$\mathcal{H}_\infty$ and Robust Estimation

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1 Introduction

The aim of these notes and the corresponding course is to present basic concepts of robust state estimation in dynamic systems, and to provide some of the existing algorithms. Special emphasis is given to robust filtering methodologies in an $\mathcal{H}_\infty$ setting as well as in a minimum variance sense.

A fundamental problem in control systems is the estimation of the state variables of a dynamic system using available noisy measurements. Over the past three decades considerable interest has been devoted to state estimation methods based on the minimization of the variance of the estimation error, i.e. the celebrated Kalman filtering approach [1]. This type of estimation assumes the knowledge of a perfect dynamic model for the signal generating system, and that the noise sources are "white processes" with known statistics, or coloured noises with known spectral density. Unfortunately, these assumptions limit the applications of minimum variance estimators as in many situations only an approximate signal model is available and/or the statistics of the noise sources are not fully known or unavailable. Also, it has been known that Kalman filter type estimators may not be robust against uncertainty in the dynamic model of the signal generating system. The frequent recurrence of filtering problems with modelling uncertainty has led to a wide interest in alternative filtering methods.

The need to handle uncertainty in filtering problems has motivated the use of a new measure of performance - the $\mathcal{H}_\infty$ norm - which has been introduced for robust control design ([12, 14, 59]). In $\mathcal{H}_\infty$ estimation the noise sources one considers are arbitrary signals with bounded energy. The estimator, which is required to produce an estimate of a given linear combination of the state variables based on the available measurements, is designed to guarantee that the transference from the noise signals to the estimation error should possess an $\mathcal{H}_\infty$ norm less than a prescribed positive value. This is equivalent to imposing an upper bound on the maximum gain of the estimation error over all frequencies. This estimation approach is very appropriate in a number of practical situations, for example; (a) when only upper bounds on the spectral density of the process and measurement noises are known; (b) when the estimation error is required to be uniformly small over all frequencies; (c) to provide robust stability for the estimation error to plant unmodelled dynamics.

$\mathcal{H}_\infty$ filtering for linear systems was first addressed in [17] where a polynomial approach was used. A solution to the $\mathcal{H}_\infty$ filtering problem via the interpolation theory was also presented in [15]. However, one of the most popular methods in the past few years is the Riccati equation approach; see, e.g. [2, 4, 26, 27, 33, 35, 39, 49, 53, 56, 57, 58]. Filters that minimize a bound on the variance of the estimation error while satisfying a prescribed $\mathcal{H}_\infty$ performance have been also proposed [6, 21, 24, 32, 37, 40]. In addition, the problem of $\mathcal{H}_\infty$ nonlinear filtering has been recently tackled in [5, 28].

Over the past few years, there have been an increasing interest in the problem of robust estimation for systems with parameter uncertainty in the dynamic model. In the context of linear systems with real norm-bounded parameter uncertainty, robust filters with a guaranteed $\mathcal{H}_\infty$ performance irrespective of the uncertainties have been developed in [9, 16, 36, 44, 45], whereas the design of robust $\mathcal{H}_\infty$ smoothers has been extensively analysed in [41] including fixed-point, fixed-lag as well as fixed-interval smoothers. In addition, design methodologies of robust $\mathcal{H}_\infty$ filters for a class of uncertain nonlinear systems have been recently proposed in [11, 28, 46, 47, 50]. On the other hand, very recently the issue of robust filtering for uncertain linear systems with a performance measure in the minimum variance sense has been attracting a lot of interest. In particular, the design of filters with an optimal guaranteed error variance has been investigated by a number of investigators [7, 20, 29, 31, 34]. These filters can be viewed as extensions of the Kalman filter in order to provide a guaranteed performance irrespective of parameter uncertainties in the system model.

In these notes we will present an overview of techniques of standard $\mathcal{H}_\infty$ filtering and robust filtering for systems with parameter uncertainty, where for the latter performance measures in the minimum variance as well as in the $\mathcal{H}_\infty$ sense will be considered. For simplicity of presentation, only continuous-time systems will be treated.

The organization of these notes is as follows. Initially, in Section 2, we present a number of fundamental concepts related to signals and systems including, the notions of $\mathcal{L}_2$ norm of a signal, $\mathcal{H}_\infty$ norm of a transfer function, and quadratic stability. We shall also review the main results on Kalman filtering theory.

In Section 3, we analyse the $\mathcal{H}_\infty$ filtering problem for linear systems where the matrices of the state space model are perfectly known, i.e. there is no uncertain parameter. We consider both finite-horizon and infinite-horizon estimation problems.

Section 4 is devoted to $\mathcal{H}_\infty$ filtering for a class of nonlinear systems described by a known linear state space model with the addition of known state-dependent nonlinearities which appear in the system dynamics as well as in the measurement model.

In Section 5 we analyse the problem of robust minimum variance filtering for linear systems subject to norm-bounded parameter uncertainty in both the state and output matrices of the state space model. Attention will be focused on the design of linear filters with an optimal guaranteed error variance, i.e. filters with an optimized upper bound on the estimation error variance for all admissible uncertainties.

Session 6 deals with a robust $\mathcal{H}_\infty$ filtering technique for linear uncertain systems of the same form as in Section 5. The robust filter is required to guarantee a prescribed performance in an $\mathcal{H}_\infty$ sense in spite of the parameter uncertainty.

We conclude these notes by discussing in Section 7 the robust version of the $\mathcal{H}_\infty$ filtering problem treated in Section 4. The class of nonlinear systems we will considered is described by a linear state space model subject to norm-bounded parameter uncertainty in both the state and output matrices and with the addition of known state-dependent nonlinearities. As in Section 6, the nonlinearities are allowed to appear in both the state and measurement equations.

1.1 Notation

$\Re$	set of real numbers.
$\Re^n$	n-dimensional real Euclidean space.
$\Re^{m\times n}$ ($C^{m\times n}$)	set of $m \times n$ real (complex) matrices.
$\mathrm{Re}(\alpha)$	real part of the complex number α.
$\triangleq$	is defined to be.
sup	supremum (similar to maximum).
$E[\cdot]$	mean or expectation of a random variable.
A^T	transpose of the matrix A.
A^*	complex conjugate transpose of the matrix A.
I	identity matrix.
$I_{n\times n}$	identity matrix of dimension $n \times n$.
$diag(A_1, A_2, ..., A_n)$	block diagonal matrix with A_i, $i = 1, 2, ..., n$ on the main diagonal.
$P \geq 0$ ($P \leq 0$)	symmetric positive (negative) semi-definite matrix $P \in \Re^{n\times n}$.
$P > 0$ ($P < 0$)	symmetric positive (negative) definite matrix $P \in \Re^{n\times n}$.
$P \geq Q$ ($P \leq Q$)	$P - Q \geq 0$ ($P - Q \leq 0$) for symmetric matrices $P, Q \in \Re^{n\times n}$.
$P > Q$ ($P < Q$)	$P - Q > 0$ ($P - Q < 0$) for symmetric matrices $P, Q \in \Re^{n\times n}$.
$\lambda_i(A)$	ith eigenvalue of matrix A.
$\sigma_i(A)$	ith singular value of matrix A, i.e. $\sigma_i(A) = \sqrt{\lambda_i(A^*A)} = \sqrt{\lambda_i(AA^*)}$.
$\sigma_{max}(A)$	maximum singular value of matrix A.
$tr(A)$	trace of the matrix A.
$\lVert\cdot\rVert$	Euclidean norm of a vector, i.e. $\lVert x\rVert = \sqrt{x^*x}$, or spectral norm of a matrix, defined by the maximum singular value.

2 Background and Fundamentals

In this section we shall introduce a number of definitions and fundamental concepts related to signals and systems which will be used throughout these notes. Moreover, we give a number of basic results we shall need further on.

2.1 Stability, Detectability and Stabilizability

Consider a linear system described by the state space model

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

$$y(t) = Cx(t) \tag{2.2}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^r$ is the input, $y(t) \in \Re^m$ is output, and A, B and C are real constant matrices. The system (2.1)-(2.2) is asymptotically stable ($\|x(t)\| \to 0$ as $t \to \infty$ for any initial state and with zero input) if and only if all the eigenvalues of A have negative real part. In this case the matrix A is said to be *Hurwitz stable*.

A way of testing the Hurwitz stability of a matrix (or equivalently, the asymptotic stability of the system (2.1)-(2.2)) is via the well known Lyapunov Lemma:

Lemma 2.1 *The matrix A is Hurwitz stable if and only if there exists a matrix $P = P^T > 0$ such that*

$$A^T P + PA < 0. \tag{2.3}$$

Note thats $V(x) = x^T Px$ is a Lyapunov function for the system (2.1).

The system (2.1)-(2.2), or the pair (C, A), is said to be *detectable* if there exists a matrix K such $A - KC$ is Hurwitz stable. (C, A) detectable means that there exists an asymptotically stable state observer for the system (2.1)-(2.2) of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)]$$

such that the estimation error $x(t) - \hat{x}(t)$ goes to zero as $t \to \infty$. The detectability of (C, A) can be tested using the following fact:

Proposition 2.1 *The pair (C, A) is detectable if and only if any of the following equivalent conditions holds:*

(a) $\text{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n,$ *for all* $\lambda = \lambda_i(A)$ *with* $Re(\lambda) \geq 0$.

(b) $Ax = \lambda x$ *and* $Cx = 0$ *for some complex number* λ *with* $Re(\lambda) \geq 0$ *implies* $x = 0$.

The system (2.1)-(2.2), or the pair (A, B), is said to be *stabilizable* if there exists a matrix J such $A - BJ$ is Hurwitz stable. (A, B) stabilizable means that there exists a state feedback control law $u(t) = -Jx(t)$ for the system (2.1) such that the resulting closed-loop system is asymptotically stable. Note that (A, B) is stabilizable if and only if (B^T, A^T) is detectable.

2.2 Norms of Signals and Systems

In the sequel we present the definitions of norms of signals and systems used throughout these notes. We will also recall few important properties of these norms.

$\mathcal{L}_2$-Norm of a Signal

We define $\mathcal{L}_2^n$ as the space of square integrable n-dimensional vector signals $u(\cdot)$ on $[0, \infty)$ with norm

$$\|u\|_2 \triangleq \left[\int_0^\infty \|u(t)\|^2 dt\right]^{\frac{1}{2}} < \infty$$

where $\|\cdot\|$ denotes the Euclidean vector norm. We shall refer to $\|u\|_2$ as the $\mathcal{L}_2$-norm of u.

We note that $\|u\|_2^2$ can be interpreted as the total energy of the signal u. The motivation for this is that if u is a scalar signal which is assumed to be the voltage across a 1Ω resistor, then $\|u\|_2^2$ coincides with the total energy dissipated in this resistor. Thus, a signal $u \in \mathcal{L}_2^n$ will be called an *energy signal*.

Observe that by Parseval's theorem, the $\mathcal{L}_2$-norm can also be calculated in the frequency domain as:

$$\|u\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^\infty \|U(j\omega)\|^2 d\omega\right]^{\frac{1}{2}}$$

where $U(j\omega)$ is the Fourier transform of $u(t)$.

$\mathcal{L}_2[0, T]$-Norm of a Signal

We define $\mathcal{L}_2^n[0, T]$ as the set of square integrable n-dimensional vector signals $u(\cdot)$ on a finite interval $[0, T]$ with norm

$$\|u\|_{[0,T]} \triangleq \left[\int_0^T \|u(t)\|^2 dt\right]^{\frac{1}{2}} < \infty.$$

We shall refer to $\|u\|_{[0,T]}$ as the $\mathcal{L}_2[0, T]$-norm of u.

Root Mean Square (RMS) Value of a Signal

We define the RMS value of a vector signal u as

$$\|u\|_{rms} \triangleq \left[\lim_{T\to\infty} \frac{1}{T}\int_0^T \|u(t)\|^2 dt\right]^{\frac{1}{2}}$$

provided this limit exists. Note that $\|\cdot\|_{rms}$ is not a norm ($\|u\|_{rms} = 0$ does not imply that $u(\cdot)$ is identically zero).

The square of the RMS value of a signal is known as the signal average power. The reason for this is that if u is a scalar signal which is assumed to be the voltage across a 1Ω resistor, then $\|u\|_{rms}^2$ is the average power dissipated in this resistor. Thus, a signal u with a finite RMS value will be called a *power signal*.

We observe that the RMS value is a measure of the average size of a signal that persists, i.e. it is a steady state measure of a signal. Such a signal does not have a finite total energy and thus the $\mathcal{L}_2$-norm is not defined. On the other hand, the $\mathcal{L}_2$-norm is appropriate for transient signals which decay to zero as time progresses. Indeed, these signals have a zero RMS value.

Note that similarly to the $\mathcal{L}_2$-norm, by Parseval's theorem, the RMS value can also be calculated in the frequency domain as:

$$\|u\|_{rms} = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \|U(j\omega)\|^2 d\omega\right]^{\frac{1}{2}}.$$

$\mathcal{L}_2$ Gain of a System

Let a dynamic system G with input $w \in \mathcal{L}_2^q$ and output $y \in \mathcal{L}_2^p$ (G may be time-varying and nonlinear). Denote by $\mathcal{G}$ the operator mapping from w to y, i.e. $y = \mathcal{G} \cdot w$. The $\mathcal{L}_2$ gain of G, denoted by $\|\mathcal{G}\|_{\mathcal{L}_2}$, is defined as the $\mathcal{L}_2$-induced norm of the operator $\mathcal{G}$, i.e.

$$\|\mathcal{G}\|_{\mathcal{L}_2} \triangleq \sup_{w}\left\{\frac{\|\mathcal{G} \cdot w\|_2}{\|w\|_2} : \ w \in \mathcal{L}_2^q,\ w \neq 0\right\}. \tag{2.4}$$

The $\mathcal{L}_2$ gain is the maximum factor by which the system can amplify the size of the input w, as measured by the $\mathcal{L}_2$-norms of the input and output signals. The $\mathcal{L}_2$ gain is therefore a worst-case performance measure in the sense of energy gain.

It follows from (2.4) that

$$\|\mathcal{G} \cdot w\|_2 \leq \|\mathcal{G}\|_{\mathcal{L}_2}\|w\|_2, \ \text{for all } w \in \mathcal{L}_2^q.$$

We also note that in the case when the system G is linear:

$$\|\mathcal{G}\|_{\mathcal{L}_2} = \sup_{\|w\|_2 \leq 1} \|\mathcal{G} \cdot w\|_2. \tag{2.5}$$

The right hand side of (2.5) corresponds to the largest $\mathcal{L}_2$-norm of the response of the system G to energy signals w such that $\|w\|_2 \leq 1$. Hence, in the case of linear systems the $\mathcal{L}_2$ gain of a system is also the worst-case response $\mathcal{L}_2$-norm.

$\mathcal{H}_\infty$ Norm of a Transfer Function

Let $G(s)$ be a proper stable transfer function matrix of a linear time-invariant system G. The $\mathcal{H}_\infty$ norm of $G(s)$, denoted as $\|G(s)\|_\infty$, is defined by

$$\|G(s)\|_\infty \triangleq \sup_{\omega \in \Re} \sigma_{max}[G(j\omega)].$$

The $\mathcal{H}_\infty$ norm can be interpreted as the maximum transfer function matrix gain on the $j\omega$ axis. In the case of a single-input single-output (SISO) system, the $\mathcal{H}_\infty$ norm is the maximum value of the gain $|G(j\omega)|$, i.e. the peak value on the Bode magnitude diagram of $G(s)$ or the distance in the complex plane from the origin to the farthest point on the Nyquist diagram of $G(s)$. The $\mathcal{H}_\infty$ norm is a measure of the worst-case response of the system and in the SISO case corresponds to the largest amplitude of the steady-state response of the system to any unit amplitude sinusoidal input.

