# Notes on
# Robust Stability and Control:
# The parametric approach*

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# 1. Preliminaries

## 1.1. Kharitonov's Theorem

(Kharitonov 78[1])

Consider the family $\mathcal{F}$ of real polynomials,

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n$$

where

$$\delta_i \in [x_i, y_i], \qquad \forall\ i \in [0, 1, 2...n]$$

Then the entire family $\mathcal{F}$ is strictly Hurwitz if and only if

$$\begin{aligned}
K^1(s) &= x_0 + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + \cdots \\
K^2(s) &= x_0 + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + \cdots \\
K^3(s) &= y_0 + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + \cdots \\
K^4(s) &= y_0 + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + \cdots
\end{aligned}$$

are strictly Hurwitz.

## 1.2. Characteristic Property

Let

$$\begin{aligned}
K^{\text{even,min}}(s) &= x_0 + y_2 s^2 + x_4 s^4 + \cdots \\
K^{\text{even,max}}(s) &= y_0 + x_2 s^2 + y_4 s^4 + \cdots \\
K^{\text{odd,min}}(s) &= x_1 s + y_3 s^3 + x_5 s^5 + \cdots \\
K^{\text{odd,max}}(s) &= y_1 s + x_3 s^3 + y_5 s^5 + \cdots
\end{aligned}$$

The Kharitonov polynomials are built as follows:

$$\begin{aligned}
K^1(s) &= K^{\text{even,min}}(s) + K^{\text{odd,min}}(s) \\
K^2(s) &= K^{\text{even,min}}(s) + K^{\text{odd,max}}(s) \\
K^3(s) &= K^{\text{even,max}}(s) + K^{\text{odd,min}}(s) \\
K^4(s) &= K^{\text{even,max}}(s) + K^{\text{odd,max}}(s)
\end{aligned}$$

For an arbitrary polynomial $\delta(s)$, define

$$\begin{aligned}
\delta^e(\omega) &= \delta^{\text{even}}(j\omega) \\
&= \delta_0 - \delta_2\omega^2 + \delta_4\omega^4 - \cdots \\
\delta^o(\omega) &= \frac{\delta^{\text{odd}}(j\omega)}{j\omega} \\
&= \delta_1 - \delta_3\omega^2 + \delta_5\omega^4 - \cdots
\end{aligned}$$

Then for every polynomial $\delta(s) \in \mathcal{F}$,

$$\begin{aligned}
K^{\text{even,min}}(\omega) \le \ \delta^e(\omega) \ &\le K^{\text{even,max}}(\omega), \quad \forall\ \omega \in [-\infty, +\infty] \\
K^{\text{odd,min}}(\omega) \le \ \delta^o(\omega) \ &\le K^{\text{odd,max}}(\omega), \quad \forall\ \omega \in [-\infty, +\infty]
\end{aligned}$$

## 1.3. The Stability Ball in Coefficient Space

$\mathcal{P}_n$: vector space of real polynomials of degree $\leq n$, provided with an arbitrary norm $\|\cdot\|$.

$\delta(s)$: an arbitrary Hurwitz (stable) polynomial

**Q?** How far is $\delta(s)$ from instability?

Let $\mathcal{B}(\delta(s), \rho(\delta))$ denote the open ball of radius $\rho(\delta))$ centered at $\delta(s)$ and $\mathcal{S}(\delta(s), \rho(\delta))$ its boundary or surface.

**Theorem** Given $\delta(s)$ of degree $n$ and having all its roots in the LHP, there exists a positive real number $\rho(\delta)$ such that:

a) every polynomial in $\mathcal{B}(\delta(s), \rho(\delta))$ is stable and of degree $n$.

b) at least one polynomial on the hypersphere $\mathcal{S}(\delta(s), \rho(\delta))$ has one of its roots on the $j\omega$ axis, or is of degree less than $n$.

c) no polynomial lying on the hypersphere has a root in the RHP.

**Remark** Of course this remains true if the LHP is replaced by an arbitrary open region $\mathcal{U}$ and the $j\omega$ axis is replaced by the boundary of $\mathcal{U}$, $\partial\mathcal{U}$.

## 1.4. Computation of $\rho(\delta)$ for the $\ell^2$-norm

**Problem:** Compute $\rho(\delta)$ in $\ell^2$ norm.

**Answer:** Define some subspaces of $\mathcal{P}_n$:

$$\begin{aligned}
\Delta_0 &:= \{\text{polynomials with a zero at the origin}\} \\
\Delta_n &:= \{\text{polynomials of degree less than } n\} \\
\Delta_\omega &:= \{\text{polynomials with a zero at } \pm j\omega\}
\end{aligned}$$

Given a stable polynomial, $\delta(s)$, define

$$\begin{aligned}
d_0 &:= \text{distance from } \delta(s) \text{ to the subspace } \Delta_0 \\
d_n &:= \text{distance from } \delta(s) \text{ to the subspace } \Delta_n \\
d_\omega &:= \text{distance from } \delta(s) \text{ to the subspace } \Delta_\omega \\
d_{\min} &:= \inf_{\omega \geq 0} d_\omega
\end{aligned}$$

**Theorem** (Soh, Berger, Dabke, 1985) The radius of the largest stability ball around a stable polynomial $\delta(s)$ is given by:

$$\rho(\delta) = \min\{d_0, d_n, d_{\min}\}$$

where

$$d_0 = |\delta_0| \qquad d_n = |\delta_n|$$

## Computation of $d_{\min}$

Separate

$$\delta(s) = \delta^{\text{even}}(s) + \delta^{\text{odd}}(s)$$

and define

$$\begin{aligned}
\delta^{e}(\omega) &:= \delta^{\text{even}}(j\omega) \\
&= \delta_0 - \delta_2\omega^2 + \delta_4\omega^4 - \cdots \\
\delta^{o}(\omega) &:= \frac{\delta^{\text{odd}}(j\omega)}{j\omega} \\
&= \delta_1 - \delta_3\omega^2 + \delta_5\omega^4 - \cdots
\end{aligned}$$

**Theorem** The distance $d_\omega$ from $\delta(s)$ to $\Delta_\omega$ is given by:

i) when $n = 2p$ (even degree)

$$d_\omega^2 = \frac{[\delta^{e}(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4p}} + \frac{[\delta^{o}(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4(p-1)}}$$

ii) when $n = 2p + 1$ (odd degree)

$$d_\omega^2 = \frac{[\delta^{e}(\omega)]^2 + [\delta^{o}(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4p}}$$

These formulas are obtained by using the Projection Theorem on the distance from $\delta$ to the subspace $\Delta_\omega$.

**Minimization Procedure for $d_{\min} = \inf_{\omega \geq 0} d_\omega$.** In order to avoid a minimization over the infinite range $[0, \infty)$, we consider the following: To every polynomial $P(s)$, associate $P^r(s) = s^n P(\frac{1}{s})$. the map that takes $P(s)$ into $P^r(s)$ is a linear isometry. $P(s)$ has a root at $j\omega$ if and only if $P^r(s)$ has a root at $\frac{1}{j\omega}$. Thus the distance from $P(s)$ to $\Delta_\omega$ is the distance from $P^r(s)$ to $\Delta_{\frac{1}{\omega}}$, which reduces the problem to two similar minimization procedures over the fixed range $[0, 1]$.

