag{59}$$

and in the case of discounting with $b > 1$

$$\mu_D(u_0^*, u_1^*, \ldots \mid x_0, b) = \\ = \max_{u_0, u_1, \ldots} \lim_{N \to \infty} \bigwedge_{t=0}^{N} b^t(\mu_C(u_t \mid x_t) \wedge \mu_G(x_{t+1})) \tag{60}$$

Evidently, we seek in fact an optimal stationary policy, possibly in the sense of $a^* : X \to U$, relating an optimal control to the current state only. Then, the problems (59) and (60) become to find an optimal stationary strategy $a_\infty^* = (a^*, a^*, \ldots)$ such that

$$\mu_D(a_\infty^* \mid x_0) = \\ = \max_{a_\infty} \lim_{N \to \infty} \bigwedge_{t=0}^{N} (\mu_C(a(x_t) \mid x_t) \wedge \mu_G(x_{t+1})) \tag{61}$$

and in the case of discounting with $b > 1$

$$\mu_D(a_\infty^* \mid x_0, b) = \\ = \max_{a_\infty} \lim_{N \to \infty} \bigwedge_{t=0}^{N} b^t(\mu_C(a(x_t) \mid x_t) \wedge \mu_G(x_{t+1})) \tag{62}$$

respectively, with the ordering "$\succeq$" between the strategies given by $a\prime_\infty \succeq a\prime\prime_\infty \iff \mu_D(a\prime_\infty \mid x_0) \geq \mu_D(a\prime\prime_\infty \mid x_0)$, and, evidently, $a_\infty^* \succeq a_\infty, \forall a_\infty$.

First, the deterministic system under control is again given by $x_{t+1} = f(x_t, u_t)$, where $x_t, x_{t+1} \in X$, and $u_t \in U$, $t = 0, 1, \ldots$. The initial state is $x_0 \in X$. At each t, u_t is subjected to a state-dependent fuzzy constraint $\mu_C(u_t \mid x_t)$, and on x_t a fuzzy goal $\mu_G(x_t)$ is imposed.

Using the fuzzy decision (58) with the discount factor, we obtain

$$\mu_D(a_\infty \mid x_0) = \\ = \lim_{N \to \infty} \bigwedge_{t=0}^{N} b^t(\mu_C(a(x_t) \mid x_t) \wedge \mu_G(f(x_t, a(x_t)))) \tag{63}$$

and the problem is to find an optimal stationary strategy $a_\infty^* = (a^*, a^*, \ldots)$ such that

$$\mu_D(a_\infty^* \mid x_0) = \max_{a_\infty} \mu_D(a_\infty \mid x_0) = \\ = \lim_{N \to \infty} \bigwedge_{t=0}^{N} b^t(\mu_C(a(x_t) \mid x_t) \wedge \mu_G(f(x_t, a(x_t)))) \tag{64}$$

Problem (64) was first formulated and solved by Kacprzyk and Staniewski (1983) who have proposed a policy iteration type algorithm to solve the problem considered in a finite number of steps. Basically, the consecutive steps of the algorithms are:

Step 1. Assume an arbitrary $a_\infty = (a, a, \ldots)$.

Step 2. Solve in the $\mu_D(a_\infty \mid s_i)$'s the following set of n equations

$$\begin{cases} \mu_D(a_\infty \mid s_1) = \\ \quad = \mu_C(a(s_1) \mid s_1) \wedge \mu_G(f(s_1, a(s_1))) \wedge \\ \quad \wedge\, b\, \mu_D(a_\infty \mid f(s_1, a(s_1))) \\ \cdots\cdots\cdots\cdots \\ \mu_D(a_\infty \mid s_n) = \\ \quad = \mu_C(a(s_n) \mid s_n) \wedge \mu_G(f(s_n, a(s_n))) \wedge \\ \quad \wedge\, b\, \mu_D(a_\infty \mid f(s_n, a(s_n))) \end{cases} \tag{65}$$

Step 3. Improve the strategy, i.e. using the $\mu_D(a_\infty \mid s_i)$'s determined in Step 2, find for each $s_i \in X$ a maximizing policy $z^* : X \longrightarrow U$ such that

$$\begin{aligned} &\mu_C(z^*(s(i) \mid s_i) \wedge \mu_G(f(s_i, z^*(s_i))) \wedge \\ &\quad \wedge b\, \mu_D(a_\infty \mid f(s_i, a(s_i))) = \\ &= \max_z \mu_C(z(s(i) \mid s_i) \wedge \mu_G(f(s_i, z(s_i))) \wedge \\ &\quad \wedge b\, \mu_D(a_\infty \mid f(s_i, a(s_i))) \end{aligned} \tag{66}$$

Step 4. If the maximizing strategy found, $z^*_\infty = (z^*, z^*, \ldots)$, is the same as the previous one, i.e. if $z^*_\infty = a_\infty$, then it is an optimal stationary strategy to be determined. Otherwise, assume $a_\infty = z^*_\infty$ and return to Step 2.