The $\mathcal{H}_\infty$ norm of a transfer function $G(s)$ coincides with the $\mathcal{L}_2$ gain of the system G. i.e.

$$\|G(s)\|_\infty = \sup_w \left\{ \frac{\|\mathcal{G}\cdot w\|_2}{\|w\|_2} : \ w \in \mathcal{L}_2^q, \ w \neq 0 \right\} \tag{2.6}$$

where $w \in \mathcal{L}_2^q$ and $z = \mathcal{G}\cdot w \in \mathcal{L}_2^p$ are, respectively, the input and output of G, and $\mathcal{G}$ is the operator mapping from w to z. It also turns out that

$$\|G(s)\|_\infty = \sup_w \left\{ \frac{\|\mathcal{G}\cdot w\|_{rms}}{\|w\|_{rms}} : \ \|w\|_{rms} \neq 0 \right\}. \tag{2.7}$$

The right hand side of (2.7) is known as the *RMS gain of the system G*. In view of the above, we have:

$$\|G(s)\|_\infty = \sup_{\|w\|_2 \leq 1} \|\mathcal{G}\cdot w\|_2 = \sup_{\|w\|_{rms} \leq 1} \|\mathcal{G}\cdot w\|_{rms} \tag{2.8}$$

and

$$\|z\|_2 \leq \|G(s)\|_\infty \|w\|_2, \text{ for all } w \in \mathcal{L}_2^q \tag{2.9}$$

$$\|z\|_{rms} \leq \|G(s)\|_\infty \|w\|_{rms}, \text{ for all } w \text{ such that } \|w\|_{rms} < \infty. \tag{2.10}$$

The $\mathcal{H}_\infty$ norm of a transfer function may be calculated by computing $\sigma_{max}[G(j\omega)]$ on a discrete grid of frequencies, and then taking the largest value obtained as the $\mathcal{H}_\infty$ norm. This procedure can be computation-intensive, in particular in the case of multi-input multi-output systems, and may miss narrow peaks in the frequency response. An alternative way of calculating the $\mathcal{H}_\infty$ norm which is based on Lemma 2.2 as below will be discussed in the sequel.

For simplicity, $G(s)$ is assumed to be strictly proper and let a state space realization (A, B, C) for $G(s)$, i.e. $G(s) = C(sI - A)^{-1}B$. Then we have the following result.

Lemma 2.2 ([8]) *Let A be a Hurwitz stable matrix and $\gamma > 0$ a given scalar. Then $\|G(s)\|_\infty < \gamma$ if and only if the Hamiltonian matrix*

$$H \triangleq \begin{bmatrix} A & \gamma^{-2}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

has no purely imaginary eigenvalues.

In view of Lemma 2.2 it follows that the $\mathcal{H}_\infty$ norm of a strictly proper stable $G(s)$ is the infimum of $\gamma > 0$ such that the matrix H has at least one eigenvalue on the imaginary axis. Therefore, Lemma 2.2 suggests the following way to compute $\|G(s)\|_\infty$. Select an initial $\gamma > 0$ and test if $\|G(s)\|_\infty < \gamma$ by calculating the eigenvalues of H. If any of these eigenvalues is purely imaginary, increase γ; otherwise decrease γ. Repeat the procedure until $\|G(s)\|_\infty$ is calculated within a desired accuracy.

$\mathcal{H}_\infty$ Norm of a System on Finite-Horizon

Let a dynamic system G with input $w \in \mathcal{L}_2^r[0, T]$ and output $z \in \mathcal{L}_2^q[0, T]$ (G may be time-varying and nonlinear). Denote by $\mathcal{G}$ the operator mapping from w to z. Motivated by (2.6), we define the $\mathcal{H}_\infty$ norm of G over the time-horizon $[0,T]$ by

$$\|G\|_\infty \triangleq \sup_w \left\{ \frac{\|\mathcal{G}\cdot w\|_{[0,T]}}{\|w\|_{[0,T]}} : \ w \in \mathcal{L}_2^r[0,T],\ w \neq 0 \right\} \tag{2.11}$$

The right hand side of (2.11) is known as the $\mathcal{L}_2[0,T]$-induced norm of the operator $\mathcal{G}$, or the *$\mathcal{L}_2[0,T]$ gain of the system G.*

In the case when the system G is linear, the $\mathcal{H}_\infty$ norm of (2.11) can be easily computed using the result of Theorem 2.2(b) as described latter on in this section after the statement of Theorem 2.2.

2.3 Riccati Equations

Many important modern techniques of filtering and control, including Kalman and $\mathcal{H}_\infty$ filters, require solving algebraic matrix equations of the form:

$$AX + XA^T + XMX + Q = 0 \tag{2.12}$$

where A, M, and Q are square real matrices with M and Q being symmetric. This equation is known as algebraic Riccati equation (ARE).

Note that if there exists a symmetric solution to (2.12), in general this solution is not necessarily unique. One solution to the ARE (2.12) which plays an important role in filtering and control theory is the *stabilizing solution.*

Definition 2.1 *A solution $X = X^T$ to the Riccati equation (2.12) is said to be a stabilizing solution if the matrix $A + XM$ is Hurwitz stable.*

It may happen that (2.12) has no stabilizing solution. However, if (2.12) possesses a stabilizing solution X, then there exists no other stabilizing solution. We now recall an important monotonicity property of algebraic Riccati equations.

Lemma 2.3 ([10]) *Consider the algebraic Riccati equation*

$$AP + PA^T + P(R - W)P + Q = 0 \tag{2.13}$$

where Q and W are symmetric positive semi-definite matrices. If (2.13) has a stabilizing solution $P_1 = P_1^T \geq 0$ for $Q = Q_1 \geq 0$, $R = R_1 \geq 0$ and $W = W_1 \geq 0$, then for any symmetric positive semi-definite matrices Q_2, R_2 and W_2 satisfying $Q_2 \leq Q_1$, $R_2 \leq R_1$ and $W_2 \geq W_1$, the ARE (2.13) with $Q = Q_2$, $R = R_2$ and $W = W_2$ has a stabilizing solution $P_2 = P_2^T \geq 0$ as well. Moreover, $P_2 \leq P_1$.

2.4 Kalman-Bucy Filter

One of the most popular techniques for state estimation over the past three decades has been the celebrated Kalman filtering. This filtering technique provides the linear minimum variance (or least mean squares error) estimator for signals in a linear system described by a state space model of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + v_1(t) &(2.14)\\ y(t) &= Cx(t) + v_2(t) &(2.15)\\ z(t) &= Lx(t) &(2.16)\end{aligned}$$

where $x(t) \in \Re^n$ is the state, $v_1(t) \in \Re^r$ is the process noise, $y(t) \in \Re^m$ is the measurement, $v_2(t) \in \Re^m$ is the measurement noise, $z(t) \in \Re^q$ is a linear combination of state variables to be estimated using past measurements, and A, B, C, D and L are known real constant matrices with appropriate dimensions. When the state $x(t)$ is to be estimated, L is set to the identity matrix.

The noise signals v_1 and v_2 are assumed to zero-mean white signals with covariance matrices as below:

$$\begin{aligned} E\left[v_1(t)v_1^T(t-\tau)\right] &= V_1\delta(t-\tau)\\ E\left[v_2(t)v_2^T(t-\tau)\right] &= V_2\delta(t-\tau); \quad V_2 > 0\\ E\left[v_1(t)v_2^T(t-\tau)\right] &= S\delta(t-\tau)\end{aligned}$$

where $\delta(t)$ denotes the Dirac delta. Also, the initial state of (2.14) is assumed to be a random variable, x_0, with a covariance matrix P_0, which is uncorrelated with v_1 and v_2. The non-singularity condition $V_2 > 0$, which means that all the components of measurement vector are noisy, is in general adopted for the sake of technical simplification. In such cases the filtering problem is said to be *non-singular*. We note that the non-singularity assumption $V_2 > 0$ can be removed, however the correponding filtering results becomes too complicated.

The time-domain problem statement is to find a linear causal estimator for $z(t)$ that minimizes the variance of estimation error:

$$E\left\{[z(t)-\hat{z}(t)]^T[z(t)-\hat{z}(t)]\right\} \tag{2.17}$$

where $\hat{z}(t)$ denotes the estimate of $z(t)$ based on the measurements $\{y(\tau),\ 0 \le \tau \le t\}$.

The linear causal filter that minimizes (2.17) for any finite $t \ge 0$ is given by:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + K(t)\left[y(t) - C\hat{x}(t)\right], \quad \hat{x}(0) = x_0 &(2.18)\\ \hat{z}(t) &= L\hat{x}(t) &(2.19)\end{aligned}$$

where the filter gain matrix $K(t)$ satisfies

$$K(t) = \left[P(t)C^T + BD^T\right]V_2^{-1} \tag{2.20}$$

where $P(t) = P^T(t) \ge 0$ is the solution of the Riccati differential equation (RDE):

$$\begin{aligned} \dot{P}(t) = \left(A - SV_2^{-1}C\right)P(t) + P(t)\left(A - SV_2^{-1}C\right)^T - P(t)C^TV_2^{-1}CP(t)\\ + V_1 - SV_2^{-1}S^T, \quad P(0) = P_0 \end{aligned} \tag{2.21}$$

The filter of (2.18)-(2.19) is known as the *Kalman filter*. We note that the RDE (2.21) is guaranteed to have a bounded solution $P(t) = P^T(t) \geq 0$ over any finite-horizon $[0, t]$.

Observe that, although the system (2.14)-(2.16) is time-invariant, the filter of (2.18)-(2.19) turns out to be time-varying. It should be also remarked that the above result still holds when the matrices A, B, C, L, S, V_1 and V_2 are piecewise continuous time-varying.

In the case where we aim to minimize the asymptotic error variance, i.e. minimize (2.17) for $t \to \infty$, the optimal estimate of z can be obtained via a stationary Kalman filter. In such situation, the filter is also required to be asymptotically stable and for that, the following assumption on the system (2.14)-(2.16) is needed:

Assumption 2.1

(a) (C, A) *is detectable;*

(b) rank $\begin{bmatrix} A - j\omega I & V_1^{\frac{1}{2}} \\ C & V_2^{\frac{1}{2}} \end{bmatrix} = n + m$, *for all* $\omega \in (-\infty, \infty)$.

The linear causal asymptotically stable filter that asymptotically minimizes (2.17) is given by (2.18)-(2.19) with $P(t)$ replaced by the constant matrix $P = P^T \geq 0$, namely the stabilizing solution to the algebraic Riccati equation:

$$\left(A - SV_2^{-1}C\right) P + P \left(A - SV_2^{-1}C\right)^T - PC^T V_2^{-1} CP + V_1 - SV_2^{-1} S^T = 0. \qquad (2.22)$$

Note that the existence of a stabilizing solution $P = P^T \geq 0$ to the ARE (2.22) is guaranteed by Assumption 2.1.

2.5 Bounded Real Lemma

An important tool in $\mathcal{H}_\infty$ performance analysis is the so-called bounded real lemma. It relates the boundedness of the $\mathcal{H}_\infty$ norm of linear systems to a certain type of Riccati equations and inequalities. In the following we shall present the strict bounded real lemma for both the cases of finite and infinite horizon.

Strict Bounded Real Lemma on Infinite-Horizon

Theorem 2.1 ([12, 60]) *Given a scalar* $\gamma > 0$, *the following statements are equivalent:*

(a) The matrix A *is Hurwitz stable and* $\|C(sI - A)^{-1}B\|_\infty < \gamma$;

(b) There exists a stabilizing solution $P = P^T \geq 0$ *to the Riccati equation*

$$AP + PA^T + \gamma^{-2} PC^T CP + BB^T = 0; \qquad (2.23)$$

(c) There exists a matrix $Q = Q^T > 0$ *such that*

$$AQ + QA^T + \gamma^{-2} QC^T CQ + BB^T < 0; \qquad (2.24)$$

(d) There exists a stabilizing solution $X = X^T \geq 0$ to the Riccati equation

$$A^T X + XA + \gamma^{-2} XBB^T X + C^T C = 0; \tag{2.25}$$

(e) There exists a matrix $Y = Y^T > 0$ such that

$$A^T Y + YA + \gamma^{-2} YBB^T Y + C^T C < 0. \tag{2.26}$$

Strict Bounded Real Lemma on Finite-Horizon

Consider the following linear time-varying system

$$(S): \quad \dot{x}(t) = A(t)x(t) + B(t)w(t), \quad x(0) = x_0 \tag{2.27}$$

$$z(t) = C(t)x(t) \tag{2.28}$$

where $x(t) \in \Re^n$ is the state, x_0 is an unknown initial state, $w(t) \in \Re^r$ is the input signal, $z(t) \in \Re^q$ is the output, and the time-varying matrices $A(t)$, $B(t)$ and $C(t)$ are assumed to be real piecewise continuous and bounded.

Assuming that $u \in \mathcal{L}_2^r[0,T]$ and $z \in \mathcal{L}_2^q[0,T]$, we define the following worst-case performance measure for the system (2.27)-(2.28) over the time-horizon $[0,T]$:

$$\mathcal{N}(R,T) \triangleq \sup \left\{ \left[\frac{\|z\|_{[0,T]}^2}{\|w\|_{[0,T]}^2 + x_0^T R x_0} \right]^{\frac{1}{2}} \right\}$$

where $R = R^T > 0$ is a given weighting matrix for the initial state and the supremum is taken over all $(w, x_0) \in \mathcal{L}_2^r[0,T] \oplus \Re^n$ such that $\|w\|_{[0,T]}^2 + x_0^T R x_0 \neq 0$.

The above performance measure is indeed the induced norm of the operator mapping from (w, x_0) to z. Thus, $\mathcal{N}(R,T)$ can be viewed as the worst-case gain from the (w, x_0) to z.

In the case where the initial state x_0 is known to be zero, the performance measure $\mathcal{N}(R,T)$ turns out to be the $\mathcal{H}_\infty$ norm of the system (S) over the time-horizon $[0,T]$, namely:

$$\|S\|_\infty \triangleq \sup_w \left\{ \frac{\|z\|_{[0,T]}}{\|w\|_{[0,T]}} : \quad w \in \mathcal{L}_2^r[0,T],\ w \neq 0:\ x_0 = 0 \right\}.$$

Next, introduce the following Riccati differential equation (RDE)

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + \gamma^{-2} Q(t)C^T(t)C(t)Q(t) + B(t)B^T(t) \tag{2.29}$$

We note that the existence of a bounded solution $Q(t)$ to (2.29) over $[0,\tau]$, $\forall\ \tau > 0$, with initial condition $Q(0) = M = M^T > 0$ (respectively, $M = M^T \geq 0$), implies that $Q(t)$ is symmetric positive definite (respectively, positive semi-definite) over $[0,\tau]$.

A version of the strict bounded real lemma for the system (2.27)-(2.28) on a finite time-horizon $[0,T]$ is given below.

Theorem 2.2 (**[23]**) *Given the system (2.27)-(2.28) and a scalar $\gamma > 0$, we have the following results:*

(a) $\mathcal{N}(R,T) < \gamma$ if an only if there exists a bounded solution $Q(t) = Q^T(t) > 0$ to the RDE (2.29) over $[0,T]$ with initial condition $Q(0) = R^{-1}$.

(b) $\|S\|_\infty < \gamma$ if and only if there exists a bounded solution $Q(t) = Q^T(t) \geq 0$ to the RDE (2.29) over $[0,T]$ with initial condition $Q(0) = 0$.

In view of Theorem 2.2 (b), the $\mathcal{H}_\infty$ norm of the system S on a finite-horizon $[0,T]$ can be calculated by a search on γ similarly to the case of the $\mathcal{H}_\infty$ norm of a transfer function. Pick an initial $\gamma > 0$ and check if $\|G\|_\infty < \gamma$ by testing if the RDE (2.29) with initial condition $Q(0) = 0$ has a bounded solution over $[0,T]$. If such solution exists, decrease γ; otherwise increase γ. Repeat the procedure until a desired accuracy is achieved.

2.6 Quadratic Stability and Robust Performance

Fundamental issues in the design of filters (and controllers) for uncertain systems, i.e. systems with significant modelling uncertainty, are *robust stability* and *robust performance*. Robust stability means that the system remains stable for a given set of uncertainty while robust performance means that both stability and performance requirements are met for a given set of uncertainty. A widely used concept of robust stability for uncertain systems with time-varying parameter uncertainty is that of *quadratic stability* which was proposed in [3]. This robust stability concept is based on a quadratic Lyapunov function and can be viewed as an extension of the stability result of Lemma 2.1.

Consider uncertain linear systems described by

$$\dot{x}(t) = [A + \Delta A(t)]x(t), \qquad x(0) = x_0 \tag{2.30}$$

where $x \in \Re^n$, A is a known real constant matrix, and $\Delta A(t)$ is an unknown time-varying matrix representing time-varying parameter uncertainties. The admissible uncertainties $\Delta A(t)$ are assumed to belong to a set $\mathcal{D}$ and satisfy certain regularity conditions such that the solution of (2.30) is well defined.