**The Monic Case:** Let

$$\delta(s) := \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_{n-1} s^{n-1} + s^n$$

Find the radius of the largest stability hypersphere in the affine space of monic polynomials of degree $n$.

**Result:** The radius of the largest stability hypersphere around $\delta(s)$ is given by

$$\rho(\delta) = \min\{|\delta_0|, \inf_{\omega \geq 0} d_\omega^m\}$$

where

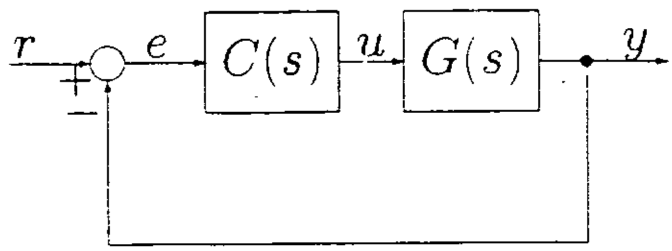
i) when $n = 2p$ (even degree)

$$d_\omega^{m2} = \frac{[\delta^e(\omega)]^2 + [\delta^o(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4(p-1)}}$$

ii) when $n = 2p + 1$ (odd degree)

$$d_\omega^{m2} = \frac{[\delta^e(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4p}} + \frac{[\delta^o(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4(p-1)}}$$

## 2. Stability Analysis in Parameter Space




Suppose that the plant transfer function $G(s)$ contains the parameter vector $\underline{P}$ and the controller is characterized by the real vector $\underline{X}$ such that

$$G(s) := G(s, \underline{P}) \qquad C(s) := C(s, \underline{X}) \qquad .$$

and let the characteristic polynomial of the closed loop system be

$$\delta(s, \underline{X}, \underline{P}) = \sum_{i=0}^{n} \delta_i(\underline{X}, \underline{P}) s^i$$

**Assumption:** $\delta_i(\underline{X}, \underline{P})$ are linear functions of the plant parameter vector $\underline{P}$. Note that this assumption always holds in SIMO or MISO systems if the parameter vector $\underline{P}$ is taken to be the list of plant transfer function coefficients.

Facts(SIMOcase): Let

$$G(s) = \frac{1}{d^p(s)} \begin{bmatrix} n_1^p(s) \\ n_2^p(s) \\ \vdots \\ n_m^p(s) \end{bmatrix} \quad \text{of order } q$$

and

$$C(s) = \frac{1}{d^c(s)} \begin{bmatrix} n_1^c(s) & n_2^c(s) & \cdots & n_m^c(s) \end{bmatrix} \quad \text{of order } r.$$

The plant parameter vector is chosen to be:

$$\underline{P} = \begin{bmatrix} n_1^p(s) & n_2^p(s) & \cdots & n_m^p(s) & d^p(s) \end{bmatrix}.$$

Then the characteristic polynomial of the closed loop system is given by $\delta(s)$ of degree $n = q + r$:

$$\delta(s) = d^c(s)d^p(s) + n_m^c(s)n_m^p(s) + \cdots + n_1^c(s)n_1^p(s).$$

## General Form of Polynomials for Robust Stability

$$\delta(s) = P_1(s)Q_1(s) + P_2(s)Q_2(s) + P_m(s)Q_m(s) + R(s)$$

where the parameters of $P_i(s)$ are allowed to perturb independently and $Q_i(s)$ are fixed. When the coefficients of $P_i(s)$ vary in intervals we refer to this as a linear interval system.

## 2.1. The Real Stability Ball in Parameter Space

Define the vector space of all $m$-tuples

$$\mathcal{P}_{n_1,n_2,\cdots,n_m} := \mathcal{P}_{n_1} \times \mathcal{P}_{n_2} \times \cdots \times \mathcal{P}_{n_m}$$

where

$$\underline{P} := \left[\, P_1(s) \;\; P_2(s) \;\; \cdots \;\; P_l(s) \,\right] \qquad P_j(s) \in \mathcal{P}_j.$$

and $\mathcal{P}_{n_1,n_2,\cdots,n_m}$ is provided with any norm $\|\cdot\|$.

For an arbitrary $m$-tuple of polynomials

$$\underline{Q} = \left[\, Q_1(s) \;\; Q_2(s) \;\; \cdots \;\; Q_m(s) \,\right]$$

and an arbitrary polynomial $R(s)$. consider the affine map

$$\begin{aligned} \mathcal{P}_{n_1,n_2,\cdots,n_m} & \stackrel{\delta_Q}{\Longrightarrow} \mathcal{P}_n \\ \underline{P} & \Longrightarrow \delta_Q(\underline{P}) \end{aligned}$$

where

$$\delta_Q(\underline{P}) = \sum_{j=1}^{m} P_j(s)Q_j(s) + R(s)$$

and $n$ is the generic degree of the resulting polynomial:

$$n = \max\{\max(n_i + d^o[Q_i(s)]), d^o[R(s)]\}$$

**Remark:** We say that $\underline{Q}$ stabilizes $\underline{P}$ if $\delta_Q(\underline{P})$ is stable.

Let $B(\underline{P}, \rho(\underline{P}))$ denote the open, real ball in the parameter space centered at the point $\underline{P}$ and of radius $\rho(\underline{P}$ and let $S(\underline{P}, \rho(\underline{P}))$ denote its boundary or surface.

**Theorem** Let $\underline{P}$ be an arbitrary element of $\mathcal{P}_{n_1, n_2, \cdots, n_m}$ stabilized by $\underline{Q}$. Then there exists a positive number $\rho(\underline{P})$ such that:

a) Every $m$-tuple in $B(\underline{P}, \rho(\underline{P}))$ is stabilized by $\underline{Q}$ and of degree $n$.

b) At least one element $\underline{P}'$ on the hypersphere $S(\underline{P}, \rho(\underline{P}))$ is such that $\delta_Q(\underline{P}')$ is unstable or of degree less than $n$.

c) If $\underline{P}'$ is any plant on the sphere such that $\delta_Q(\underline{P}')$ is unstable, then the unstable roots of $\delta_Q(\underline{P}')$ can only be pure imaginary or zero.

## 2.2. Computation of $\rho(\underline{P})$ for the $\ell^2$ - norm

Induced $\ell^2$ - norm on $\mathcal{P}_{n_1,n_2,\cdots,n_m}$:

$$[[\underline{P}]]_2^2 := \|P_1\|_2^2 + \|P_2\|_2^2 + \cdots + \|P_m\|_2^2$$

where $\|\cdot\|_2$ is the $\ell^2$ - norm on $\mathcal{P}_{n_j}$ for all $j$.

For an arbitrary $m$-tuple

$$\underline{Q} = \left[\, Q_1(s) \;\; Q_2(s) \;\; \cdots \;\; Q_m(s) \,\right]$$

and an arbitrary $R(s)$, consider the affine map

$$\mathcal{P}_{n_1,n_2,\cdots,n_m} \stackrel{\delta_Q}{\Longrightarrow} \mathcal{P}_n$$

$$\underline{P} \Longrightarrow \delta_Q(\underline{P})$$

where

$$\delta_Q(\underline{P}) = \sum_{j=1}^{m} P_j(s)Q_j(s) + R(s)$$

and $n$ is the generic degree of the resulting polynomial.