As shown by Kacprzyk and Staniewski (1983), an optimal stationary strategy exists, and it found by the above algorithm.

5.2 Stochastic system under control

The stochastic system is a Markov chain whose dynamics is described by the conditional probability $p(x_{t+1} \mid x_t, u_t) \in [0,1]$. At each t, u_t is subjected to $\mu_C(u_t \mid x_t)$, and $\mu_G(x_{t+1})$ is imposed on x_{t+1}.

The fuzzy decision is assumed here to be without the discount factor, and the problem is now to find $u^*_0, u^*_1, \ldots$ such that

$$\begin{aligned} &\mathrm{E}\, \mu_D(u^*_0, u^*_1, \ldots \mid x_0) = \\ &= \max_{u_0, u_1, \ldots} \mathrm{E}\, \mu_D(u_0, u_1, \ldots \mid x_0) = \\ &= \max_{u_0, u_1, \ldots} \mathrm{E}(\lim_{N \longrightarrow \infty} \bigwedge_{t=0}^{N-1} (\mu_C(u_t \mid x_t) \wedge \mu_G(x_{t+1}))) \end{aligned} \tag{67}$$

Notice that we apply here Kacprzyk and Staniewski's (1980) formulation of the control of a stochastic system in which the maximization of the expected value of the fuzzy decision is involved. Bellman and Zadeh's (1970) approach whose essence is the maximization of the probability of attaining the fuzzy goal under fuzzy constraints looses somewhat its relevance in the case of an infinite termination time.

We are evidently interested in finding an optimal stationary strategy $a^*_\infty = (a^*, a^*, \ldots)$ such that

$$\mathrm{E}\, \mu_D(a^*_\infty \mid x_0) = \max_{a_\infty} \mathrm{E}\, \mu_D(a_\infty \mid x_0) \tag{68}$$

with a natural ordering between the stationary strategies given as $a\prime_\infty \succeq a\prime\prime_\infty \iff \mathrm{E}\, \mu_D(a\prime_\infty \mid x_0) \geq \mathrm{E}\, \mu_D(a\prime\prime \mid x_0), \forall x_0 \in X$ and, evidently, $a^*_\infty \succeq a_\infty, \forall a_\infty$.

The solution of (68) virtually consists of:

- the determination of $\mathrm{E}\, \mu_D(a_\infty \mid x_0)$, and
- the determination of an optimal stationary strategy a^*_∞.

It is easy to see that both these two issues are nontrivial. We will not discuss them in detail for lack of space, and refer the interested reader to either the source Kacprzyk and Staniewski's (1983) paper or to Kacprzyk's (1983c) book.

The basic results can be summarized as:

- an optimal stationary strategy exists,
- the (one step) strategy improvement (similar to that in the policy iteration algorithm for the deterministic system) gives better and better strategies, and
- an optimal stationary strategy should be sought among equivalent strategies obtained as a final result of the above strategy improvement.

These facts imply a policy iteration type (sub)optimization algorithm for solving the problem considered. It yields as a result some unimprovable strategies. An optimal stationary strategy is among those unimprovable strategies found. If the number of such unimprovable strategies is not too high, we can employ full enumeration to find an optimal one, otherwise some heuristic technique is evidently preferable. Notice finally that the (stationary) strategies are here functions of the state and a "summary" of the trajectory.

Now we will proceed to the case of a fuzzy system under control.

5.3 Fuzzy system under control

At each t, U_t is subjected to $\mu_{\overline{C}}(U_t \mid X_t)$, and the fuzzy goal is the same for each t, $\mu_{\overline{G}}(X_t)$.

We seek an optimal stationary policy a^*_∞ such that

$$\begin{aligned} &\mu_D(a^*_\infty \mid X_0) = \\ &= \max_{a_\infty} \mu_D(a_\infty \mid X_0) = \\ &= \lim_{N \to \infty} \bigwedge_{t=0}^{N-1} b^t(\mu_{\overline{C}}(a \mid X_t) \wedge \mu_{\overline{G}}(X_{t+1})) \end{aligned} \tag{69}$$

where $b > 1$ is a discount factor.