Definition 2.2 *The system (2.30) is said to be quadratically stable if there exists a matrix $P = P^T > 0$ such that*

$$[A + \Delta A(t)]^T P + P[A + \Delta A(t)] < 0 \tag{2.31}$$

for all $\Delta A(t)$ belonging to $\mathcal{D}$.

We observe that quadratic stability implies uniform asymptotic stability for all admissible uncertainties. Indeed, it is easy to see that $V(x) = x^T Px$ is a Lyapunov function for the system (2.30) for all admissible $\Delta A(t)$.

In the robust filtering problems treated later in these notes we will consider uncertainties $\Delta A(t)$ which are norm-bounded and of the form

$$\Delta A(t) = HF(t)E \tag{2.32}$$

where $F(t)$ is an unknown real time-varying matrix satisfying

$$\|F(t)\| \leq 1, \text{ for all } t \geq 0 \tag{2.33}$$

and where E and H are known real constant matrices of appropriate dimensions which specify how the uncertain parameters in $F(t)$ affect the matrix A. The above form of $\Delta A(t)$ is a matrix generalization of the scalar case where the magnitude of $\Delta A(t)$ is known to be bounded, say $|\Delta A(t)| \leq \alpha$. Indeed, such $\Delta A(t)$ is of the form of (2.32)-(2.33) with E and H being any scalars such that $HE = \alpha$.

The quadratic stability of uncertain linear systems with $\Delta A(t)$ as in (2.32)-(2.33) can be easily ascertained via an $\mathcal{H}_\infty$ norm condition as described in the theorem that follows.

Theorem 2.3 ([22]) *Consider the uncertain system (2.30) with ΔA as in (2.32). Then this system is quadratically stable for all uncertainty matrices $F(t)$ satisfying (2.33) if an only if $\|E(sI - A)^{-1}H\|_\infty < 1$.*

We now present a robust performance result for the following class of nonlinear systems:

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + Gg[x(t)] + Bw(t) \tag{2.34}$$

$$y(t) = Cx(t) \tag{2.35}$$

where $x(t) \in \Re^n$, $y(t) \in \Re^m$, $w(t) \in \Re^r$ and belongs to $\mathcal{L}_2^r$, A, B, C and G are known real constant matrices of appropriate dimensions, $\Delta A(t)$ is a norm-bounded uncertain matrix as in (2.32)-(2.33), and $g(\cdot) : \Re^n \to \Re^{n_g}$ is a known nonlinear function satisfying the following assumption:

Assumption 2.2 *There exists a known constant matrix W_g such that for any $x \in \Re^n$*

$$\|g(x)\| \leq \|W_g x\|.$$

Then we have the following robust performance result in an $\mathcal{H}_\infty$ setting:

Lemma 2.4 ([42]) *Consider the system (2.34)-(2.35) satisfying Assumption 2.2 and let $\gamma > 0$ be a given scalar. If for same scalar $\varepsilon > 0$ there exists a stabilizing solution $P = P^T \geq 0$ to the Riccati equation*

$$AP + PA^T + P\left(\gamma^{-2}C^TC + \varepsilon E^TE + W_g^TW_g\right) + BB^T + \frac{1}{\varepsilon}HH^T + GG^T = 0 \tag{2.36}$$

then the system (2.34) is globally uniformly asymptotically stable and under zero-initial conditions.

$$\|y\|_2 < \gamma\|w\|_2$$

for all non-zero $w \in \mathcal{L}_2^r$ and for all uncertain matrices $F(t)$ satisfying (2.33).

In the case where there is no parameter uncertainty in the system (2.34)-(2.35), the above result reduces to the following:

Corollary 2.1 ([42]) *Consider the system (2.34)-(2.35) satisfying Assumption 2.2 and with $\Delta A(t) \equiv 0$, and let $\gamma > 0$ be a given scalar. If there exists a stabilizing solution $P = P^T \geq 0$ to the Riccati equation*

$$AP + PA^T + P\left(\gamma^{-2}C^TC + W_g^TW_g\right) + BB^T + GG^T = 0 \qquad (2.37)$$

then this system is globally uniformly asymptotically stable and under zero initial conditions,

$$\|y\|_2 < \gamma\|w\|_2$$

for all non-zero $w \in \mathcal{L}_2^r$.

We conclude this section by recalling an important matrix inequality:

Lemma 2.5 ([48]) *Let A, E, F, H and Q be real matrices of appropriate dimensions, with F allowed to be time-varying and Q being symmetric. Then there exists a matrix $P = P^T > 0$ such that*

$$[A + HFE]P + P[A + HFE]^T + Q < 0$$

for all matrices F satisfying $\|F\| \leq 1$ if and only if there exists a scalar $\varepsilon > 0$ such that

$$AP + PA^T + \varepsilon PE^TEP + \frac{1}{\varepsilon}HH^T + Q < 0.$$

3 $\mathcal{H}_\infty$ Linear Filtering

This section deals with the design of state estimators with an $\mathcal{H}_\infty$ perfomance measure and where the only uncertainty in the system model is in the form of a bounded energy noise signal, i.e. there is no uncertainty in the matrices of the system state space model. We consider both finite-horizon and infinite-horizon estimation problems. Moreover, we consider two situations for the system initial state: one corresponds to the case when the initial state is unknown, while in the other the initial state is assumed to be zero. The latter case will also correspond to the situation where we are concerned with steady state filtering, i.e. under the assumption that measurements have been continuing for a sufficient long time so that the effect of the initial condition has become zero.

3.1 Problem Formulation

Consider linear systems described by a state space model of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \quad x(0) = x_0 && (3.1) \\ y(t) &= Cx(t) + Dw(t) && (3.2) \\ z(t) &= Lx(t) && (3.3) \end{aligned}$$

where $x(t) \in \Re^n$ is the state, x_0 is an unknow initial state, $w(t) \in \Re^r$ is the noise signal, $y(t) \in \Re^m$ is the measurement, and $z(t) \in \Re^q$ is a linear combination of state variables to be estimated over the horizon $[0,T]$, $T > 0$, using the measurements $y(t)$. The noise w is assumed to be an arbitrary signal in $\mathcal{L}_2^r[0,T]$ and A, B, C, D and L are known real constant matrices with appropriate dimensions. When the matrix L is the identity, the state $x(t)$ is to be estimated.

For the sake of technical simplification, we shall adopt the following assumption:

Assumption 3.1

$$V \triangleq DD^T > 0.$$

Note that Assumption 3.1 is similar to the standard assumption in non-singular Kalman filter which considers that all the components of the measurement vector are noisy. The case of a singular matrix V can be treated using a technique proposed in [38] for solving singular $\mathcal{H}_\infty$ control problems.

We observe that the case where the input and measurement noise signals are different, say v_1 and v_2, respectively, is a particular case of (3.1)-(3.2) where the matrices B and D are replaced by $[B \quad 0]$ and $[0 \quad D]$, respectively.

The above signal estimation setting is quite general and encompasses a number of typical situations which arise in the areas of control engineering and signal processing. For example, consider the filtering problem with a signal generating mechanism as shown in Figure 3.1. The signals $v(t)$ and $n(t)$ are energy bounded noise sources and $s(t)$ is the signal to be estimated using the noisy measurement, $y(t)$. Both the signal generator and measurement subsystem are described by linear state-space models with the signal generator model being strictly proper and the measurement system is assumed square. In this situation, it is easy to see that this filtering problem can be recast as a filtering problem similar to those analysed in these notes.

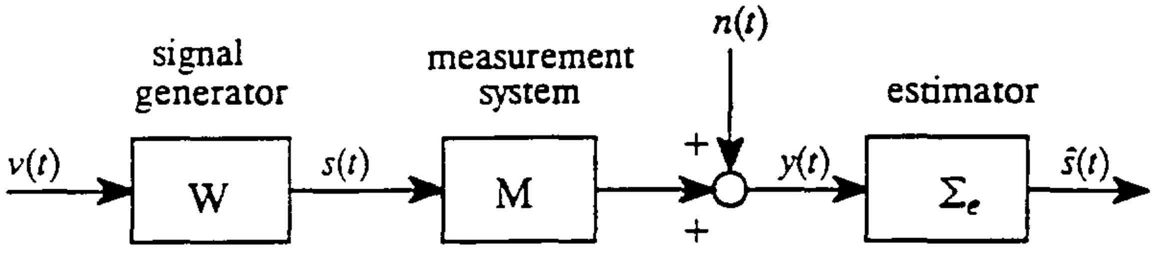

Figure 3.1 Signal Generating Mechanism.

Here we are concerned with obtaining an estimate $\hat{z}(t)$ of $z(t)$ over a horizon $[0,T]$ via a linear causal filter $\mathcal{F}$ using the measurements $\mathcal{Y}_t = \{y(\tau),\ 0 \le \tau \le t\}$ and where no *a priori* estimate of the initial state of (3.1) is assumed. The filter is required to provide a uniformly small estimate error, $e(t) = z(t) - \hat{z}(t)$, $\forall\ t \in [0,T]$, for all $w \in \mathcal{L}_2^r[0,T]$ and $x_0 \in \Re^n$. We shall consider the following worst-case performance measure:

$$J_1(R,T) \triangleq \sup \left\{ \left[\frac{\|z-\hat{z}\|^2_{[0,T]}}{\|w\|^2_{[0,T]} + x_0^T R x_0} \right]^{\frac{1}{2}} \right\} \tag{3.4}$$

where $R = R^T > 0$ is a given weighting matrix for the initial state and the supremum is taken over all $(w, x_0) \in \mathcal{L}_2^r[0,T] \oplus \Re^n$ such that $\|w\|^2_{[0,T]} + x_0^T R x_0 \neq 0$. The weighting matrix R is a measure of the uncertainty in x_0 relative to the uncertainty in w. A 'large' value of R indicates that the initial state is likely to be very close to zero.

The above index of performance is indeed the induced norm of the operator from the 'noise' signal, (w, x_0), to the estimation error, $z - \hat{z}$. Thus, $J_1(R,T)$ can be interpreted as the worst-case gain from (w, x_0) to $z - \hat{z}$.

In the case where the initial state x_0 is known to be zero, the performance measure of (3.4) is replaced by:

$$J_0(T) \triangleq \sup_{0 \neq w \in \mathcal{L}_2^r[0,T]} \left\{ \frac{\|z - \hat{z}\|_{[0,T]}}{\|w\|_{[0,T]}} \right\}. \tag{3.5}$$

In (3.5), T is allowed to be ∞ and in this case by the notation $\mathcal{L}_2^r[0,T]$ and $\|\cdot\|_{[0,T]}$ we mean $\mathcal{L}_2^r$ and $\|\cdot\|_2$, respectively. We observe that the index of performance of (3.5) can be viewed as the limit of (3.4) as the smallest eigenvalue of weighting matrix R tends to infinity. The reason for this is because for such matrix R, the initial state x_0 will be forced to be zero. Also, note that the performance measure $J_0(\infty)$ coincides with $\|G_{ew}(s)\|_\infty$, where $G_{ew}(s)$ is the transfer function matrix from w to the estimation error, $e = z - \hat{z}$.

The filtering problems we address in this section are as follows:

Given a prescribed level of 'noise' attenuation $\gamma > 0$ and an initial state weighting matrix $R = R^T > 0$, find a linear causal filter $\mathcal{F}$ such that:

- *In the finite-horizon case,*

  $J_1(R,T) < \gamma$, *or alternatively,* $J_0(T) < \gamma$ *when* $x_0 = 0$.

- *In the infinite-horizon case,*

  The filter is asymptotically stable, and under zero initial conditions, $\|G_{ew}(s)\|_\infty < \gamma$.

The above estimation problems, will be referred to as *standard $\mathcal{H}_\infty$ filtering* and the resulting filters, which are commonly known as *$\mathcal{H}_\infty$ sub-optimal filters*, are said to achieve a level of noise attenuation γ. These filters are designed to guarantee that the worst-case gain from (w, x_0) to the estimation error $z - \hat{z}$ is less γ. We note that in the infinite-horizon case, an $\mathcal{H}_\infty$ filter ensures that the 'maximum gain' in frequency from the noise source to the estimation error is bounded by γ. An *$\mathcal{H}_\infty$ optimal filter* is the one which minimizes either $J_1(R,T)$ (alternatively, $J_0(T)$) or $\|G_{ew}(s)\|_\infty$, i.e. it minimizes the worst-case gain from noise to the estimation error.

The standard $\mathcal{H}_\infty$ problem can be interpreted as a minimax problem where the estimator strategy $\hat{z}(\cdot)$ is determined such that the payoff function

$$\mathcal{J} \triangleq \|z - \hat{z}\|^2_{[0,T]} - \gamma^2 \left[\|w\|^2_{[0,T]} + x_0^T R x_0\right]$$

satisfies $\mathcal{J} < 0$ for all $w \in \mathcal{L}_2[0,T]$ and $x_0 \in \Re^n$ subject to the constraint $\|w\|^2_{[0,T]} + x_0^T R x_0 \neq 0$.

We can view the $\mathcal{H}_\infty$ estimation problem as a dynamic, two-persons, zero-sum game. In this game the first player, say a statistician, is required to find an estimation strategy $\hat{z}(t) = \mathcal{F}(\mathcal{Y}_t)$, so that the cost function $\mathcal{J}$ will be minimized. The statistician opponent, say nature, is looking for the worst possible noise signal and initial state which comply with the given measurements and will maximize $\mathcal{J}$. The $\mathcal{H}_\infty$ estimator will corresponds to the statistician' strategy which is indeed designed to handle the worst possible x_0 and w.

Observe that unlike the traditional minimum variance filtering approach, e.g. the celebrated Kalman filtering method, $\mathcal{H}_\infty$ filtering treats the noise signals as deterministic disturbance and thus no *a priori* knowledge of the noise statistics is required, except of the boundedness assumption of its energy or average power. This make the $\mathcal{H}_\infty$ filtering approach suitable for applications where little knowledge about the noise statistics is available.

3.2 $\mathcal{H}_\infty$ Filters

We first note that the standard $\mathcal{H}_\infty$ filtering problem can be solved via several different approaches, including interpolation theory approach, game theoretical approach, and Riccati equation approach. In the sequel, we will present a solution to the standard $\mathcal{H}_\infty$ filtering problem using a Riccati equation approach. To this end, introduce the following matrix Riccati differential equation (RDE):

$$\begin{aligned}\dot{P}(t) &= \left(A - BD^TV^{-1}C\right)P(t) + P(t)\left(A - BD^TV^{-1}C\right)^T \\ &\quad + P(t)\left(\gamma^{-2}L^TL - C^TV^{-1}C\right)P(t) + B\left(I - D^TV^{-1}D\right)B^T. \qquad (3.6)\end{aligned}$$

It can be easily shown that the existence of a bounded solution $P(t)$ to (3.6) over $[0,\tau]$, $\forall\, \tau > 0$, with initial condition $P(0) = M = M^T > 0$ (respectively, $M = M^T \geq 0$), implies that $P(t)$ is symmetric positive definite (respectively, positive semi-definite) over $[0,\tau]$.

First, we shall present a solution to the problem of $\mathcal{H}_\infty$ filtering on finite-horizon.

Theorem 3.1 *Consider the system (3.1)-(3.3) satisfying Assumption 2.1 and let $\gamma > 0$ be a given constant. Then we have the following results:*

(a) Assume that x_0 is unknown and let $R = R^T > 0$ be a given initial state weighting matrix. Then, there exists a linear causal filter such that $J_1(R,T) < \gamma$ if and only if there exists a bounded solution $P(t)$ to the RDE (3.6) over $[0,T]$ with initial condition $P(0) = R^{-1}$.

(b) Assuming that $x_0 = 0$, there exists a linear causal filter such that $J_0(T) < \gamma$ if and only if there exists a bounded solution $P(t)$ to the RDE (3.6) over $[0,T]$ with initial condition $P(0) = 0$.