Define

$$\Pi_0 := \delta_Q^{-1}(\Delta_0) \qquad d_0^p := d(\underline{P}, \Pi_0)$$
$$\Pi_n := \delta_Q^{-1}(\Delta_n) \qquad d_n^p := d(\underline{P}, \Pi_n)$$

and

$$\begin{aligned} \Pi_\omega &:= \delta_Q^{-1}(\Delta_\omega) \\ d_\omega^p &:= d(\underline{P}, \Pi_\omega) \\ d_{\min}^p &:= \inf_{\omega \geq 0} d_\omega^p \end{aligned}$$

**Theorem** (Biernacki, Hwang, Bhattacharyya 87[2])

$$\rho(\underline{P}) = \min(d_0^p, d_n^p, d_{\min}^p)$$

**Remark:** Computation of $d_\omega^p$ is the major difficulty of the problem.

## 2.3. Computation of $d^p_\omega$

(Chapellat, Bhattacharyya. Keel 88[3])

a) If $Q_k(j\omega) = 0, \quad \forall\, k$ and $R(j\omega) \neq 0$. then $d^p_\omega = +\infty$.

b) Otherwise,

$$d^{p\,2}_\omega = \frac{\lambda_1^2[[\underline{Z}_1]]^2 + \lambda_2^2[[\underline{Z}_2]]^2 - 2\lambda_1\lambda_2 << \underline{Z}_1, \underline{Z}_2 >>}{[[\underline{Z}_1]]^2[[\underline{Z}_2]]^2 - << \underline{Z}_1, \underline{Z}_2 >>^2}$$

where

$$\lambda_1 := \delta^e_Q(\omega) \qquad \lambda_2 := \delta^o_Q(\omega)$$

$$\begin{aligned}
\underline{Z}_1 &:= [Q^e_1(\omega)R_1(s) + Q^o_1(\omega)T_1(s), \cdots \\
&\qquad \cdots, Q^e_m(\omega)R_m(s) + Q^o_m(\omega)T_m(s)] \\
\underline{Z}_2 &:= [Q^e_1(\omega)T_1(s) - \omega^2 Q^o_1(\omega)R_1(s), \cdots \\
&\qquad \cdots, Q^e_m(\omega)T_m(s) - \omega^2 Q^o_m(\omega)R_m(s)]
\end{aligned}$$

1) when $n_j = 2l$ (even degree)

$$\begin{aligned}
R_j(s) &= \begin{cases} s - \omega^2 s^3 + \cdots + (-1)^{l-1}\omega^{2l-2}s^{2l-1} \\ 0 \qquad \text{if} \quad q = 0 \end{cases} \\
T_j(s) &= 1 - \omega^2 s^2 + \cdots + (-1)^l \omega^{2l} s^{2l}.
\end{aligned}$$

2) when $n_j = 2l + 1$ (odd degree)

$$\begin{aligned}
R_j(s) &= s - \omega^2 s^3 + \cdots + (-1)^l \omega^{2l} s^{2l+1} \\
T_j(s) &= 1 - \omega^2 s^2 + \cdots + (-1)^l \omega^{2l} s^{2l}
\end{aligned}$$

## 2.4. Illustrative Example

$$G(s) = \frac{n_p(s)}{d_p(s)} = \frac{s}{1 - s + 4s^2 + s^3}$$

$$C(s) = \frac{n_c(s)}{d_c(s)} = \frac{3}{1+s} \qquad \text{stabilizing controller}$$

Then the characteristic polynomial is

$$\delta(s) = 3.s + (1+s)(1 - s + 4s^2 + s^3) = 1 + 3s + 3s^2 + 5s^3 + s^4$$

$$\begin{array}{ll} Q_1^{e}(\omega) = n^{ce}(\omega) = 3 & Q_1^{o}(\omega) = n^{co}(\omega) = 0 \\ Q_2^{e}(\omega) = d^{ce}(\omega) = 1 & Q_2^{o}(\omega) = d^{co}(\omega) = 1 \\ n_1 = 1 & n_2 = 3 \end{array}$$

$$\left\{ \begin{array}{lcl} d^{pe}(\omega) & = & 1 - 4\omega^2 \\ d^{po}(\omega) & = & -1 - \omega^2 \end{array} \right. \qquad \left\{ \begin{array}{lcl} \lambda_1 & = & \delta^{e}(\omega) \\ & = & 1 - 3\omega^2 + \omega^4 \\ \lambda_2 & = & \delta^{o}(\omega) \\ & = & 3 - 5\omega^2 \end{array} \right.$$

$$\left\{ \begin{array}{lcl} R_1(s) & = & s \\ R_2(s) & = & s - \omega^2 s^3 \end{array} \right. \qquad \left\{ \begin{array}{lcl} T_1(s) & = & 1 \\ T_2(s) & = & 1 - \omega^2 s^2 \end{array} \right.$$

$$\begin{array}{rcl} \underline{Z}_1 & = & [3s, 1 + s - \omega^2 s^2 - \omega^2 s^3] \\ & & \Longrightarrow \quad [[\underline{Z}_1]]^2 = 11 + 2\omega^4 \\ \underline{Z}_2 & = & [3, 1 - \omega^2 s - \omega^2 s^2 + \omega^4 s^3] \\ & & \Longrightarrow \quad [[\underline{Z}_2]]^2 = 10 + 2\omega^4 + \omega^8 \end{array}$$

$$\begin{array}{rcl} << \underline{Z}_1, \underline{Z}_2 >> & = & 1 - \omega^2 + \omega^4 - \omega^6 \\ & = & (1 - \omega^2)(1 + \omega^4) \end{array}$$

Thus, we have with $t = \omega^2$

$$d_\omega^{p\,2} = \frac{95 - 332t + 321t^2 - 58t^3 + 18t^4 + 4t^5 + 17t^6}{109 + 2t + 39t^2 + 4t^3 + 12t^4 + 2t^5 + t^6}$$

The derivative of this rational function has only one positive root at

$$t^* := 0.573438$$

for which we have $d_\omega^{p\,2}(t^*) \approx 0.01626$.

## 2.5. Relationship Between Kharitonov's Theorem and the Real Stability Ball

**Assumption:** The family $\mathcal{F}$ of interval polynomials $\delta(s)$ with coefficients in the box

$$\mathcal{B} := [x_0, y_0] \times [x_1, y_1] \times \cdots \times [x_n, y_n]$$

is entirely stable.

Define

$$\begin{array}{ccc} \mathcal{F} & \stackrel{\rho}{\Longrightarrow} & \mathcal{R}^+ \\ \delta(s) & \Longrightarrow & \rho(\delta). \end{array}$$

**Question:** Is there a point in $\mathcal{F}$ which is the nearest to instability? Equivalently, has the function $\rho$ a minimum and is there a precise point in $\mathcal{F}$ where it is reached?

**Theorem** (Chapellat and Bhattacharyya 89[4]) The function $\rho$ has a minimum reached at one of the four Kharitonov polynomials associated with $\mathcal{F}$.