This problem was first formulated and solved by Kacprzyk and Staniewski (1982), and a policy iteration type algorithm was proposed analogous to that for the deterministic system. By the approximation by reference fuzzy states and controls, (69) was transformed into one with an auxiliary deterministic system to use the policy iteration type algorithm. Thus, if $A(.)$ means this approximation, then the auxiliary deterministic system representing the fuzzy system (25) is given by

$$\overline{X}_{t+1} = A(F(\overline{X}_t, \overline{U}_t)), t = 0, 1, \ldots \qquad (70)$$

and, e.g., $\mu_{\overline{C}}(\overline{U}_t \mid \overline{X}_t) = e((\overline{C}(\overline{X}_t), \overline{U}_t)$ and $\mu_{\overline{G}}(\overline{X}_t) = e(\overline{G}, \overline{X}_t)$, where $e(.,.)$ is a degree of equality.

We seek an optimal stationary policy a^*_∞ such that

$$\begin{aligned} &\mu_D(a^*_\infty) = \\ &= \max_{a_\infty} \lim_{N\to\infty} \textstyle\bigwedge_{t=0}^{N} b^t(\mu_{\overline{C}}(a \mid \overline{X}_t) \wedge \\ &\qquad \wedge \mu_{\overline{G}}(\overline{X}_{t+1})) \end{aligned} \qquad (71)$$

An a^*_∞ solving (71) is determined by a policy iteration type algorithm (Kacprzyk and Staniewski, 1982) whose essence is:

Step 1. Choose an arbitrary $a_\infty = (a, a, \ldots)$.

Step 2. Solve in $\mu_D(a_\infty \mid \overline{S}_i)$'s, $i = 1, \ldots, r$:

$$\begin{aligned} &\mu_D(a_\infty \mid \overline{S}_i) = \\ &= \mu_{\overline{C}}(a(\overline{S}_i) \mid \overline{S}_i) \wedge \mu_{\overline{G}}(A(F(\overline{S}_i, a(\overline{S}_i)))) \wedge \\ &\quad \wedge b\mu_D(a_\infty \mid A(F(\overline{S}_i, a(\overline{S}_i)))) \end{aligned} \qquad (72)$$

Step 3. Improve a_∞, i.e. find a z^* maximizing $\mu_{\overline{C}}(z(\overline{S}_i) \mid \overline{S}_i) \wedge \mu_{\overline{G}}(A(F(\overline{S}_i,\ z(\overline{S}_i)))) \wedge b\mu_D(a_\infty \mid A(F(\overline{S}_i, a(\overline{S}_i))))$.

Step 3. If z^* found in Step 3 is the same as previously, it is an optimal stationary policy sought. Otherwise, assume $a_\infty = z^*$ and return to Step 2.

We obtain (in a finite number of steps!) an a^*_∞ for the reference fuzzy states only. To be able to determine U^*_t for an arbitrary X_t, a simple method was proposed by Kacprzyk and Staniewski (1982): a^*_∞ is represented by IF $(\overline{X}_t = \overline{S}_1)$ THEN $(\overline{U}^* = \overline{C}_{u1})$ ELSE ...ELSE IF $(\overline{X}_t = \overline{S}_r)$ THEN $(\overline{U}^* = \overline{C}_{ur})$; this forms a fuzzy relation, R^*, constituting an *optimal fuzzy stationary policy*. Then, $U^*_t = X_t \circ R^*$, $t = 0, 1, \ldots$, where "$\circ$" is the max-min composition. Then, if needed, an optimal nonfuzzy control may be determined, e.g., as an u^*_t which maximizes $\mu_{U^*_t}(u_t)$. Evidently, we can also use the interpolative reasoning scheme presented for the fuzzy system and the fixed and specified termination time.

6 CONCLUDING REMARKS

This concludes our short exposition of a (prescriptive) approach to (optimal) control under fuzziness. The purpose was to recall the previous results which appeared too early, long before the recent wave of interest in fuzzy control triggered by relevant applications. We think that the models presented – which go beyond of what is commonly meant as fuzzy control – can form a basis of a "new generation" of fuzzy control in which the trajectory to be followed will be more rationally determined by some optimal control type tools.

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