(c) When condition (a), or (b) is satisfied, a suitable filter for both the above cases is given by:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K(t)\left[y(t) - C\hat{x}(t)\right], \qquad \hat{x}(0) = 0 \qquad (3.7)$$

$$\hat{z}(t) = L\hat{x}(t) \qquad (3.8)$$

where the filter gain matrix $K(t)$ satisfies

$$K(t) = \left[P(t)C^T + BD^T\right]V^{-1}. \qquad (3.9)$$

(d) When condition (a), or (b) is satisfied, every linear causal filter that achieves a level of noise attenuation γ is of the form:

$$
\begin{aligned}
\dot{\xi}(t) &= A\xi(t) + K(t)[y(t) - C\xi(t)] - \gamma^{-2}P(t)L^T r(t), \qquad \xi(0) = 0 &(3.10)\\
\hat{z}(t) &= L\xi(t) + r(t) &(3.11)\\
r(t) &= \mathcal{W}[y(t) - C\xi(t)] &(3.12)
\end{aligned}
$$

where $\mathcal{W}(\cdot)$ is any linear causal operator from $\mathcal{L}_2^m[0,T]$ to $\mathcal{L}_2^q[0,T]$ with an $\mathcal{L}_2[0,T]$-induced norm smaller than γ.

Proof. For simplicity of presentation, we shall only give the proof of the sufficiency of the results (a) and (b). The other proofs can be found in [26, 27].

We first consider the proof of the sufficiency of (a). Let $P(t)$ be a bounded solution to (3.6) over $[0,T]$ with initial condition $P(0) = R$. In view of (3.9), it is easy to verify that the RDE (3.6) can be rewritten in the form

$$
\begin{aligned}
\dot{P}(t) &= [A - K(t)C]\,P(t) + P(t)\,[A - K(t)C]^T + \gamma^{-2}P(t)L^T LP(t)\\
&\quad + [B - K(t)D]\,[B - K(t)D]^T, \qquad P(0) = R^{-1}. \qquad (3.13)
\end{aligned}
$$

Next, letting $\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$ and considering (3.1)-(3.3) and (3.7)-(3.8), we have that the estimation error, $z(t) - \hat{z}(t)$, can be described by the following state space model

$$
\begin{aligned}
\dot{\tilde{x}}(t) &= [A - K(t)C]\tilde{x}(t) + [B - K(t)D]w(t), \qquad \tilde{x}(0) = x_0\\
z(t) - \hat{z}(t) &= L\tilde{x}(t).
\end{aligned}
$$

Hence, in view of Theorem 2.2 (a), (3.13) implies that $J_1(R,T) < \gamma$.

The proof of the sufficiency of (b) is similar the above proof except that now $P(0) = 0$ and Theorem 2.2 (b) is used *in lieu* of Theorem 2.2 (a).

$\nabla\nabla\nabla$

Remark 3.1 It should be observed that the results of Theorem 3.1 still hold in the case where the matrices A, B, C, D and L of the system (3.1)-(3.3) are bounded, piecewise continuous time-varying. □

Remark 3.2 Note the similarity of the above $\mathcal{H}_\infty$ filtering result with the Kalman filtering. As $\gamma \to \infty$ the Riccati equation (3.6) becomes the Riccati equation of the finite-horizon Kalman filter for the system (3.1)-(3.2) subject to the assumption that the noise signal $w(t)$ is a zero-mean white process with an identity power spectrum density matrix. Furthermore, the estimator of (3.7)-(3.9) becomes the corresponding Kalman filter for the system (3.1)-(3.3). □

We now analyse the $\mathcal{H}_\infty$ filtering problem on infinite-horizon. Since in this case the filter is required to be asymptotically stable, we shall make the following assumption for the system (3.1)-(3.2):

Assumption 3.2

(a) (C, A) *is detectable;*

(b) $rank \begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} = n + m,$ *for all* $\omega \in (-\infty, \infty)$.

It should be noted that Assumption 3.2 (a) is necessary for the existence of an asymptotically stable filter for the system of (3.1)-(3.2). Assumption 3.2 (b) is equivalent to requiring that the transfer function matrix $C(sI - A)^{-1}B + D$ has no transmission zeros on the imaginary axis. We observe that Assumption 3.2 is a standard assumption in infinite-horizon Kalman filtering.

A complete solution to the $\mathcal{H}_\infty$ filtering on infinite-horizon is provided by the next theorem: see [26, 27] for the proof.

Theorem 3.2 *Let the system (3.1)-(3.3) with* $x_0 = 0$ *and satisfying Assumptions 3.1 and 3.2. Given a constant* $\gamma > 0$, *the following statements are equivalent:*

(a) There exist a linear causal, asymptotically stable filter such that $\|G_{ew}(s)\|_\infty < \gamma$.

(b) There exists a stabilizing solution $P = P^T \geq 0$ *to the algebraic Riccati equation*

$$\left(A - BD^TV^{-1}C\right)P + P\left(A - BD^TV^{-1}C\right)^T + P\left(\gamma^{-2}L^TL - C^TV^{-1}C\right)P$$
$$+ B\left(I - D^TV^{-1}D\right)B^T = 0. \tag{3.14}$$

When any of the above conditions is satisfied, we have that:

(i) A suitable time-invariant filter is given by (3.7)-(3.9) with $P(t)$ *replaced by the constant matrix* P *as in (b).*

(ii) Every proper linear time-invariant filter that achieves a level of noise attenuation γ *is of the form (3.10)-(3.12), with* $\mathcal{W}(\cdot)$ *replaced by any proper stable transfer function matrix* $W(s)$ *satisfying* $\|W(s)\|_\infty < \gamma$.

∇∇∇

Remark 3.3 Similarly to the case of finite-horizon $\mathcal{H}_\infty$ filtering, as $\gamma \to \infty$, the result of Theorem 3.2 reduces to the well known infinite-horizon Kalman filtering result for the system (3.1)-(3.3) under the assumption that the noise signal $w(t)$ is a zero-mean white process with an identity power spectrum density matrix. □

3.3 An Example

Consider the following second order resonant system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & -1+\delta \\ 1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} -0.4545 \\ 0.9090 \end{bmatrix} w(t) \\ y(t) &= [0 \quad 100]\, x(t) + v(t) \\ z(t) &= [0 \quad 100]\, x(t) \end{aligned}$$

where δ is an uncertain parameter satisfying $|\delta| \leq 0.3$.

Both stationary Kalman filter and $\mathcal{H}_\infty$ filter are designed for the nominal system that has been chosen to correspond to $\delta = 0$. These filters are of the following form:

$$\dot{\hat{x}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -0.5 \end{bmatrix} \hat{x}(t) + K\left[y(t) - [0 \quad 100]\hat{x}(t)\right], \quad \hat{x}(0) = 0$$
$$\hat{z}(t) = [0 \quad 100]\,\hat{x}(t)$$

For the Kalman filter design, the noise sources $w(t)$ and $v(t)$ were assumed to be uncorrelated, zero-mean, white signals with unit power spectra densities. The resulting filter gain is given by

$$K = K_K = [0.447 \quad 0.909]^T.$$

For the $\mathcal{H}_\infty$ filter design, we take $\gamma = 1.1 = 0.83\,\text{dB}$ which yields

$$K = K_\infty = [1.0350 \quad 2.1807]^T.$$

We then apply the two filters to the above system, with $\delta = 0.3$ and $\delta = -0.3$. The frequence response magnitude of the transfer function from $[w \quad v]^T$ to $e = z - \hat{z}$ for both filters, denoted by $[G_{ew} \quad G_{ev}]$, are shown in Figures 2.1 and 2.2. From the results in these figures, we make the following observations:

- The Kalman filter is more sensitive to parameter changes than the standard $\mathcal{H}_\infty$ filter;
- The $\mathcal{H}_\infty$ filter performs better than the Kalman filter for both $\delta = 0.3$ and $\delta = -0.3$. We note that the $\mathcal{H}_\infty$ filter provides not only a smaller maximum frequency gain but also a smaller estimation error variance;
- The magnitude of $[G_{ew} \quad G_{ev}]$ are worsened for both designs when the parameter uncertainty δ exists;

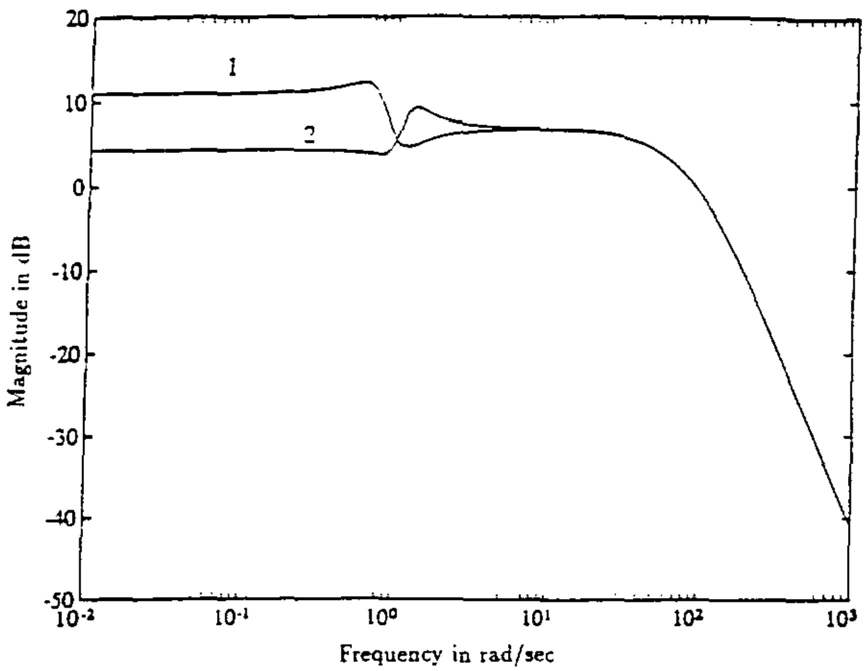

Figure 3.2 Kalman Filter: spectrum $10log(\|G_{ew}(j\omega)\|^2 + \|G_{ev}(j\omega)\|^2)$
(curve 1: $\delta = 0.3$, curve 2: $\delta = -0.3$)

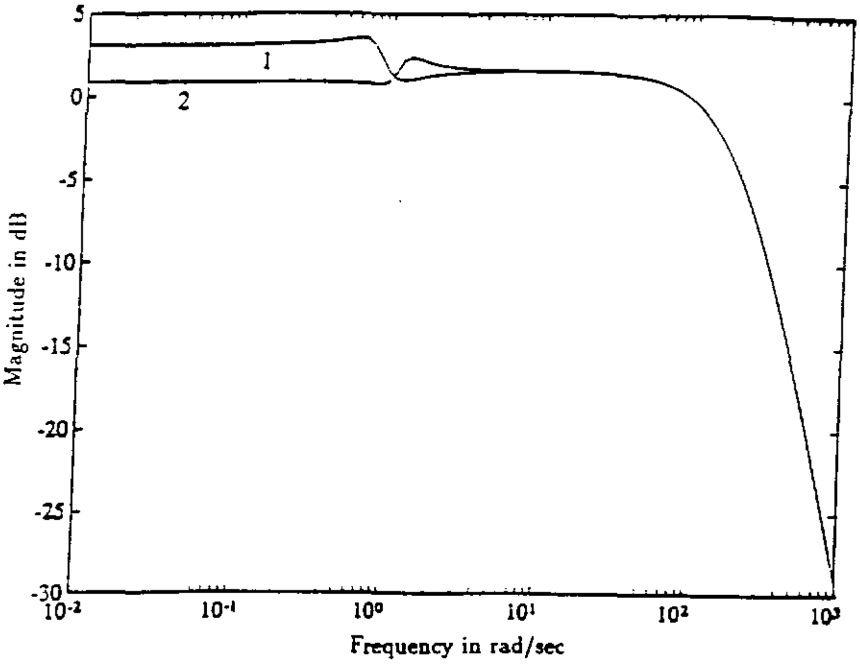

Figure 3.3 $\mathcal{H}_\infty$ Filter: spectrum $10log(\|G_{ew}(j\omega)\|^2 + \|G_{ev}(j\omega)\|^2)$
(curve 1: $\delta = 0.3$, curve 2: $\delta = -0.3$)

Based on the above observations, there is a need to consider the parameter uncertainty in the design procedure in order to obtain a more robust filter. This problem will be analysed in Sections 5-7.

4 $\mathcal{H}_\infty$ Filtering for a Class of Nonlinear Systems

In this section we consider the problem of $\mathcal{H}_\infty$ filtering for a class of nonlinear systems. The class of systems considered here is described by a linear state space model with the addition of known state-dependent nonlinearities satisfying global Lipschitz conditions which appear in both the state and output equations. The problem we address is the design of nonlinear filters such that the estimation error is globally asymptotically stable and the induced $\mathcal{L}_2$ norm of the operator mapping from the noise to the estimation error is within a prescribed bound.

4.1 Problem Formulation

Consider a nonlinear system of the form:

$$\dot{x}(t) = Ax(t) + Gg[x(t)] + Bw(t) \tag{4.1}$$

$$y(t) = Cx(t) + Hh[x(t)] + Dw(t) \tag{4.2}$$

$$z(t) = Lx(t) \tag{4.3}$$

where $x(t) \in \Re^n$ is the state, $w(t) \in \Re^r$ is a noise signal which is assumed to belong to $\mathcal{L}_2^r$, $y(t) \in \Re^m$ is the measurement, $z(t) \in \Re^q$ is a linear combination of state variables to be estimated, $g(\cdot) : \Re^n \rightarrow \Re^{n_g}$ and $h(\cdot) : \Re^n \rightarrow \Re^{n_h}$ are known nonlinear functions, and A, B, C, D and L are known real constant matrices of appropriate dimensions.

We shall make the following assumptions for the system (4.1)-(4.3):

Assumption 4.1

There exist known constant matrices W_g and W_h such that for any x_1 and $x_2 \in \Re^n$

$$\begin{aligned} \|g(x_1) - g(x_2)\| &\leq \|W_g(x_1 - x_2)\| \\ \|h(x_1) - h(x_2)\| &\leq \|W_h(x_1 - x_2)\|. \end{aligned}$$

Assumption 4.2

(a) (C, A) is detectable;

(b) $DD^T + HH^T > 0$.

Observe that Assumption 4.2 is a standard assumption in non-singular $\mathcal{H}_\infty$ filtering for the linear part of system (4.1)-(4.3). Assumption 4.2(b) can be viewed as a non-singularity assumption for the $\mathcal{H}_\infty$ filtering problem for system (4.1)-(4.3). We note that when there is no nonlinearity in the output equation (4.2), i.e. $H = 0$, Assumption 4.2(b) reduces to $DD^T > 0$, which is a standard assumption in the non-singular $\mathcal{H}_\infty$ filtering for the linear part of the system (4.1)-(4.3).

In this section we are concerned with designing a nonlinear filter $\mathcal{F}$ to provide an estimate $\hat{z}(t)$ of $z(t)$, based on $\{y(\tau);\ 0 \leq \tau \leq t\}$ and with a uniformly small estimation error $z(t) - \hat{z}(t)$ for all $w \in \mathcal{L}_2^r$. For the sake of simplification of the presentation, attention will be focused on the design of a stationary filter on infinite-horizon, and thus the initial state of (4.1) is assume to be zero. More specifically, the $\mathcal{H}_\infty$ filtering problem we shall address is as follows:

Given a prescribed level of noise attenuation $\gamma > 0$, find a causal filter $\mathcal{F}$ such that the filtering error is globally uniformly asymptotically stable and subject to zero initial conditions, $\|z - \hat{z}\|_2 < \gamma\|w\|_2$ for all non-zero $w \in \mathcal{L}_2^r$.

4.2 A $\mathcal{H}_\infty$ Nonlinear Filter

A nonlinear filter that solves the $\mathcal{H}_\infty$ filtering problem for the system (4.1)-(4.3) is provided below.