**Remark:** The result is independent of a type of norm used.

## 3. Generalization of K-Theorem (CB Theorem)

### 3.1. Linear Interval Control Systems

**Motivation:** In control problems we need to consider the situation where plant parameters are subject to uncertainty whereas controller parameters are fixed. Very often the characteristic polynomial of the closed loop system is a linear combination of uncertain polynomials multiplied by fixed polynomials. Thus we formulate the following problem:

Let $\underline{P} = [P_1(s), P_2(s) \cdots, P_m(s)]$ be an $m$-tuple of polynomials subject to uncertainty. In particular assume that each $P_i(s)$ is an interval polynomial with

$$p_{i,j} \in [\alpha_{i,j}, \beta_{i,j}], \quad i = 1, \cdots, m; j = 0, \cdots, d^o(P_j).$$

Let $\underline{Q} = [Q_1(s) Q_2(s) \cdots Q_m(s)]$ be a given $m$-tuple of fixed polynomials and consider the family $\mathcal{F}$ of the form

$$\delta(s) = P_1(s)Q_1(s) + P_2(s)Q_2(s) + \cdots + P_m(s)Q_m(s).$$

**Question:** Find a necessary and sufficient condition under which all polynomials $\delta(s)$ in $\mathcal{F}$ are stable.

## 3.2. Kharitonov's Theorem does not apply

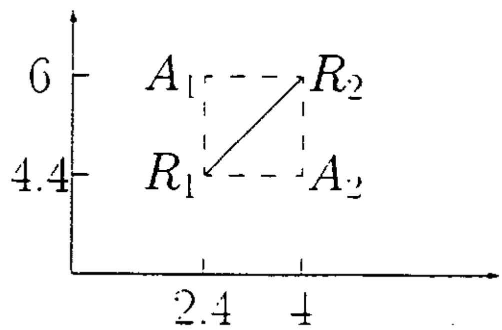
$$G(s) = \frac{n^p}{d^p} = \frac{s}{1 - s + \alpha s^2 + s^3}, \qquad \alpha \in [3.4, 5]$$

and has a nominal value $\alpha^0 = 4$. The controller $C(s) = \frac{3}{s+1}$ stabilizes the nominal plant with

$$\delta_4(s) = 1 + 3s + 3s^2 + 5s^3 + s^4.$$

For a perturbed plant the characteristic polynomial is

$$\delta_\alpha(s) = 1 + 3s + (\alpha - 1)s^2 + (\alpha + 1)s^3 + s^4$$




**Note:** $\delta_{A_1}(s) = 1 + 3s + 2.4s^2 + 6s^3 + s^4$ is unstable. Therefore using Kharitonov's theorem here does not allow us to conclude the stability of the entire family of closed loop systems. And yet if one checks the values of the Hurwitz determinants along the segment $[R_1, R_2]$ one finds:

$$H^\alpha = \begin{vmatrix} \alpha + 1 & 3 & 0 & 0 \\ 1 & \alpha - 1 & 1 & 0 \\ 0 & \alpha + 1 & 3 & 0 \\ 0 & 1 & \alpha - 1 & 1 \end{vmatrix}$$

and

$$\begin{cases} H_1^\alpha &= \alpha + 1 \\ H_2^\alpha &= \alpha^2 - 4 \\ H_3^\alpha &= 2\alpha^2 - 2\alpha - 13 \\ H_4^\alpha &= H_3^\alpha \end{cases}$$

are all positive for $\alpha \in [3.4, 5]$.

**Possible Solution:** Applying the Edge Theorem. However. the result is dependent on the order of the plant and does not reduce to Kharitonov-like theorem where the number of polynomials to be checked is independent of the order the of the plant.

## 3.3. Notation

- $P_i(s)$ - an interval polynomial.
- $K_i^j(s), j = 1, 2, 3, 4$ - four Kharitonov polynomials associated with the family of interval polynomials corresponding to $P_i(s)$.

Then we can define the following family of $m$ tuples of polynomial segments called the CB segments: For any fixed integer $l \in [1, m]$, set

$$P_i(s) = K_i^k(s), \quad \text{for} \quad i \neq l; k = 1, 2, 3, 4$$

and for $l$, suppose that $P_l(s)$ varies in one of the four *Kharitonov segments*

$$S_l := \begin{cases} [K_l^1(s), K_l^2(s)] & \text{or} \\ [K_l^1(s), K_l^3(s)] & \text{or} \\ [K_l^2(s), K_l^4(s)] & \text{or} \\ [K_l^3(s), K_l^4(s)] \end{cases}$$

Now consider the family of all $m$ tuple polynomial segments of the form

$$\underline{P}_\lambda =$$
$$(K_1^{j_1}(s), \cdots, K_{l-1}^{j_{l-1}}(s), (1-\lambda)K_l^1(s) + \lambda K_l^2(s),$$
$$K_{l+1}^{j_{l+1}}(s), \cdots, K_m^{j_m}(s))$$

The set of all such segments as $l$ varies over $[1, m]$ constitute the CB segments. In the most general case when all parameters vary independently there are $m4^m$ such segments.

Define

- $\mathcal{K}_m$ - the finite set of all possible $m$ - tuples where each polynomial $P_i(s)$ is equal to one of the four corresponding Kharitonov polynomials. $\mathcal{K}_m$ contains $4^m$ $m$ - tuples.

Let $\underline{Q} = (Q_1(s), \cdots, Q_m(s))$ be arbitrary. We say that $\underline{Q}$ stabilizes another given $m$ - tuple

$$\underline{R} = (R_1(s), \cdots, R_m(s))$$

if the polynomial

$$Q_1(s)R_1(s) + Q_2(s)R_2(s) + \cdots + Q_m(s)R_m(s) \quad .$$

is stable.

**Remark:** Similarly, we say that $\underline{Q}$ stabilizes the segment above if $\underline{Q}$ stabilizes $\underline{P}_\lambda$ for all $\lambda \in [0, 1]$.

## 3.4. CB Theorem

(Chapellat and Bhattacharyya 89[5])

For any given $m$ - tuple

$$\underline{Q} = (Q_1(s), \cdots, Q_m(s))$$

I) $\underline{Q}$ stabilizes the entire family $\mathcal{F}$ of $m$ - tuples if and only if $\underline{Q}$ stabilizes every CB segment $\underline{P}_\lambda$.

II) Moreover, if for each polynomial $Q_i(s)$, $Q_i(s)$ is either an even or odd. then it is enough that $\underline{Q}$ stabilizes the finite set of $m$ - tuples $\mathcal{K}_m$.

III) Finally. stabilizing the finite set $\mathcal{K}_m$ is not sufficient when the polynomials $Q_i(s)$ do not satisfy the restrictions of II).

**Remark1:** Clearly for $m = 1$ and $Q_1(s) = 1$ (which is even) the CB Theorem reduces to Kharitonov's Theorem.

**Remark2:** The condition II of this Theorem has recently been extended as follows:

**Vertex Lemma** (Bhattacharyya 91[6]) Under the conditions of the above Theorem suppose that the controller polynomials are of the form

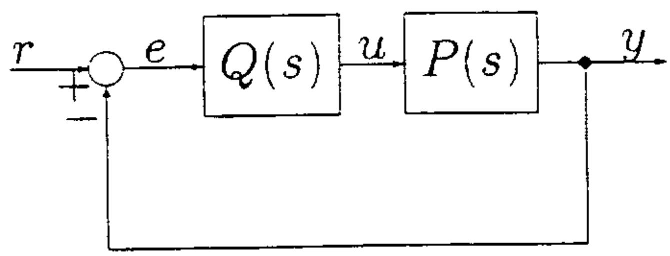
$$Q_i(s) = \underbrace{A(s)}_{\text{antiHurwitz}} .s^t(as+b). \underbrace{P(s)}_{\text{odd or even}}$$

then for robust stability it is sufficient to check the K-vertices $\mathcal{K}_m$.