Theorem 4.1 *Consider the system (4.1)-(4.3) satisfying Assumptions 4.1 and 4.2. Given a precribed level of noise attenuation $\gamma > 0$, the $\mathcal{H}_\infty$ filtering problem for (4.1)-(4.3) is solvable if for some $\varepsilon > 0$ there exists a stabilizing solution $P = P^T \geq 0$ to the ARE*

$$\begin{aligned} &\left(A - BD^T\tilde{V}^{-1}C\right)P + P\left(A - BD^T\tilde{V}^{-1}C\right)^T + P\left(\gamma^{-2}L^TL + W_g^TW_g + W_h^TW_h - C^T\tilde{V}^{-1}C\right)P \\ &\quad + B\left(I - D^T\tilde{V}^{-1}D\right)B^T + GG^T = 0 \end{aligned} \tag{4.4}$$

where

$$\tilde{V} = DD^T + HH^T. \tag{4.5}$$

Moreover, a suitable nonlinear filter is given by:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Gg[\hat{x}(t)] + K\left[y(t) - C\hat{x}(t) - Hh[\hat{x}(t)]\right], \quad \hat{x}(0) = 0 \tag{4.6}$$

$$\hat{z}(t) = L\hat{x}(t) \tag{4.7}$$

where

$$K = \left(PC^T + BD^T\right)\tilde{V}^{-1}. \tag{4.8}$$

Proof. With the filter (4.6)-(4.7) and letting $\tilde{x} \triangleq x - \hat{x}$, we obtain the following state space equations for the estimation error:

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) + (\bar{G} - K\bar{H})\,\xi(x,\hat{x}) + (B - KD)w(t), \quad \tilde{x}(0) = 0 \tag{4.9}$$

$$z(t) - \hat{z}(t) = L\tilde{x}(t) \tag{4.10}$$

where

$$\xi(x,\hat{x}) = \begin{bmatrix} g(x) - g(\hat{x}) \\ h(x) - h(\hat{x}) \end{bmatrix},$$

$$\bar{G} = [G \quad 0], \quad \bar{H} = [0 \quad H].$$

Also, note that by Assumption 4.1

$$\| \xi(x,\hat{x}) \| \leq \| W\tilde{x} \|$$

where

$$W \triangleq \left[W_g^T \quad W_h^T\right]^T$$

Next, considering (4.8), it is easy to verify that the ARE (4.4) can be rewritten as

$$(A - KC)P + P(A - KC)^T + P\left(\gamma^{-2}L^TL + W^TW\right)P + (B - KD)(B - KD)^T$$
$$+ (\bar{G} - K\bar{H})(\bar{G} - K\bar{H})^T = 0. \tag{4.11}$$

Hence, in view of Corollary 2.1, (4.11) implies that the estimation error system (4.9)-(4.10) is globally uniformly asymptotically stable and $\|z - \hat{z}\|_2 < \|w\|_2$ for all non-zero $w \in \mathcal{L}_2^r$. ∇∇∇

We note that, when there is no nonlinear term in (4.1) and (4.2), the matrices G, H, W_g and W_h should be set to zero. Under these conditions, the filter of (4.6)-(4.7) recovers the linear stationary $\mathcal{H}_\infty$ filter discussed in Section 3.

5 Robust Minimum Variance Filtering

This section is concerned with the robust minimum variance filtering problem for linear systems subject to norm-bounded parameter uncertainty in both the state and output matrices. The problem addressed is the design of stationary linear filters which yield an optimized upper bound on the estimation error variance for all admissible uncertainties. This filtering methodology can be viewed as an extension of the well known steady state Kalman filter, in the sense that it provides a guaranteed performance irrespective of parameter uncertainties in the system model.

5.1 Problem Formulation

Throughout this section we consider linear uncertain systems described by a state space model of the form

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + Bw(t) \tag{5.1}$$

$$y(t) = [C + \Delta C(t)]x(t) + Dw(t) \tag{5.2}$$

$$z(t) = Lx(t) \tag{5.3}$$

where $x(t) \in \Re^n$ is the state, $y(t) \in \Re^m$ is the measurement, $w(t) \in \Re^r$ is a zero-mean white noise signal with an identity power spectrum density matrix, $z(t) \in \Re^q$ is a linear combination of state variables to be estimated, A, B, C and D are known constant matrices that describe the nominal system of (5.1)-(5.3), and $\Delta A(t)$ and $\Delta C(t)$ are unknown matrices representing time-varying parameter uncertainties. The admissible uncertainties are assumed to be of the form

$$\Delta A(t) = H_1 F(t) E, \qquad \Delta C(t) = H_2 F(t) E \tag{5.4}$$

where $F(t) \in \Re^{i \times j}$ is an unknown time-varying matrix with Lebesgue measurable elements satisfying

$$\|F(t)\| \leq 1, \ \ \forall\, t \geq 0 \tag{5.5}$$

and E, H_1 and H_2 are known real constant matrices of appropriate dimensions which specify how the elements of the nominal matrices A and C are affected by the uncertain parameters in $F(t)$.

We observe that the case where the input and measurement noise signals are uncorrelated zero-mean white signals, say $v_1(t)$ and $v_2(t)$, respectively, with identity power spectrum density matrices, is a particular case of (5.1)-(5.2) where $w(t) = [v_1^T(t) \quad v_2^T(t)]^T$ and the matrices B and D are replaced by $[B \quad 0]$ and $[0 \quad D]$, respectively.

It is assumed that the initial state of (5.1) is a zero-mean random variable, x_0, which is uncorrelated with $w(t)$ for all $t \geq 0$. We shall also adopt the following assumption for the system (5.1)-(5.2):

Assumption 5.1

(a) The system (5.1) is quadratically stable;

(b) $[D \quad H_2]$ *is of full row rank.*

It should be noted that due to the presence of time-varying parameter uncertainty, Assumption 5.1 (a) is required in order to guarantee the uniform asymptotic stability of the estimation error dynamics. The reason for this is because of the parameter uncertainty, the estimation error dynamics is driven by the state of the system of (5.1) and thus the quadratic stability of the latter system is required for the boundedness of the estimation error. Note that the Hurwitz stability of the nominal state matrix A is a necessary condition for Assumption 5.1 (a) to hold.

Assumption 5.1 (b) means that the robust $\mathcal{H}_\infty$ filtering problem is non-singular. Observe that if the parameter uncertainty in the output matrix disappears, i.e. $H_2 = 0$, Assumption 5.1 (b) reduces to $DD^T > 0$, which is a standard assumption in the Kalman filtering problem for the nominal system of (5.1)-(5.3).

An important consequence of Assumption 5.1 (a) is that the uniform asymptotic stability of the estimation error dynamics is ensured by the asymptotic stability of the filter. Note that his assumption also implies the detectability of the system (5.1)-(5.2).

Our aim is the design of a stationary robust linear estimator for z with a guaranteed performance in the sense of the mean squares error. More specifically, we are concerned with finding an *nth* order asymptotically stable estimator for z of the form

$$\dot{\hat{x}}(t) = A_e\hat{x}(t) + Ky(t), \qquad \hat{x}(0) = 0 \tag{5.6}$$

$$\hat{z}(t) = L_e\hat{x}(t) \tag{5.7}$$

where $\hat{z}$ is the estimate of z and A_e, K, and L_e are constant matrices to be determined in order to ensure that, asymptotically, the worst-case error variance

$$\sup_{\|F\|\le 1} E\left\{[(z-\hat{z}) - E(z-\hat{z})]^T[(z-\hat{z}) - E(z-\hat{z})]\right\}$$

will satisfy a known upper bound, where $E[\cdot]$ denotes the mean or expectation. Moreover, this upper bound is required to be as small as possible.

In view of the system (5.1)-(5.3) and estimator (5.6)-(5.7), the estimation error, $e(t) = z(t) - \hat{z}(t)$, can be described by the following state-space equations:

$$\dot{\xi}(t) = [A_c + H_cF(t)E_c]\xi(t) + B_cw(t), \qquad \xi(0) = \xi_0 \tag{5.8}$$

$$e(t) = L_c\xi(t) \tag{5.9}$$

where

$$\xi = \begin{bmatrix} \hat{x} \\ x - \hat{x} \end{bmatrix}, \quad \xi_0 = \begin{bmatrix} 0 \\ x_0 \end{bmatrix},$$

$$A_c = \begin{bmatrix} A_e + KC & KC \\ A - A_e - KC & A - KC \end{bmatrix}, \quad B_c = \begin{bmatrix} KD \\ B - KD \end{bmatrix},$$

$$H_c = \begin{bmatrix} KH_2 \\ H_1 - KH_2 \end{bmatrix}, \quad E_c = [E \quad E], \quad L_c = [L - L_e \quad L_e].$$

Note that since x_0 and $w(t)$ have zero means, it follows from (5.8)-(5.9) that the means of $\xi(t)$ and $e(t)$, denoted by $\bar{\xi}(t)$ and $\bar{e}(t)$, respectively, satisfy:

$$\dot{\bar{\xi}}(t) = [A_c + H_cF(t)E_c]\bar{\xi}(t), \qquad \bar{\xi}(0) = 0 \tag{5.10}$$

$$\bar{e}(t) = L_c\bar{\xi}(t). \tag{5.11}$$

This implies that $\bar{\xi}(t) = 0$ and $\bar{e}(t) = 0$ for all $t \ge 0$, i.e. the signals $\xi(t)$ and $e(t)$ have zero means for all $t \ge 0$.

In connection with the estimation error system of (5.8)-(5.9) we introduce the following Riccati equation

$$A_cP + PA_c^T + \varepsilon PE_c^TE_cP + B_cB_c^T + \frac{1}{\varepsilon}H_cH_c^T = 0 \tag{5.12}$$

where ε is a positive parameter to be chosen. Hence, we have the following result.

Lemma 5.1 *Consider the system (5.1)-(5.3) satisfying Assumption 5.1 and let a filter of the form (5.6)-(5.7) be given. Assume that, for some scalar $\varepsilon > 0$, there exists a stabilizing solution $P = P^T \geq 0$ to the ARE (5.12). Then, the given filter and the system (5.8) are asymptotically stable, and the asymptotic covariance matrix of ξ satisfies the bound*

$$E\left[\xi(t)\xi^T(t)\right] \leq P \tag{5.13}$$

for all admissible uncertainties.

Proof. First, we note that the matrix $A_c + H_cF(t)E_c$ is similar to

$$\tilde{A}_c(t) \triangleq \begin{bmatrix} A + \Delta A(t) & 0 \\ K[C + \Delta C(t)] & A_e \end{bmatrix}.$$

Indeed, it is easy to verify that $A_c + H_cF(t)E_c = T^{-1}\tilde{A}_c(t)T$ where

$$T = \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix}.$$

For a given filter of the form (5.6)-(5.7), assume that there exists a positive semi-definite stabilizing solution P to (5.12) for some $\varepsilon > 0$. By Lemma 2.3, it follows that there exists a stabilizing solution $Q = Q^T \geq 0$ to the Riccati equation

$$A_cQ + QA_c^T + \varepsilon QE_c^TE_cQ + \frac{1}{\varepsilon}H_cH_c^T = 0.$$

In view of Theorem 2.3, this implies that the system

$$\dot{\eta}(t) = [A_c + H_cF(t)E_c]\eta(t), \quad \eta(0) = \eta_0$$

is quadratically stable for any time-varying matrix $F(t)$ of appropriate dimensions satisfying the bound $\| F(t) \| \leq 1$ for any $t \geq 0$. Thus, we conclude that the system (5.8) is asymptotically stable. Furthermore, since $A_c + H_cF(t)E_c$ is similar to $\tilde{A}_c(t)$, we also have that the matrix A_e is Hurwitz stable, i.e. the given filter is asymptotically stable.

Now, as $\xi(t)$ has zero mean for any $t \geq 0$, we readily find from (5.8) and (5.10) that the covariance matrix of $\xi(t)$, namely $P_\xi(t) \triangleq E[\xi(t)\xi^T(t)]$, satisfies

$$\begin{aligned} \dot{P}_\xi(t) &= [A_c + H_cF(t)E_c]P_\xi(t) + P_\xi(t)[A_c + H_c(t)F(t)E_c]^T + B_cB_c^T, \quad \forall\, t \geq 0; \\ &\qquad P_\xi(0) = diag\,\{0, X_0\} \end{aligned} \tag{5.14}$$

where X_0 is the covariance matrix of x_0. Observe that the exponential stability of the system (5.8) guarantees the boundedness of the asymptotic covariance matrix of $\xi(t)$.

Next, using Lemma 2.5, it follows from (5.12) that

$$[A_c + H_c F(t) E_c] P + P[A_c + H_c(t) F(t) E_c]^T + B_c B_c^T \leq 0. \tag{5.15}$$

Hence, subtracting (5.14) from (5.15) it follows that $\tilde{P}(t) \triangleq P - P_\xi(t)$ satisfies the differential inequality

$$\dot{\tilde{P}}(t) \geq [A_c + H_c F(t) E_c] \tilde{P}(t) + \tilde{P}(t)[A_c + H_c(t) F(t) E_c]^T. \tag{5.16}$$

Finally, since the system (5.8) is asymptotically stable, we obtain from (5.16) that. asymptotically, $\tilde{P}(t) \geq 0$, i.e. $P_\xi(t) \leq P$ as $t \to \infty$ and for all admissible uncertainties.

$\nabla\nabla\nabla$

In view of Lemma 5.1, it follows that any filter of the form (5.6)-(5.7) for which the ARE (5.12) has a positive semi-definite stabilizing solution P for some $\varepsilon > 0$, will guarantee that the asymptotic estimation error variance satisfies the bound

$$E\left[e^T(t) e(t)\right] \leq tr\left[L_c P L_c^T\right] \tag{5.17}$$

for all admissible uncertainties, where $tr(\cdot)$ stands for the matrix trace.

In the sequel we will derive a filter which minimizes the bound on the asymptotic error variance in (5.17).

5.2 The Optimal Robust Filter

We begin by introducing the following Riccati equations:

$$AS + SA^T + \varepsilon S E^T E S + BB^T + \frac{1}{\varepsilon} H_1 H_1^T = 0 \tag{5.18}$$

and

$$\left(A - \hat{B}\hat{D}^T \hat{V}^{-1} C\right) Y + Y \left(A - \hat{B}\hat{D}^T \hat{V}^{-1} C\right)^T + Y\left(\varepsilon E^T E - C^T \hat{V}^{-1} C\right) Y$$
$$+ \hat{B}\left(I - \hat{D}^T \hat{V}^{-1} \hat{D}\right) \hat{B}^T = 0 \tag{5.19}$$

where

$$\hat{B} = \begin{bmatrix} B & \frac{1}{\sqrt{\varepsilon}} H_1 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D & \frac{1}{\sqrt{\varepsilon}} H_2 \end{bmatrix}, \quad \hat{V} = \hat{D}\hat{D}^T \tag{5.20}$$

and ε is a positive scalar to be chosen.

At this point we wish to note that the Riccati equations (5.12), (5.18) and (5.19) are closely related. Indeed, it happens that the existence of a positive semi-definite stabilizing solution to (5.12) guarantees the existence of a positive semi-definite stabilizing solution to (5.18) and (5.19) as shown below.

Lemma 5.2

(i) If for a given filter of (5.6)-(5.7) and for some scalar $\varepsilon > 0$ the ARE (5.12) has a stabilizing solution $P = P^T \geq 0$, then there exists a stabilizing solution $S = S^T \geq 0$ to the ARE (5.18) for the same ε;

(ii) If for some scalar $\varepsilon > 0$ the ARE (5.18) has a stabilizing solution $S = S^T \geq 0$, then there exists a stabilizing solution $Y = Y^T \geq 0$ to the ARE (5.19) for the same ε. Moreover, $Y \leq S$.

Proof. (i) Let $P = P^T \geq 0$ be the stabilizing solution to (5.12) for a given filter (5.6)-(5.7) and for some $\varepsilon > 0$. First, introduce the non-singular transformation matrix

$$M \triangleq \begin{bmatrix} I_n & I_n \\ I_n & 0 \end{bmatrix}.$$

and denote $\hat{P} \triangleq MPM^T$. Premultiplying and postmultiplying (5.12) by M and M^T, respectively, it follows that $\hat{P}$ satisfies the Riccati equation

$$\hat{A}_c\hat{P} + \hat{P}\hat{A}_c^T + \varepsilon\hat{P}\hat{E}_c^T\hat{E}_c\hat{P} + \frac{1}{\varepsilon}\hat{H}_c\hat{H}_c^T + \hat{B}_c\hat{B}_c^T = 0 \tag{5.21}$$

where

$$\hat{A}_c = MA_cM^{-1}, \quad \hat{B}_c = MB_c, \quad \hat{E}_c = E_cM^{-1}, \quad \hat{H}_c = MH_c.$$

Also, we observe that

$$\hat{A}_c + \varepsilon\hat{P}\hat{E}_c^T\hat{E}_c = M\left(A_c + \varepsilon PE_c^TE_c\right)M^{-1}$$

and then the matrix $\hat{A}_c + \varepsilon\hat{P}\hat{E}_c^T\hat{E}_c$ is similar to $A_c + \varepsilon PE_c^TE_c$. Since P is the stabilizing solution to (5.12), this implies that $\hat{P}$ is the stabilizing solution to (5.21).

Now, partitioning $\hat{P}$ conform with M as below:

$$\hat{P} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^T & \hat{P}_{22} \end{bmatrix}$$

it is easy to verify from (5.21) that $\hat{P}_{11}$ satisfies the ARE (5.18). It remains to be shown that $\hat{P}_{11}$ is the stabilizing solution to (5.18), i.e. the matrix $A + \varepsilon\hat{P}_{11}E^TE$ is Hurwitz stable. To this end, first we note that the matrix $\hat{A}_c + \varepsilon\hat{P}\hat{E}_c^T\hat{E}_c$ is of the form

$$\hat{A}_c + \varepsilon\hat{P}\hat{E}_c^T\hat{E}_c = \begin{bmatrix} A + \varepsilon\hat{P}_{11}E^TE & 0 \\ * & * \end{bmatrix}$$

where '*' denotes entries which are irrelevant. Since $\hat{P}$ is the stabilizing solution of (5.21), the above matrix is Hurwitz stable, which implies $A + \varepsilon\hat{P}_{11}E^TE$ is Hurwitz stable as well.

(ii) First, we observe that by Theorem 2.1, the existence of a stabilizing solution $S = S^T \geq 0$ to (5.18) for some $\varepsilon > 0$ guarantees that

$$\| \sqrt{\varepsilon}E(sI - A)^{-1}\hat{B} \|_\infty < 1. \tag{5.22}$$

Next, in connection with the ARE (5.19) we define the system

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + \hat{B}\tilde{w} \\ \tilde{y} &= C\tilde{x} + \hat{D}\tilde{w} \\ \tilde{z} &= \sqrt{\varepsilon}E\tilde{x} \end{aligned}$$

where $\tilde{w}$ is a noise signal belonging to $\mathcal{L}_2[0, \infty)$, $\tilde{y}$ is the measurement, $\tilde{z}$ is a linear combination of the state variables to be estimated, and ε is the same as in (5.18). Using Theorem 3.2, we have that the existence of a stabilizing solution $Y = Y^T \geq 0$ to (5.19) is a necessary and sufficient condition for the existence of a linear causal time-invariant filter to estimate $\tilde{z}$, based on $\tilde{y}$, such that $\| G_{\tilde{e}\tilde{w}}(s) \|_\infty < 1$, where $G_{\tilde{e}\tilde{w}}(s)$ is the transfer matrix from the noise signal, $\tilde{w}$, to the estimation error, $\tilde{e} = \tilde{z} - \tilde{z}_e$, with $\tilde{z}_e$ being the estimate of $\tilde{z}$.

Now, considering that the transfer matrix from $\tilde{w}$ to $\tilde{z}$, denoted by $G_{\tilde{z}\tilde{w}}(s)$, is given by

$$G_{\tilde{z}\tilde{w}}(s) = \sqrt{\varepsilon} E(sI - A)^{-1}\hat{B}$$

it follows from (5.22) that with the estimate $\tilde{z}_e(t) \equiv 0$, $\| G_{\tilde{e}\tilde{w}}(s) \|_\infty = \| G_{\tilde{z}\tilde{w}}(s) \|_\infty < 1$. This implies the existence of a stabilizing solution $Y = Y^T \geq 0$ to (5.19) for the same ε as in (5.18).

In order to show that $Y \leq S$, we first note that the ARE (5.19) can be rewritten as

$$AY + YA^T + \varepsilon YE^TEY + BB^T + \frac{1}{\varepsilon}H_1H_1^T - \left(YC^T + \hat{B}\hat{D}^T\right)\hat{V}^{-1}\left(YC^T + \hat{B}\hat{D}^T\right)^T = 0.$$

Hence, comparing the above equation with (5.18) it follows from Lemma 2.3 that $Y \leq S$. This completes the proof. ∇∇∇

We now present the optimal robust filter, in the sense of minimizing the upper bound on the asymptotic error variance in (5.17). For details of the proof see [34].

Theorem 5.1 *Consider the system (5.1)-(5.3) satisfying Assumption 5.1. Then there exists an asymptotically stable filter of the form (5.6)-(5.7) that minimizes the bound on the asymptotic error variance in (5.17) if and only if for some $\varepsilon > 0$ the ARE (5.18) has a stabilizing solution $S = S^T \geq 0$. Under this condition, the optimal filter is given by*

$$\dot{\hat{x}}(t) = (A + \varepsilon YE^TE)\hat{x}(t) + K[y(t) - C\hat{x}(t)] \quad (5.23)$$

$$\hat{z}(t) = L\hat{x}(t) \quad (5.24)$$

where

$$K = \left(YC^T + BD^T + \frac{1}{\varepsilon}H_1H_2^T\right)\hat{V}^{-1} \quad (5.25)$$

and $Y = Y^T \geq 0$ is the stabilizing solution of the ARE (5.19). Moreover, this filter guarantees that asymptotically

$$\sup_{\|F\|\leq 1} E\left[e^T(t)e(t)\right] \leq tr(LYL^T). \quad (5.26)$$

∇∇∇

We note that when there is no parameter uncertainty in the system (5.1)-(5.2), we have that $H_1 = 0$, $H_2 = 0$, and $E = 0$, and in this situation the robust filter of Theorem 5.1 recovers the stationary Kalman filter for the nominal system of (5.1)-(5.3).

In view of Theorems 2.1 and 2.3, it can be easily established that the quadratic stability of the system (5.1) implies the existence of a positive semi-definite stabilizing solution to (5.18) for some $\varepsilon > 0$. Hence, as long as Assumption 5.1 holds, the filter of Theorem 5.1 is guaranteed to exist.

In order to calculate the robust filter of (5.23)-(5.24) we need to search for a scalar $\varepsilon > 0$ for which the Riccati equation (5.18) has a positive semi-definite stabilizing solution. Note that by Theorem 2.1 and Lemma 2.2, this is equivalent of searching for an $\varepsilon > 0$ such that the matrix

$$M_\varepsilon = \begin{bmatrix} A & BB^T + \varepsilon^{-1} H_1 H_1^T \\ -\varepsilon E^T E & -A^T \end{bmatrix}$$

has no purely imaginary eigenvalues. Once such ε is found, we then calculate the stabilizing solution $Y = Y^T \geq 0$ to (5.19) for this ε and then the robust filter is readily obtained from (5.23)-(5.25). Note that the existence of the solution Y is guaranteed by Lemma 5.2(ii).

The search for a suitable $\varepsilon > 0$ can be carried out as follows. Pick a first guess for ε and computes the eigenvalues of M_ε. If none of these eigenvalues are purely imaginary, we have found a suitable value ε; otherwise decrease ε and repeat the procedure. The latter search procedure follows from the fact that if the ARE (5.18) has a positive semi-definite stabilizing solution for some $\varepsilon = \bar{\varepsilon} > 0$ then, by Lemma 2.3, (5.18) also has a positive semi-definite stabilizing solution for any $\varepsilon \in (0, \bar{\varepsilon}]$.

Remark 5.1 Although the stabilizing solution of the ARE (5.18) plays no role in the calculation of the filter of (5.23)-(5.24), in order for this filter to provide a bound on the error variance it does not suffice to find a positive semi-definite stabilizing solution to (5.19) for a suitable $\varepsilon > 0$. Observe that it is also required to verify if for this ε the ARE (5.18) has a positive semi-definite stabilizing solution. It may happen that there exist values of $\varepsilon > 0$ for which (5.19) has a positive semi-definite stabilizing solution Y but not (5.18). For such values of ε, the resulting filter cannot guarantee that $tr(LYL^T)$ is a bound on the error variance for all admissible uncertainties. This situation is illustrated via an example in the next sub-section. □

Remark 5.2 The filter of Theorem 5.1 minimizes the bound on the asymptotic error variance in (5.17) for a fixed $\varepsilon > 0$. However, since different values of ε give rise to different values for the optimal error variance bound, $tr[LY(\varepsilon)L^T]$, we can still minimize this bound with respect to the parameter ε.

We know from Lemma 2.3 that if the ARE (5.18) has a positive semi-definite stabilizing solution for some $\varepsilon = \bar{\varepsilon} > 0$, then (5.18) also has a positive semi-definite stabilizing solution for any $\varepsilon \in (0, \bar{\varepsilon}]$. This implies that if the robust filter of Theorem 5.1 can be found for a given $\varepsilon > 0$, then there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$ the robust filter is guaranteed to exists. Observe that ε^* is the largest ε such that the ARE (5.18) admits a stabilizing solution $S = S^T \geq 0$. This allows us to carry out the minimization of the upper bound on the estimation error variance with respect to ε, namely

$$\min_{\varepsilon \in (0, \varepsilon^*]} \left\{ tr\left[LY(\varepsilon)L^T\right] : Y(\varepsilon) = Y^T(\varepsilon) \geq 0 \textit{ is the stabilizing solution of } (5.19) \right\}.$$

In the case where the stabilizing solution, Y, of (5.19) is positive definite for any ε in the interval $(0, \varepsilon^*]$, by Theorem 5 of [30] it follows that $tr[LY(\varepsilon)L^T]$ is a convex function of ε on $(0, \varepsilon^*]$. Thus, in this situation any local minimum will also be a global minimum of $tr[LY(\varepsilon)L^T]$

and efficient numerical convex optimization methods can be used to perform the above minimization problem. We observe, that by a standard result in algebraic Riccati equations (see, e.g. Theorem 3.2 of [43]), the stabilizing solution of (5.19) for any ε in the interval $(0, \varepsilon^*]$ is guaranteed to be positive definite if and only if the pair $(A, [B \;\; H_1])$ is controllable. □

6 Robust $\mathcal{H}_\infty$ Filtering for Uncertain Linear Systems

In Section 3 we have studied the problem of $\mathcal{H}_\infty$ filtering for linear systems where the only modelling uncertainty is in the form of a bounded energy noise signal. In practice, very often the model used to describe the signal generating mechanism is inexact. For example, the model may be obtained by linearizing a nonlinear system around its operating points. Also, there may exist unknown parameter and/or parameter variations. In such situations it is highly desirable to have filter design techniques which can take into account all possible uncertainties.

In this section we consider the problem of $\mathcal{H}_\infty$ filtering for linear systems subject to parameter uncertainty. The parameter uncertainty allowed is time-varying norm-bounded and appears in both the state and output matrices. The problem addressed is the design of an asymptotically stable linear filter such that the induced $\mathcal{L}_2$ norm of the operator mapping from the noise to the filtering error is kept within a prescribed bound for all admissible parameter uncertainties. The above problem is referred to as *robust $\mathcal{H}_\infty$ filtering*.

6.1 Problem Formulation

We consider uncertain linear systems of the same form as in Section 5, namely:

$$\dot{x}(t) = [A + \Delta A(t)]\, x(t) + Bw(t) \tag{6.1}$$

$$y(t) = [C + \Delta C(t)]\, x(t) + Dw(t) \tag{6.2}$$

$$z(t) = Lx(t) \tag{6.3}$$

where $x(t) \in \Re^n$ is the state, $y(t) \in \Re^m$ is the measurement, $w(t) \in \Re^r$ is a noise signal which is assumed to belong to $\mathcal{L}_2^r$, $z(t) \in \Re^q$ is a linear combination of state variables to be estimated, A, B, C, D and L are known real constant matrices that describe the nominal system of (6.1)-(6.3), and $\Delta A(t)$ and $\Delta C(t)$ are unknown matrices representing time-varying parameter uncertainties. The admissible uncertainties are assumed to be of the same form as in (5.4)-(5.5), i.e.

$$\Delta A(t) = H_1 F(t) E, \quad \Delta C(t) = H_2 F(t) E \tag{6.4}$$

where $F(t) \in \Re^{i \times j}$ is an unknown time-varying matrix with Lebesgue measurable elements satisfying

$$\|F(t)\| \leq 1, \;\; \forall\, t \geq 0 \tag{6.5}$$

and E, H_1 and H_2 are known real constant matrices of appropriate dimensions.

Similarly to the problem of robust minimum variance filtering in Section 5, we shall assume that Assumption 5.1 holds.

Our objective in this section is the design of linear filters with a prescribed $\mathcal{H}_\infty$ performance irrespective of the parameter uncertainty. For the sake of simplicity of the presentation, attention will be focused on the design of stationary filters, and thus we shall assume that the initial state of (6.1) is zero. We are concerned with obtaining an estimate $\hat{z}(t)$ of $z(t)$ via an asymptotically stable linear filter $\mathcal{F}$:

$$\dot{\hat{x}}(t) = A_e\hat{x}(t) + Ky(t), \quad \hat{x}(0) = 0 \tag{6.6}$$

$$\hat{z}(t) = L_e\hat{x}(t) \tag{6.7}$$

where A_e, K, and L_e are constant matrices to be determined such that the estimation error, $z - \hat{z}$, is uniformly small for all $w \in \mathcal{L}_2$ and for all $F(t)$ satisfying (6.5). To be more precise, the problem of robust $\mathcal{H}_\infty$ filtering is as follows:

Given a prescribed level of noise attenuation $\gamma > 0$, find an asymptotically stable linear filter $\mathcal{F}$ such that under zero initial conditions, $\|z - \hat{z}\|_2 < \gamma\|w\|_2$ for all non-zero $w \in \mathcal{L}_2^-$ and for all admissible uncertainties.

Note that when there is no parameter uncertainty in the system (6.1)-(6.5), we have that $E = 0$, $H_1 = 0$ and $H_2 = 0$, and the above filtering problem becomes the standard $\mathcal{H}_\infty$ filtering problem for the nominal system of (6.1)-(6.3) which has been analysed in Section 3.

6.2 Robust $\mathcal{H}_\infty$ Filter

Similarly to the robust minimum variance filtering problem of Section 5, the state space equations of the estimation error, $e = z - \hat{z}$, in terms of the state variables of (6.1) and (6.6) are as follows:

$$\dot{\xi}(t) = [A_c + H_cF(t)E_c]\xi(t) + B_cw(t), \quad \xi(0) = 0 \tag{6.8}$$

$$e(t) = L_c\xi(t) \tag{6.9}$$

where

$$\xi = \begin{bmatrix} \hat{x}^T & (x - \hat{x})^T \end{bmatrix}^T,$$

$$A_c = \begin{bmatrix} A_e + KC & KC \\ A - A_e - KC & A - KC \end{bmatrix}, \quad B_c = \begin{bmatrix} KD \\ B - KD \end{bmatrix},$$

$$H_c = \begin{bmatrix} KH_2 \\ H_1 - KH_2 \end{bmatrix}, \quad E_c = [E \quad E], \quad L_c = [L - L_e \quad L_e].$$

In connection with the estimation error of (6.8)-(6.9) we introduce the following Riccati equation

$$A_cP + PA_c^T + P\left(\gamma^{-2}L_c^TL_c + \varepsilon E_c^TE_c\right)P + B_cB_c^T + \frac{1}{\varepsilon}H_cH_c^T = 0 \tag{6.10}$$

where ε is a positive parameter to be chosen. Hence, we have the following result.

Lemma 6.1 *Consider the system (6.1)-(6.3) satisfying Assumption 5.1 and let $\gamma > 0$ be a given scalar. Assume that for a given filter of the form (6.6)-(6.7) and for some scalar $\varepsilon > 0$ the ARE (6.10) admits a stabilizing solution $P = P^T \geq 0$. Then the following results hold:*

(i) The given filter is asymptotically stable;

(ii) The error system (6.8)-(6.9) is asymptotically stable and satisfies

$$\|z - \hat{z}\|_2 < \gamma\|w\|_2 \tag{6.11}$$

for all non-zero $w \in \mathcal{L}_2^r$ and for all admissible uncertainties.

Proof. The proof of the asymptotic stability of the filter and error system (6.8)-(6.9) parallels that of a similar result in Lemma 5.1.