Also can show: If the $Q_i(s)$ do not satisfy the above form it is <u>not</u> enough to stabilize the vertices.

## 3.5. Robust Control of Interval Systems: Scalar systems

Here we specialize the CB Theorem to the case of a single input single output interval system.




$$Q(s) = \frac{Q_1(s)}{Q_2(s)} \qquad P(s) = \frac{N(s)}{D(s)}$$

$$\begin{aligned} N(s) &= n_p s^p + n_{p-1} s^{p-1} + \cdots + n_1 s + n_0 \\ D(s) &= d_q s^q + d_{q-1} s^{q-1} + \cdots + d_1 s + d_0 \end{aligned}$$

•

$$\begin{aligned} \mathcal{N}(s) &= \{N(s) \ : \ n_i \in [n_i^-, n_i^+], \quad i = 0, 1, \cdots, p\} \\ \mathcal{D}(s) &= \{D(s) \ : \ d_i \in [d_i^-, d_i^+], \quad i = 0, 1, \cdots, q\} \end{aligned}$$

$$\mathcal{P}(s) := \{\frac{N(s)}{D(s)} \ : \ (N(s), D(s)) \in \mathcal{N}(s) \times \mathcal{D}(s)\}$$

- $\mathcal{P}(s)$ - interval system
- $\mathcal{N}(s) \times \mathcal{D}(s)$ - uncertainty set

**Definition:**

- Kharitonov vertex polynomial sets

$$\begin{aligned}\mathcal{K}_{\mathcal{N}}(s) &:= \{K_n^i(s), i = 1,2,3,4\}\\ \mathcal{K}_{\mathcal{D}}(s) &:= \{K_d^i(s), i = 1,2,3,4\}\end{aligned}$$

- Kharitonov segment sets

$$\begin{aligned}\mathcal{S}_{\mathcal{N}}(s) &:= [\lambda K_n^i(s) + (1-\lambda)K_n^j(s) \ : \ \lambda \in [0,1],\\ &\quad (i,j) \in \{(1,2),(1,3),(2,4),(3,4)\}]\\ \mathcal{S}_{\mathcal{D}}(s) &:= [\mu K_d^i(s) + (1-\mu)K_d^j(s) \ : \ \mu \in [0,1],\\ &\quad (i,j) \in \{(1,2),(1,3),(2,4),(3,4)\}]\end{aligned}$$

- CB subset of the uncertainty set

$$(\mathcal{N}(s) \times \mathcal{D}(s))_{\text{CB}} := \{(N(s), D(s)) \ :$$

$$\begin{aligned}&N(s) \in \mathcal{K}_{\mathcal{N}}(s), D(s) \in \mathcal{S}_{\mathcal{D}}(s) \text{ or}\\ &N(s) \in \mathcal{S}_{\mathcal{N}}(s), D(s) \in \mathcal{K}_{\mathcal{D}}(s)\}\end{aligned}$$

**Note:** Each element of $(\mathcal{N} \times \mathcal{D})_{\mathrm{CB}}$ is of the form

$$(n(s), \lambda d_1(s) + (1-\lambda)d_2(s)) \text{ or}$$
$$(\mu n_1(s) + (1-\mu)n_2(s), d(s))$$

and leads to transfer functions of the form

$$\frac{n(s)}{\lambda d_1(s) + (1-\lambda)d_2(s)} \text{ and } \frac{\mu n_1(s) + (1-\mu)n_2(s)}{d(s)}$$

i.e., one parameter families of transfer function

$$\mathcal{P}_{\mathrm{CB}}(s) := \{\frac{N(s)}{D(s)} \ : \ (N(s), D(s)) \in (\mathcal{N}(s) \times \mathcal{D}(s))_{\mathrm{CB}}\}$$

The system is stable for fixed $P(s)$, $Q(s)$ iff

$$\Pi(s) = Q_2(s)D(s) + Q_1(s)N(s)$$

has all its $n = q +$ degree$[Q_2(s)]$ roots in the open LHP. Let

$$\mathbf{\Pi}(s) = \{\Pi(s) \ : \ (N(s), D(s)) \in \mathcal{N}(s) \times \mathcal{D}(s)\}.$$

The system is robustly stable iff each polynomial in $\mathbf{\Pi}(s)$ is of degree $n$ (degree $D(s)$ remains invariant as $D(s)$ ranges over $\mathcal{D}(s)$) and has all its roots in the open LHP.

## 3.6. CB Theorem

The system is stable for all $P(s) \in \mathcal{P}(s)$ iff it is stable for all $P(s) \in \mathcal{P}_{\mathrm{CB}}(s)$.

## 3.7. Example

$$\begin{aligned} G(s) &= \frac{n(s)}{d(s)} \\ &= \frac{s^3 + \alpha s^2 - 2s + \beta}{s^4 + 2s^3 - s^2 + \gamma s + 1} \end{aligned}$$

where

$$\alpha \in [-1, -2], \quad \beta \in [0.5, 1] \quad \gamma \in [0, 1]$$

$$\begin{aligned} K_n^1(s) &= K_n^2(s) = 0.5 - 2s - s^2 + s^3 \\ K_n^3(s) &= K_n^4(s) = 1 - 2s - 2s^2 + s^3 \\ K_d^1(s) &= K_d^3(s) = 1 - s^2 + 2s^3 + s^4 \\ K_d^2(s) &= K_d^4(s) = 1 + s - s^2 + 2s^3 + s^4 \end{aligned}$$

We need to check the following set of polynomials:

$$\begin{aligned} \underline{P}_{\lambda,1} &= (0.5(1+\lambda) - 2s - (1+2\lambda)s^2 + s^3, \\ &\qquad 1 - s^2 + 2s^3 + s^4) \\ \underline{P}_{\lambda,2} &= (0,5(1+\lambda) - 2s + (1+2\lambda)s^2 + s^3, \\ &\qquad 1 + s - s^2 + 2s^3 + s^4) \\ \underline{P}_{\lambda,3} &= (0.5 - 2s - s^2 + s^3, 1 + \lambda s - s^2 + 2s^3 + s^4) \\ \underline{P}_{\lambda,4} &= (1 - 2s - 2s^2 + s^3, 1 + \lambda s - s^2 + 2s^3 + s^4) \end{aligned}$$

# 4. Uncertainties of Mixed Type

## 4.1. Characterization of $H_\infty$ Norm of Transfer Functions

**Notation:** $\mathbf{H} := \{s \in \mathbb{C} \; : \; \mathrm{Re}(s) > 0\}$

$H_\infty^{m \times p}$ is the space of matrix-valued functions $F(s)$ that are bounded and analytic in $H$, with norm

$$\|F\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(F(j\omega))$$

**Multivariable Case:** $G(s) \in H_\infty(\mathbf{H}^{m \times p})$,

- assume that $p \geq m$.
- $G(s) = N(s)D^{-1}(s)$ be a right coprime description of $G(s)$ over the ring of polynomial matrices, with $D(s)$ column-reduced.

**Problem:** Find a characterization of the $H_\infty$ norm of a transfer matrix in terms of the stability of a parametrized family of polynomials.

(Chapellat, Dahleh, Bhattacharyya 90[7])

**Lemma** $\|G\|_\infty < 1$ iff

a) $\|G(\infty)\| < 1$.

b) $\det\left( D(s) + U \begin{bmatrix} I \\ 0 \end{bmatrix} N(s) \right)$ is Hurwitz for all unitary matrices $U$ in $\mathbb{C}^{p \times p}$.

## Special Case: SISO

$g(s) = n(s)/d(s)$ a rational function in $H_\infty(\mathbf{H})$

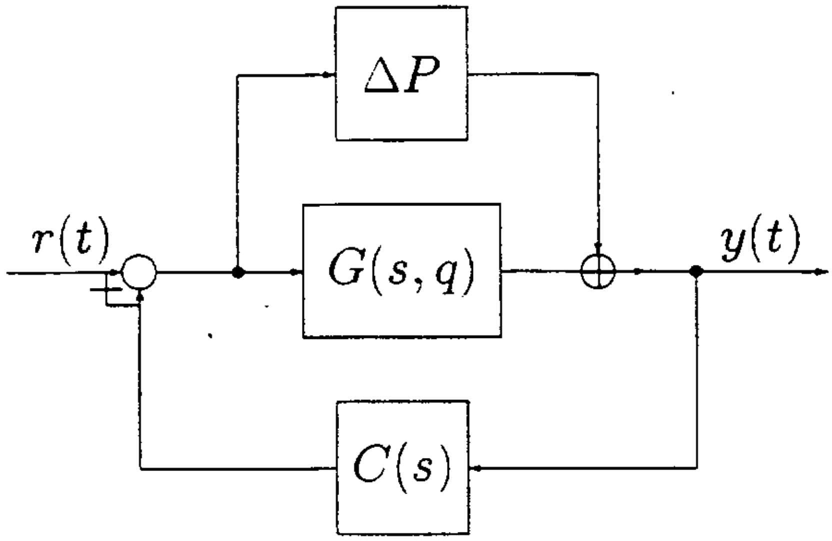
$\deg[d(s)] = q$

**Lemma** $\|g\| < 1$ iff

a) $|n_q| < |d_q|$.

b) $d(s) + e^{j\theta} n(s)$ is Hurwitz for all $\theta$ in $[0, 2\pi)$.

## 4.2. Simultaneous Unstructured and Parametric Uncertainties




- $q$ is an uncertain parameter
- $\Delta P$ represents unstructured perturbation

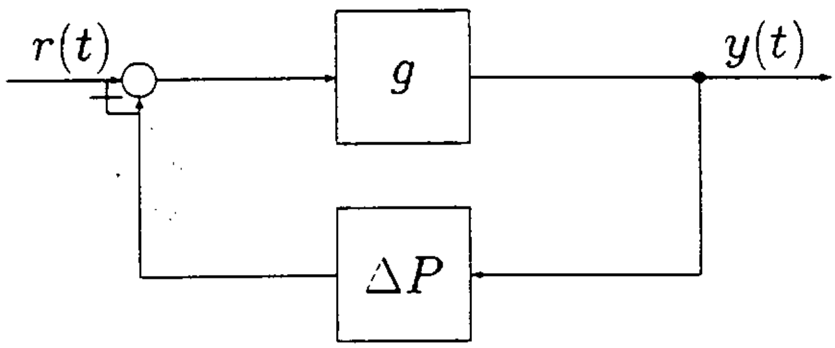
$$G(s) = \frac{N(s)}{D(s)} \qquad N(s) \in \mathcal{N}(s), D(s) \in \mathcal{D}(s)$$

$$\begin{aligned} \mathcal{G} &= \{G(s) \mid (N(s), D(s)) \in \mathcal{N}(s) \times \mathcal{D}(s)\} \\ \mathcal{G}_{\text{CB}} &= \{G(s) \mid (N(s), D(s)) \in (\mathcal{N}(s) \times \mathcal{D}(s))_{\text{CB}}\} \subset \mathcal{G} \end{aligned}$$

## 4.3. Computation of Unstructured Stability Margins

### 4.3.1. *Standard Small Gain Theorem*

(Francis 85[8])
Let $g$ be a stable transfer function. The closed loop in the figure below is stable for all $\Delta P$ such that $\|\Delta P\|_\infty \leq \alpha$ iff $\|g\| < \frac{1}{\alpha}$.




**Problem:** How can we compute in a non-conservative way the norm of the smallest destabilizing perturbation in the presence of parametric uncertainty in the forward loop transfer function $g(s)$?

- Parametric uncertainty is modeled by interval systems: Let $\mathcal{G}$ be a family of SISO plants with transfer function $g(s) = \frac{a(s)}{b(s)}$ where
  - $a(s)$ belongs to a family of real interval polynomials $\mathcal{A}$ and $b(s)$ belongs to a stable family of real interval polynomials $\mathcal{B}$,
  - we define
    - $\mathcal{K}_{\mathcal{A}}, \mathcal{K}_{\mathcal{B}}$ - sets of K-polynomials
    - The K-systems of $\mathcal{G}$

$$\mathcal{G}_{\mathcal{K}} = \{g(s) \mid \frac{K^i_{\mathcal{A}}(s)}{K^j_{\mathcal{B}}(s)} \; : \; i, j = 1, 2, 3, 4\}$$

**Problem:** Compute the global maximum of the $H_\infty$ norm over the entire family $\mathcal{G}$.

This problem was solved by Chapellat, Dahleh and Bhattacharyya (1990)

**Theorem** (Extreme-Point Property) $\|g\|_\infty < 1$ for all $g(s) \in \mathcal{G}$ iff it is the case for the 16 elements of $\mathcal{G}_{\mathcal{K}}$.

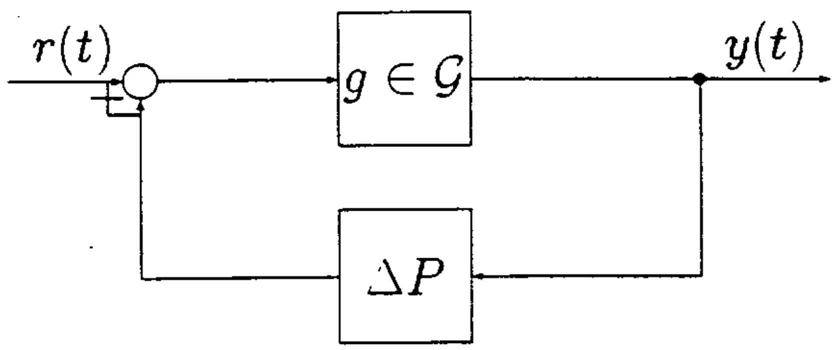
**Remark:** In other words the maximum $H_\infty$ norm over a family of interval systems occurs on the vertices that correspond to the K-systems.