In order to establish (6.11), we first note that by Theorem 2.1 the existence of a stabilizing solution $P = P^T \geq 0$ to (6.10) implies that there exists a matrix $Q = Q^T > 0$ satisfying the inequality

$$A_cQ + QA_c^T + Q\left(\gamma^{-2}L_c^TL_c + \varepsilon E_c^TE_c\right)Q + B_cB_c^T + \frac{1}{\varepsilon}H_cH_c^T < 0$$

which by Lemma 2.5 leads to

$$[A_c + H_cF(t)E_c]Q + Q[A_c + H_cF(t)E_c]^T + \gamma^{-2}QL_c^TL_cQ + B_cB_c^T < 0. \tag{6.12}$$

Premultiplying and postmultiplying (6.12) by γQ^{-1}, implies that $Z \triangleq \gamma^2Q^{-1}$ satisfies:

$$[A_c + H_cF(t)E_c]^TZ + Z[A_c + H_cF(t)E_c] + \gamma^{-2}ZB_cB_c^TZ + L_c^TL_c < 0. \tag{6.13}$$

Next introduce

$$J \triangleq \int_0^\infty \left[e^T(t)e(t) - \gamma^2w^T(t)w(t)\right]dt.$$

Since the system (6.8) is asymptotically stable and $\xi(0) = 0$, by completing the squares using (6.8)-(6.9) it can be easily shown that

$$\begin{aligned} J &= \int_0^\infty \left[e^Te - \gamma^2w^Tw + \frac{d}{dt}(\xi^TZ\xi)\right]dt \\ &= \int_0^\infty \xi^T\left\{[A_c + H_cF(t)E_c]^T Z + Z\,[A_c + H_cF(t)E_c] + \gamma^{-2}ZB_cB_c^TZ + L_c^TL_c\right\}\xi dt \\ &\quad -\gamma^2\int_0^\infty \left([w - \gamma^{-2}B_c^TZ\xi\right)^T\left(w - \gamma^{-2}B_c^TZ\xi\right)dt. \end{aligned}$$

Finally, by considering (6.13), the above implies that J is negative whenever w is non-zero. Thus, we conclude that $\|e\|_2 < \gamma\|w\|_2$ for all non-zero $w \in \mathcal{L}_2^r$ and for all admissible uncertainties. ∇∇∇

In view of Lemma 6.1, it follows that any filter of the form (6.6)-(6.7) for which the ARE (6.10) has a positive semi-definite stabilizing solution for some $\varepsilon > 0$ will solve the robust $\mathcal{H}_\infty$ filtering problem. A filter that satisfies the conditions of Lemma 6.1, and thus solves the problem of robust $\mathcal{H}_\infty$ filtering, is presented in the following.

Theorem 6.1 *Consider the system (6.1)-(6.3) satisfying Assumption 5.1. Given a prescribed level of noise attenuation $\gamma > 0$, the robust $\mathcal{H}_\infty$ filtering problem is solvable if for some $\varepsilon > 0$ the following conditions are satisfied:*

(a) There exists a stabilizing solution $Y = Y^T \geq 0$ to the ARE

$$\left(A - \hat{B}\hat{D}^T\hat{V}^{-1}C\right)Y + Y\left(A - \hat{B}\hat{D}^T\hat{V}^{-1}C\right)^T + Y\left(\gamma^{-2}L^TL + \varepsilon E^TE - C^T\hat{V}^{-1}C\right)Y + \hat{B}\left(I - \hat{D}^T\hat{V}^{-1}\hat{D}\right)\hat{B}^T = 0 \tag{6.14}$$

where

$$\hat{B} = \begin{bmatrix} B & \frac{1}{\sqrt{\varepsilon}}H_1 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D & \frac{1}{\sqrt{\varepsilon}}H_2 \end{bmatrix}, \quad \hat{V} = \hat{D}\hat{D}^T. \tag{6.15}$$

(b) There exists a stabilizing solution $X = X^T \geq 0$ to the ARE

$$AX + XA^T + \varepsilon XE^TEX + \gamma^{-2}YL^TLY + BB^T + \frac{1}{\varepsilon}H_1H_1^T = 0 \tag{6.16}$$

where $Y = Y^T \geq 0$ is the stabilizing solution of the ARE (6.14).

When conditions (a) and (b) are satisfied, a suitable robust $\mathcal{H}_\infty$ filter is given by:

$$\dot{\hat{x}}(t) = (A + \varepsilon YE^TE)\hat{x}(t) + K\left[y(t) - C\hat{x}(t)\right], \quad \hat{x}(0) = 0 \tag{6.17}$$

$$\hat{z}(t) = L\hat{x}(t) \tag{6.18}$$

where

$$K = \left(YC^T + BD^T + \frac{1}{\varepsilon}H_1H_2^T\right)\hat{V}^{-1} \tag{6.19}$$

and $Y = Y^T \geq 0$ is the stabilizing solution of the ARE (6.14).

Proof. First, using Lemma 2.3 it can be easily verified that $X \geq Y$. Next, with the filter of (6.16)-(6.17) and considering the AREs (6.14) and (6.16), it can be shown using straightforward but tedious manipulations that the matrix

$$P \triangleq \begin{bmatrix} X - Y & 0 \\ 0 & Y \end{bmatrix} \geq 0$$

is the stabilizing solution of (6.10). Hence, the desired result follows from Lemma 6.1.

∇∇∇

We observe that when there is no parameter uncertainty in the system (6.1)-(6.2), i.e. $H_1 = 0$, $H_2 = 0$ and $E = 0$, the robust filter of Theorem 6.1 recovers the infinite-horizon $\mathcal{H}_\infty$ filter for the nominal system of (6.1)-(6.3) as presented in Theorem 3.2.

Remark 6.1 Note the similarity of the above robust $\mathcal{H}_\infty$ filtering result with the robust minimum variance filtering of Section 5. Indeed, as $\gamma \to \infty$ the result of Theorem 6.1 recovers that of Theorem 5.1.

Similarly to the robust minimum variance filtering result of Theorem 5.1, the stabilizing solution of the ARE (6.16) plays no role in the calculation of the filter of (6.17)-(6.18). In fact this filter is in terms of the stabilizing solution of the ARE (6.14) only. However, in order for the filter (6.17)-(6.18) to guarantee a level of noise attenuation γ for all admissible uncertainties, it does not suffice to find a positive semi-definite stabilizing solution to (6.14) for a suitable $\varepsilon > 0$. It is also required to verify if for this ε the ARE (6.16) has a positive semi-definite stabilizing solution as well. It may happen that there exist values of $\varepsilon > 0$ for which (6.14) has a positive semi-definite stabilizing solution but not (6.16). For such values of ε, the resulting filter is not guaranteed to provide the desired $\mathcal{H}_\infty$ performance for all admissible uncertainties. □

7 Robust $\mathcal{H}_\infty$ Filtering for Uncertain Nonlinear Systems

This section is devoted to the robust version of the $\mathcal{H}_\infty$ nonlinear filtering problem treated in Section 4. The class of nonlinear systems we will considered is described by a linear state space model subject to time-varying norm-bounded parameter uncertainty in both the state and output matrices and with the addition of known state-dependent nonlinearities. As in Section 6, the nonlinearities are Lipschitzian and are allowed to appear in both the state and measurement equations. We will study the design of nonlinear filters such that the estimation error is globally asymptotically stable and the $\mathcal{L}_2$ gain from the noise to the estimation error is within a prescribed bound for the whole set of admissible systems.

7.1 Problem Formulation

Consider uncertain nonlinear systems of the form:

$$\dot{x}(t) = [A + \Delta A(t)]\,x(t) + Gg\,[x(t)] + Bw(t) \tag{7.1}$$

$$y(t) = [C + \Delta C(t)]\,x(t) + Hh\,[x(t)] + Dw(t) \tag{7.2}$$

$$z(t) = Lx(t) \tag{7.3}$$

where $x(t) \in \Re^n$ is the state, $w(t) \in \Re^r$ is a noise signal which is assumed to belong to $\mathcal{L}_2^r$, $y(t) \in \Re^m$ is the measurement, $z(t) \in \Re^q$ is a linear combination of state variables to be estimated, $g(\cdot) : \Re^n \rightarrow \Re^{n_g}$ and $h(\cdot) : \Re^n \rightarrow \Re^{n_h}$ are known nonlinear functions, and A, B, C, D and L are known real constant matrices of appropriate dimensions that together with $g(\cdot)$ and $h(\cdot)$ describe the nominal system of (7.1)-(7.3). The matrices $\Delta A(t)$ and $\Delta C(t)$ represent time-varying norm-bounded parameter uncertainties in A and C, respectively. The admissible uncertainties are assumed to be of the same form as in (5.4)-(5.5), i.e.

$$\Delta A(t) = H_1 F(t) E, \quad \Delta C(t) = H_2 F(t) E \tag{7.4}$$

where $F(t) \in \Re^{i \times j}$ is an unknown time-varying matrix with Lebesgue measurable elements satisfying

$$\|F(t)\| \leq 1, \;\; \forall\, t \geq 0 \tag{7.5}$$

and E, H_1 and H_2 are known real constant matrices of appropriate dimensions.

Similarly to the problem of $\mathcal{H}_\infty$ filtering for the nominal system of (7.1)-(7.3) analysed in Section 4, it is assumed that the nonlinear functions $g(\cdot)$ and $h(\cdot)$ satisfy Assumption 4.1. We shall also adopt the following assumption:

Assumption 7.1

(a) (C, A) *is detectable;*

(b) $DD^T + H_1H_1^T + HH^T > 0.$

We note that Assumption 7.1 is standard in non-singular $\mathcal{H}_\infty$ filtering for the nominal linear part of the system (7.1)-(7.3). Similarly to Assumption 4.2 (b), Assumption 7.1 (b) can be viewed as a non-singularity condition for the $\mathcal{H}_\infty$ filtering problem for system (7.1)-(7.3). Observe that if the parameter uncertainty and the nonlinearity in the output equation (7.2) disappear, it turn out that $H_2 = 0$ and $H = 0$, and Assumption 4.2 (b) reduces to $DD^T > 0$, which is a standard assumption in $\mathcal{H}_\infty$ filtering for linear systems without parameter uncertainty.

It should be noted that nonlinear models of the form (7.1)-(7.2) can be used to represent many important physical systems. A typical example is a power system modelled in the form of a single machine-infinite bus. The parameter uncertainty in the linear terms can be regarded as the variation of the operating point of the nonlinear system.

In this section we will analyse the design of a nonlinear filter $\mathcal{F}$ for estimating $z(t)$ with a prescribed $\mathcal{H}_\infty$ performance for the whole set of admissible systems, using the measurements $\mathcal{Y}_t = \{y(\tau);\ 0 \le \tau \le t\}$. As in the previous sections, we will consider the design of a stationary filter with a guaranteed noise attenuation on infinite-horizon. Letting $z_e(t) = \mathcal{F}\{\mathcal{Y}_t\}$ denote the estimate of $z(t)$, the robust $\mathcal{H}_\infty$ filtering problem we shall address is as follows:

Given a prescribed level of noise attenuation $\gamma > 0$, find a causal filter $\mathcal{F}$ such that the filtering error is globally uniformly asymptotically stable, and subject to zero initial conditions, $\|z - z_e\|_2 < \gamma\|w\|_2$ for all non-zero $w \in \mathcal{L}_2^r$ and for all admissible uncertainties.

7.2 A Robust $\mathcal{H}_\infty$ Filter

In the following we present a methodology for designing a nonlinear filter that solves the robust $\mathcal{H}_\infty$ filtering problem for the system (7.1)-(7.3).

Theorem 7.1 *Consider the system (7.1)-(7.3) satisfying Assumptions 4.1 and 7.1. Given a prescribed level of noise attenuation $\gamma > 0$, the robust $\mathcal{H}_\infty$ filtering problem for (7.1)-(7.3) is solvable if for some $\varepsilon > 0$ the following conditions are satisfied:*

(a) There exists a stabilizing solution $Y = Y^T \ge 0$ to the ARE

$$\left(A - \hat{B}\hat{D}^T\bar{V}^{-1}C\right)Y + Y\left(A - \hat{B}\hat{D}^T\bar{V}^{-1}C\right)^T + Y\left(\gamma^{-2}L^TL + \varepsilon E^TE + W_g^TW_g + W_h^TW_h - C^T\bar{V}^{-1}C\right)Y + \hat{B}\left(I - \hat{D}^T\bar{V}^{-1}\hat{D}\right)\hat{B}^T + GG^T = 0 \qquad (7.6)$$

where

$$\hat{B} = \begin{bmatrix} B & \frac{1}{\sqrt{\varepsilon}} H_1 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D & \frac{1}{\sqrt{\varepsilon}} H_2 \end{bmatrix}, \tag{7.7}$$

$$\bar{V} = DD^T + \frac{1}{\varepsilon} H_2 H_2^T + HH^T. \tag{7.8}$$

(b) There exists a stabilizing solution $X = X^T \geq 0$ to the ARE

$$\left(A - YW_g^T W_g\right) X + X\left(A - YW_g^T W_g\right)^T + X\left(\varepsilon E^T E + W_g^T W_g\right) X + Y\left(\gamma^{-2} L^T L + 2W_g^T W_g + W_h^T W_h\right) Y + BB^T + \frac{1}{\varepsilon} H_1 H_1^T + 2GG^T = 0 \tag{7.9}$$

where $Y = Y^T \geq 0$ is the stabilizing solution of the ARE (7.6).

When conditions (a) and (b) are satisfied, a suitable robust nonlinear $\mathcal{H}_\infty$ filter is given by:

$$\dot{\hat{x}}(t) = (A + \varepsilon Y E^T E)\hat{x}(t) + Gg[\hat{x}(t)] + K\left[y(t) - C\hat{x}(t) - Hh[\hat{x}(t)]\right], \quad \hat{x}(0) = 0 \tag{7.10}$$

$$\hat{z}(t) = L\hat{x}(t) \tag{7.11}$$

where

$$K = \left(YC^T + BD^T + \frac{1}{\varepsilon} H_1 H_2^T\right) \bar{V}^{-1} \tag{7.12}$$

and $Y = Y^T \geq 0$ is the stabilizing solution of the ARE (7.6).

Proof. With the filter (7.10)-(7.11) and defining $\tilde{x} \triangleq x - \hat{x}$, we obtain from (7.1)-(7.2) that

$$\dot{\tilde{x}}(t) = [A - KC + (H_1 - KH_2)F(t)E]\,\tilde{x}(t) + [-\Delta A_e + (H_1 - KH_2)F(t)E]\,\hat{x}(t) + G\left[g[x(t)] - g[\hat{x}(t)]\right] - KH\left[h[x(t)] - h[\hat{x}(t)]\right] + (B - KD)w(t) \tag{7.13}$$

where

$$\Delta A_e = \varepsilon Y E^T E.$$

Hence, by considering (7.10)-(7.11) and (7.13), a state space representation of the estimation error, $z - \hat{z}$, in terms of $\hat{x}$ and $\tilde{x}$ is as follows:

$$\dot{\eta}(t) = [A_c + H_c F(t) E_c]\,\eta(t) + G_c g_c[x(t), \hat{x}(t)] + B_c w(t), \quad \eta(0) = 0 \tag{7.14}$$

$$z(t) - \hat{z}(t) = L_c \eta(t) \tag{7.15}$$

where

$$\eta = [\hat{x}^T \quad \tilde{x}^T]^T,$$

$$A_c = \begin{bmatrix} A + \Delta A_e & KC \\ -\Delta A_e & A - KC \end{bmatrix},$$

$$B_c = \begin{bmatrix} KD \\ B - KD \end{bmatrix}, \quad H_c = \begin{bmatrix} KH_2 \\ H_1 - KH_2 \end{bmatrix},$$

$$E_c = [E \quad E], \quad L_c = [0 \quad L],$$

$$g_c[x(t),\hat{x}(t)] = \begin{bmatrix} g[x(t)] \\ g[x(t)] - g[\hat{x}(t)] \\ h[x(t)] - h[\hat{x}(t)] \end{bmatrix}, \quad G_c = \begin{bmatrix} G & 0 & 0 \\ 0 & G & -KH \end{bmatrix}.$$

Note that by Assumption 4.1

$$\|g_c(x,\hat{x})\| \leq \|\hat{W}\eta\| \tag{7.16}$$

where

$$\hat{W} = \begin{bmatrix} W_g^T & 0 & 0 \\ 0 & W_g^T & W_h^T \end{bmatrix}^T.$$

Next, using Lemma 2.3 it can be easily verified that $X \geq Y$, where $Y = Y^T \geq 0$ and $X = X^T \geq 0$ are the stabilizing solutions of (7.6) and (7.9), respectively. Also, letting

$$P \triangleq \begin{bmatrix} X - Y & 0 \\ 0 & Y \end{bmatrix} \geq 0$$

and considering the AREs (7.6) and (7.9), it can be shown using straightforward matrix manipulations that P is the stabilizing solution of the Riccati equation:

$$A_c P + P A_c^T + P\left(\gamma^{-2} L_c^T L_c + \varepsilon E_c^T E_c + \hat{W}^T \hat{W}\right) + B_c B_c^T + \frac{1}{\varepsilon} H_c H_c^T + G_c G_c^T = 0. \tag{7.17}$$

Finally, considering Lemma 2.4, we conclude from (7.17) that the estimation error system (7.14)-(7.15) is globally uniformly asymptotically stable and $\|z - \hat{z}\|_2 < \|w\|_2$ for all non-zero $w \in \mathcal{L}_2^r$ for all admissible uncertainties. $\nabla\nabla\nabla$

We note that, when there is no nonlinear term in (7.1) and (7.2), the matrices G, H, W_g and W_h should be set to zero. Under these conditions, the filter of (7.10)-(7.11) recovers the robust linear $\mathcal{H}_\infty$ filter anlaysed in Section 6.