### *4.3.2. Robust Small Gain Theorem*

(Chapellat, Dahleh and Bhattacharyya 1990[7])

Given the interval family $\mathcal{G}$ of stable proper systems. The closed loop in the figure below remains stable for all stable perturbation $\Delta P$ such that $\|\Delta P\|_\infty < \alpha$ where

$$\alpha = \frac{1}{\max_{g \in \mathcal{G}_K} \|g\|_\infty}$$




### *4.3.3. Example*

$$g(s) = \frac{n_0 + n_1 s + n_2 s^2 + n_3 s^3}{d_0 + d_1 s + d_2 s^2 + d_3 s^3}$$

where

$$\begin{array}{llll} n_0 \in [1,2] & n_1 \in [-3,1] & n_2 \in [2,4] & n_3 \in [1,3] \\ d_0 \in [1,3] & d_1 \in [2,4] & d_2 \in [6,7] & d_3 \in [1,3] \end{array}$$

$\mathcal{G}_{\mathcal{K}}$ consists of the following 16 rational functions:

$$g_1(s) = \frac{1-3s+4s^2+3s^3}{1+2s+7s^2+2s^3}, \quad g_2(s) = \frac{1-3s+4s^2+3s^3}{1+4s+7s^2+s^3},$$
$$g_3(s) = \frac{1-3s+4s^2+3s^3}{3+2s+6s^2+2s^3}, \quad g_4(s) = \frac{1-3s+4s^2+3s^3}{3+4s+6s^2+s^3},$$
$$g_5(s) = \frac{1+s+4s^2+s^3}{1+2s+7s^2+2s^3}, \quad g_6(s) = \frac{1+s+4s^2+s^3}{1+4s+7s^2+s^3},$$
$$g_7(s) = \frac{1+s+4s^2+s^3}{3+2s+6s^2+2s^3}, \quad g_8(s) = \frac{1+s+4s^2+s^3}{3+4s+6s^2+s^3},$$
$$g_9(s) = \frac{2-3s+2s^2+3s^3}{1+2s+7s^2+2s^3}, \quad g_{10}(s) = \frac{2-3s+2s^2+3s^3}{1+4s+7s^2+s^3},$$
$$g_{11}(s) = \frac{2-3s+2s^2+3s^3}{3+2s+6s^2+2s^3}, \quad g_{12}(s) = \frac{2-3s+2s^2+3s^3}{3+4s+6s^2+s^3},$$
$$g_{13}(s) = \frac{2+s+2s^2+s^3}{1+2s+7s^2+2s^3}, \quad g_{14}(s) = \frac{2+s+2s^2+s^3}{1+4s+7s^2+s^3},$$
$$g_{15}(s) = \frac{2+s+2s^2+s^3}{3+2s+6s^2+2s^3}, \quad g_{16}(s) = \frac{2+s+2s^2+s^3}{3+4s+6s^2+s^3}.$$

The maximum is achieved at $g_3$ and $\|g_3\|_\infty = 5.002$. The entire family of systems remains stable under any unstructured perturbations of $H_\infty$ norm less than

$$\alpha = \frac{1}{5.002} = 0.19992.$$

## 4.4. Computation of the Structured Margin

### 4.4.1. *Converse Problem:*

A bound is fixed on the level of unstructured perturbations that are to be tolerated and a structured stability margin is sought.

- Start with a nominal stable system

  $$g^o(s) = \frac{n_0^0 + n_1^o s + \cdots + n_p^o s^p}{d_0^o + d_1^o s + \cdots + d_q^o s^q}$$

  which satisfies $\|g^o\|_\infty = \alpha$.

- A bound $\frac{1}{\beta} < \frac{1}{\alpha}$ is then fixed on the desired level of unstructured perturbations.

- Fix the structure of the parametric perturbations: allow the parameters $n_i$, $d_j$ of the plant to vary in intervals of the form

  $$n_i \in [n_i^o - \epsilon\nu_i, n_i^o + \epsilon\nu_i] \quad d_i \in [n_i^o - \epsilon\mu_i, d_i^o + \epsilon mu_i]$$

  where the weights $\nu_i$, $\mu_j$ are fixed and nonnegative.

- Maximize a weighted $\ell_\infty$ ball around the parameters of $g^o(s)$.

- For each $\epsilon$ we get a family of interval systems $\mathcal{G}(\epsilon)$ and its associated K-systems $\mathcal{G}_K(\epsilon)$.

**Remark:** The structured stability margin is then given by the largest $\epsilon$, say $\epsilon_{\max}$, for which every system $g(s)$ in the corresponding interval family $\mathcal{G}(\epsilon_{\max})$ satisfies $\|g\|_\infty \le \beta$.

### 4.4.2. *Iterative Procedure:*

- **First:** Obtain an upper bound $\epsilon_1$ for $\epsilon_{max}$ by letting $\epsilon_1$ be the smallest number such that the interval family,

$$\{d(s) = d_0 + \cdots + d_q s^q \ : \ d_j \in [d_j^o - \epsilon_1 \mu_j, d_j^o + \epsilon_1 \mu_j]\}$$

contains an unstable polynomial.

- **Second:** Use the following bisection algorithm

1. SET LBOUND = 0, UBOUND = $\epsilon_1$
2. SET $\epsilon = \frac{UBOUND - LBOUND}{2}$
3. Update $\mathcal{G}_{\mathcal{K}}(\epsilon)$
4. IF the 16 systems in $\mathcal{G}_{\mathcal{K}}(\epsilon)$ have $H_\infty$ norm $\leq \beta$

   THEN SET LBOUND = $\epsilon$

   ELSE SET UBOUND = $\epsilon$
5. IF $|UBOUND - LBOUND|$ is small enough

   THEN EXIT

   ELSE GOTO 2

### *4.4.3. Example*

- The nominal system:

$$g^o(s) = \frac{1-s}{1+3s+s^2}.$$

- $\|g(s)\|_\infty = 1$.
- Fix the bound on the unstructured margin to be equal to $\frac{1}{\beta} = \frac{1}{2}$.
- The perturbed system is of the form

$$g_{a,b,c,d}(s) = \frac{1+a-(1+b)s}{1+c+(3+d)s+s^2}$$

where $|a|, |b|, |c|, |d| \leq \epsilon$.