Remark 7.1 We observe that the existence of a matrix X satisfying condition (b) of Theorem 7.1 will imply the global asymptotic stability of the uncertain system (7.1) for all admissible uncertainties (see, e.g. [42]). Note that due to the existence of parameter uncertainty in the system (7.1)-(7.2), the requirement of robust global asymptotic stability of (7.1) is needed in order to ensure the boundedness of the estimation error dynamics for all admissible uncertainties. □

Remark 7.2 Note that similarly to the robust $\mathcal{H}_\infty$ filtering for uncertain linear systems analysed in Section 6, the filter of (7.10)-(7.11) depends on the stabilizing solution of (7.6) but not on the stabilizing solution of (7.9). However, it should be remarked that the filter (7.10)-(7.11) can only guarantee the global asymptotic stability of the estimation error and a level of noise attenuation γ for all admissible uncertainties if both the AREs (7.6) and (7.9) have a positive semi-definite stabilizing solution for the same scalar $\varepsilon > 0$. □

References

[1] B.D.O. Anderson and J.B. Moore, *Optimal Filtering*, Prentice-Hall, Englewood Cliffs, NJ, 1979.

[2] R.N. Banavar and J.L. Speyer, "A linear-quadratic game approach to estimation and smoothing," *Proc. 1991 American Control Conf.*, pp. 2818-2822, Boston, MA. June 1991.

[3] B.R. Barmsh, "Stabilization of uncertain systems via linear control," *IEEE Trans. Automat. Control*, Vol. 28, pp. 848-850, 1983.

[4] T. Basar, "Optimal performance level for minimax filters, predictors and smoothers," *Systems & Control Letters*, Vol. 16, pp. 309-317, 1991.

[5] N. Berman and U. Shaked, "$\mathcal{H}_\infty$ nonlinear filtering," *Int. J. Robust and Nonlinear Control - Special Issue on $\mathcal{H}_\infty$ and Robust Estimation*, Vol. 4, 1994, to appear.

[6] D.S. Bernstein and W.M. Haddad, "Steady-state Kalman filtering with an $\mathcal{H}_\infty$ error bound," *Systems & Control Letters*, Vol. 12, pp. 9-16, 1989.

[7] P. Bolzern, P. Colaneri and G. De Nicolao, "Optimal robust state estimation for uncertain linear systems," *Proc. IFAC Symp. on Robust Control Design*, Rio de Janeiro, Brazil, Sept. 1994.

[8] S.P. Boyd, V. Balakhrishnan and P. Kabamba, " A bisection method for computing the $\mathcal{H}_\infty$ norm of a transfer matrix and related problems," *Math. Control, Signal and Systems*, Vol. 2, pp. 207-219, 1989.

[9] C.E. de Souza, U. Shaked and M. Fu, "Robust $\mathcal{H}_\infty$ filtering with parameteric uncertainty and deterministic input signals," *Proc. 31st IEEE Conf. on Decision and Control*, pp. 2305-2210, Tucson, AZ, Dec. 1992.

[10] C.E. de Souza and L. Xie, "On the properties of indefinite algebraic Riccati equations," Technical Report EE8952, Department of Electrical and Computer Engineering. The University of Newcastle, Australia, 1989.

[11] C.E. de Souza, L. Xie and Y. Wang, "$\mathcal{H}_\infty$ filtering for a class of uncertain nonlinear systems," *Systems & Control Letters*, Vol. 20, pp. 419-426, 1993.

[12] J.C. Doyle, K. Glover, P.P. Khargonekar and B.A. Francis, "State space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Control*, Vol. 34, pp. 831-847, 1989.

[13] A. El Sayed, M.J. Grimble and D. Ho, "On $\mathcal{H}_\infty$ filtering in multivariable discrete-time systems," *Proc. IEEE Symp. on Circuits and Systems*, Helsinki, Finland, 1988.

[14] B.A. Francis, *A Couse in $\mathcal{H}_\infty$ Control Theory*, Lecture notes in Control and Information Sciences, Vol. 88, Springer-Verlag, New York, 1987.

[15] M. Fu, "Interpolation approach to $\mathcal{H}_\infty$ estimation and the interconnections to loop transfer recovery," *Systems & Control Letters*, Vol. 17, pp. 29-36, 1991.

[16] M. Fu, C.E. de Souza and L. Xie, "$\mathcal{H}_\infty$ estimation for uncertain systems," *Int. J. Robust and Nonlinear Control*, Vol. 2, pp. 87-105, 1992.

[17] M.J. Grimble, "$\mathcal{H}_\infty$ design of optimal linear filters," in *Linear Circuits, Systems and Signal Processing: Theory and Application*, C.I. Byrnes, C.F. Martin and R.E. Saeks (Eds.), pp. 533-540, North-Holland, Amsterdam, The Netherlands, 1988.

[18] M.J. Grimble, D. Ho and A. El Sayed, "$\mathcal{H}_\infty$ robust linear estimator," *Proc. IFAC Symp. on Adaptive Systems in Control and Signal Processing*, pp. 159-164, Glasgow, Scotland. April 1989.

[19] M.J. Grimble and A. El Sayed, "Solution of the $\mathcal{H}_\infty$ optimal linear filtering problem for discrete-time systems," *IEEE Trans. Acoust., Speech and Signal Processing*, Vol. 38, pp. 1092-1104, 1990.

[20] W.M. Haddad and D.S. Bernstein, "Robust, reduced-order, nonstrictly proper state estimation via the optimal projection equations with Petersen-Hollot bounds," *Systems &.Control Letters*, Vol. 9, pp. 423-431, 1987.

[21] W.M. Haddad, D.S. Bernstein and D. Mustafa, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ regulation and estimation: The discrete-time case," *Systems & Control Letters*, Vol. 16, pp. 235-247, 1991.

[22] P.P. Khargonekar, I.R. Petersen and K. Zhou, "Robust stabilization of uncertain linear systems: quadratic stabilizability and $\mathcal{H}_\infty$ control," *IEEE Trans. Automat. Control*, Vol. 35, pp. 356-361, 1990.

[23] P.P. Khargonekar, K.M. Nagpal and K.R. Poola, "$\mathcal{H}_\infty$ control with transients," *SIAM J. Control and Optimiz.*, Vol. 29, pp. 1373-1393, 1991.

[24] P.P. Khargonekar and M.A. Rotea, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering," *Proc. 31st IEEE Conf. on Decision and Control*, pp. 2305-2210, Tucson, AZ, Dec. 1992.

[25] S.H. Lee, "Robust $\mathcal{H}_2/\mathcal{H}_\infty$ state estimation for systems with real parameter uncertainties," *Proc. 1994 American Control Conf.*, pp. 2217-2221, Baltimore, MD, June 1994.

[26] D.J.N. Limebeer and U. Shaked, "New results on $\mathcal{H}_\infty$ optimal filtering," *Proc. 9th Int. Symp. on the Mathematical Theory of Networks and Systems*, pp. 317-322, Kobe, Japan, June 1991.

[27] K.M. Nagpal and P.P. Khargonekar, "Filtering and smoothing in an $\mathcal{H}_\infty$ setting," *IEEE Trans. Automat. Control*, Vol. 36, pp. 152-166, 1991.

[28] S.K. Nguang and M. Fu, "$\mathcal{H}_\infty$ filtering for known and unknown nonlinear systems," *Proc. IFAC Symp. on Robust Control Design*, Rio de Janeiro, Brazil, Sept. 1994.

[29] I.R. Petersen and D.C. McFarlane, "Robust state estimation for uncertain systems," *Proc. 30th IEEE Conf. on Decision and Control*, pp. 2630-2631, Brighton, England, Dec. 1991.

[30] I.R. Petersen and D.C. McFarlane, "Optimizing the guaranteed cost in the control of uncertain systems, in *Robustness of Dynamical Systems with Parameter Uncertainties, Monte Verità"*, M. Mansour, S. Balemi and W. Truol (eds.), Birkhäuser Verlag, Basel, Switzerland, 1992.

[31] I.R. Petersen and D.C. McFarlane, "Optimal guaranteed cost filtering for uncertain discrete-time systems," *Proc. IFAC Symp. on Robust Control Design*, Rio de Janeiro, Brazil, Sept. 1994.

[32] H.P. Rotstein, M. Sznaier and M. Idan, "$\mathcal{H}_2/\mathcal{H}_\infty$ filtering: Theory and an aerospace application," *Proc. 1994 American Control Conf.*, pp. 1791-1795, Baltimore, MD, June 1994.

[33] U. Shaked, "$\mathcal{H}_\infty$-minimum error state estimation of linear stationary processes," *IEEE Trans. Automat. Control*, Vol. 35, pp. 554-558, 1990.

[34] U. Shaked and C.E. de Souza, "Robust $\mathcal{H}_2$ filtering," *Proc. IFAC Symp. on Robust Control Design*, Rio de Janeiro, Brazil, Sept. 1994.

[35] U. Shaked and Y. Theodor, "$\mathcal{H}_\infty$-optimal estimation: A tutorial," *Proc. 31st IEEE Conf. on Decision and Control*, pp. 2278-2286, Tucson, TX, Dec. 1992.

[36] P. Shi, C.E. de Souza and L. Xie, "Robust $\mathcal{H}_\infty$ filtering for uncertain systems with sampled-data measurements," *Proc. 32nd IEEE Conf. on Decision and Control*, pp. 793-798, San Antonio, TX, Dec. 1993.

[37] D. Simon and H. El-Sherief, " Hybrid $\mathcal{H}_2/\mathcal{H}_\infty$ estimation for phase-locked loop filter design," *Proc. 1994 American Control Conf.*, pp. 2212-2216, Baltimore, MD, June 1994.

[38] A.A. Stoorvogel, *The $\mathcal{H}_\infty$ control problem: A state space approach*, Prentice-Hall, London, England, 1992.

[39] W. Sun, K.M. Napal and P.P. Khargonekar, "$\mathcal{H}_\infty$ control and filtering for sampled-data systems," *IEEE Trans. Automat. Control*, Vol. 38, pp. 1162-11175, 1993.

[40] Y. Theodor and U. Shaked, "A dynamic game approach to mixed $\mathcal{H}_\infty/\mathcal{H}_2$ estimation and control," *Int. J. Robust & Nonlinear Control - Special Issue on $\mathcal{H}_\infty$ and Robust Estimation*, Vol. 4, 1994, to appear.

[41] Y. Theodor, U. Shaked and C.E. de Souza, "A game theory approach to robust discrete-time $\mathcal{H}_\infty$ estimation," *IEEE Trans. Signal Processing*, Vol. 43, 1994, to appear.

[42] Y. Wang, L. Xie and C.E. de Souza, "Robust control of a class of uncertain nonlinear systems," *Systems & Control Letters*, Vol. 19, pp. 139-149, 1992.

[43] L. Xie and C.E. de Souza, "State feedback $\mathcal{H}_\infty$ optimal control problems for non-detectable systems," *Systems & Control Letters*, Vol. 13, pp. 315-319, 1989.

[44] L. Xie, C.E. de Souza and M.D. Fragoso, "$\mathcal{H}_\infty$ filtering for linear periodic systems with parameter uncertainty," *Systems & Control Letters*, Vol. 17, pp. 343-350, 1991.

[45] L. Xie, C.E. de Souza and M. Fu, "$\mathcal{H}_\infty$ estimation for linear discrete-time uncertain systems," *Int. J. Robust and Nonlinear Control*, Vol. 2, pp. 111-123, 1992.

[46] L. Xie, C.E. de Souza and Y. Wang, "Robust control and filtering of discrete-time uncertain dynamical systems," *Proc. 2nd IFAC Workshop on Systems Structure and Control*, pp. 160-163, Prague, Czech Republic, Sept. 1992.

[47] L. Xie, C.E. de Souza and Y. Wang, " Worst-case filter design for a class of uncertain dynamical systems," *Proc. 31th IEEE Conf. on Decision and Control*, pp. 1230-1233, Tucson, AZ, Dec. 1992.

[48] L. Xie, M. Fu and C.E. de Souza , "$\mathcal{H}_\infty$ control and quadratic stabilization of systems with parameter uncertainty via output feedback," *IEEE Trans. Automat. Control*, Vol. 37, pp. 1253-1256, 1992.

[49] L. Xie and C.E. de Souza, "$\mathcal{H}_\infty$ state estimation for linear periodic systems," *IEEE Trans. Automat. Control*, Vol. 38, pp. 1704-1707, 1993.

[50] L. Xie, C.E. de Souza and Y. Wang, "Robust filtering for discrete-time uncertain nonlinear systems in an $\mathcal{H}_\infty$ setting," *Int J. Robust and Nonlinear Control - Special Issue on $\mathcal{H}_\infty$ and Robust Estimation*, Vol. 4, 1994, to appear.

[51] L. Xie and Y.C. Soh, "Robust Kalman filtering for uncertain systems," *Systems & Control Letters*, Vol. 22, pp. 123-129, 1994.

[52] L. Xie, Y.C. Soh and C.E. de Souza, "Robust Kalman filtering for uncertain discrete-time systems," *IEEE Trans. Automat. Control*, Vol. 39, 1994, to appear.

[53] I. Yaesh and U. Shaked, "Game theory approach to optimal linear estimation in the minimum $\mathcal{H}_\infty$ norm sense," *Proc. 28th IEEE Conf. on Decision and Control*, pp. 421-425, Tampa, FL, Dec. 1989.

[54] I. Yaesh and U. Shaked, "A transfer function approach to the problems of discrete time $\mathcal{H}_\infty$ optimal control and filtering," *IEEE Trans. Automat. Control*, Vol. 36, pp. 1267-1271, 1991.

[55] I. Yaesh and U. Shaked, "$\mathcal{H}_\infty$ optimal estimation – The discrete time case," *Proc. 9th Int. Symp. on the Mathematical Theory of Networks and Systems*, pp. 261-267, Kobe, Japan, June 1991.

[56] I. Yaesh and U. Shaked, "Game theory approach to optimal linear state estimation and its relation to $\mathcal{H}_\infty$ norm estimation," *IEEE Trans. Automat. Control*, Vol. 37, pp. 828-831, 1992.

[57] I. Yaesh and U. Shaked, "Game theory approach to state estimation of linear discrete time processes and its relation to $\mathcal{H}_\infty$-optimal estimation," *Int. J. Control*, Vol. 55, pp. 1443-1452, 1992.

[58] I. Yaesh and U. Shaked, "Game theory approach to finite-time horizon $\mathcal{H}_\infty$-optimal estimation," *IEEE Trans. Automat. Control*, Vol. 38, pp. 957-963, 1993.

[59] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximations," *IEEE Trans. Automat. Control*, Vol. 26, pp. 301-320, 1981.

[60] K. Zhou and P.P. Khargonekar, "An algebraic Riccati equation approach to optimization," *Systems & Control Letters*, Vol. 11, pp. 85-91, 1988.