**Problem:** Find the largest $\epsilon$ such that

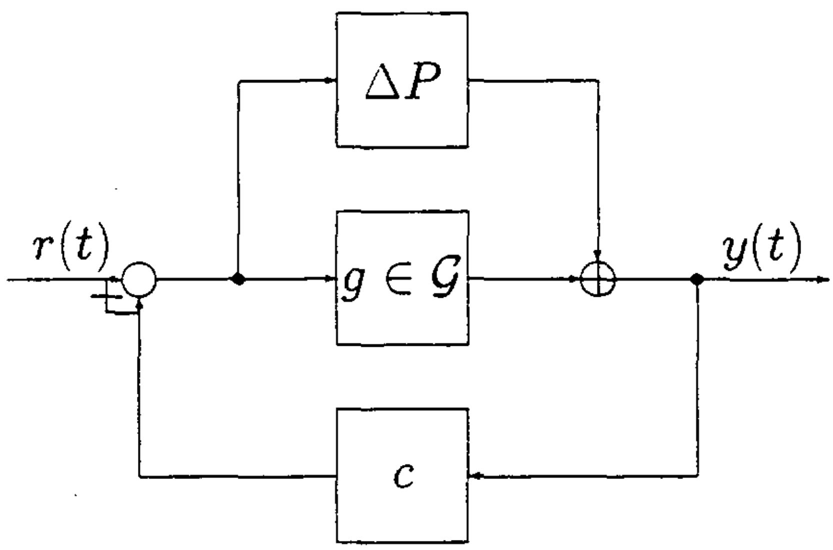
$$\|g_{a,b,c,d}\|_\infty \leq 2 \quad \text{for all} \quad (a,b,c,d).$$

**Solution:** analytically $\epsilon_{\max} = \frac{1}{3}$ and the extremal systems are

$$\frac{\frac{4}{3}-\frac{2}{3}s}{\frac{2}{3}+\frac{10}{3}s+s^2}, \frac{\frac{4}{3}-\frac{4}{3}s}{\frac{2}{3}+\frac{10}{3}s+s^2}, \frac{\frac{4}{3}-\frac{2}{3}s}{\frac{2}{3}+\frac{8}{3}s+s^2}, \frac{\frac{4}{3}-\frac{4}{3}s}{\frac{2}{3}+\frac{8}{3}s+s^2}.$$

## 4.5. Small Gain Theorem for Control Systems

- $\mathcal{G}$ be a family of strictly proper SISO interval plants.
- $c(s) = \frac{n_c(s)}{d_c(s)}$ is a stabilizing controller for the entire family.




**Problem:** Determine the amount of unstructured perturbations that can be tolerated by this family of interval plants

**Solution:** (Chapellat, Dahleh, Bhattacharyya 90[7])

- We must find the maximum of the $H_\infty$ norm of the closed loop transfer function $c(s)(1+g(s)c(s))^{-1}$ over all elements $g(s) \in \mathcal{G}$.

**Remark:** Note that $c(s)(1+g(s)c(s))^{-1}$ is no longer an interval family.

**Theorem**

$$\max_{g\in\mathcal{G}} \|c(s)(1+c(s)g(s))^{-1}\|_\infty = \max_{g\in\mathcal{G}_{\mathrm{CB}}} \|c(s)(1+c(s)g(s))^{-1}\|_\infty.$$

**Theorem** The closed loop is stable for all $\|\Delta P\|_\infty < \alpha$ iff

$$\alpha \leq \alpha^* = \frac{1}{\max_{g\in\mathcal{G}_{\mathrm{CB}}} \|c(s)(1+c(s)g(s))^{-1}\|_\infty}.$$

Computation is reduced to (at most) 32 $H_\infty$ computations along line segments.

**Theorem** (Synthesis Theorem) Let $\mathcal{G}$ be a family of interval plants of fixed degree and let $\alpha > 0$ be given. There exists a linear time invariant controller that stabilizes $\mathcal{G}$ and that satisfies

$$\sup_{g\in\mathcal{G}} \|c(s)(1+g(s)c(s))^{-1}\|_\infty \leq \alpha$$

iff such a controller exists for $\mathcal{G}_{\mathrm{CB}}$.

## 4.6. Example

$$g_{\beta,\gamma} = \frac{n_p(s)}{d_p(s)} = \frac{\beta s}{1 - s + \gamma s^2 + s^3}$$

where $\beta \in [1, 2]$ and $\gamma \in [3.4, 5]$.

- The controller: $c(s) = \frac{3}{s+1}$.
- The transfer function that is seen by the perturbation is given by $c(s)(1 + g_{\beta,\gamma}(s)c(s))^{-1} =$

$$\frac{3(1 - s + \gamma s^2 + s^3}{1 + 3\beta s + (\gamma - 1)s^2 + (\gamma + 1)s^3 + s^4}$$

- By the previous result, we need to find the maximum $H_\infty$ norm of four one-parameter families of rational functions:

$$\begin{aligned}
r_\lambda(s) &= \frac{3(1 - s + \lambda s^2 + s^3)}{1 + 3s + (\lambda - 1)s^2 + (\lambda + 1)s^3 + s^4}, \\
r_\mu(s) &= \frac{3(1 - s + \mu s^2 + s^3)}{1 + 6s + (\mu - 1)s^2 + (\mu + 1)s^3 + s^4}, \\
r_\nu(s) &= \frac{3(1 - s + 3.4s^2 + s^3)}{1 + 3\nu s + 2.4s^2 + 4.4s^3 + s^4}, \\
r_\xi(s) &= \frac{3(1 - s + 5s^2 + s^3)}{1 + 3\xi s + 4s^2 + 6s^3 + s^4},
\end{aligned}$$

  where $\lambda \in [3.4, 5], \mu \in [3.4, 5], \nu \in [1, 2], \xi \in [1, 2]$

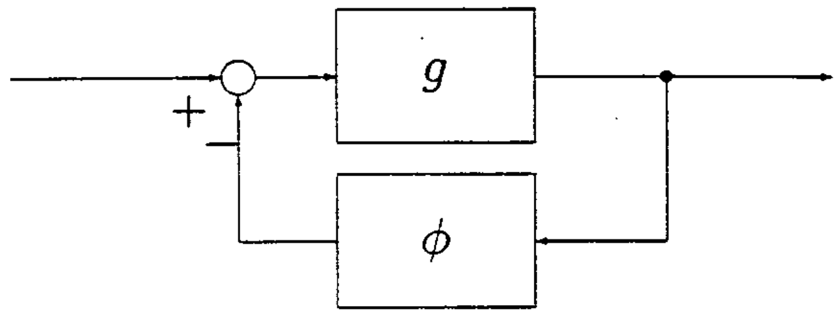
- Performing the maximization we get

$$\max_{\beta\in[1,2],\gamma\in[3.4,5]} \|c(s)(1 + g_{\beta,\gamma}(s)c(s))^{-1}\|_\infty = 34.14944.$$

  where the maximum is in fact achieved for $\beta = 1$, and $\gamma = 3.4$.

# 5. Nonlinear Feedback Perturbations

In this section we deal with nonlinear feedback perturbations of systems containing parameter uncertainty. In the classical Lur'e problem, a stable linear time-invariant system is connected by feedback to a memoryless time-varying nonlinearity.




*Class of allowable nonlinearities:* sector bounded feedback gains $\phi(t,\sigma)$ satisfies $\phi(t,0)$ for all $t \geq 0$,

$$0 \leq \sigma\phi(t,\sigma) \leq k\sigma^2.$$

**Theorem** If $g(s)$ is a stable transfer function and $\phi$ belongs to the sector $[0,k]$, then a sufficient condition for absolute stability is

$$\mathrm{Re}[\frac{1}{k} + g(j\omega)] > 0 \quad \text{for all} \quad \omega \in \mathbb{R}.$$

(i.e., $\frac{1}{k} + g(s)$ is SPR)

## 5.1. Strict Positive Realness

A proper transfer function $g(s)$ is said to be SPR if

1) $g(s)$ has no poles in the closed RHP and

2) $\text{Re}[g(j\omega)] > 0$ for all $\omega \in [-\infty, +\infty]$.

## 5.2. SPR Property of Interval Systems

(The following results were given in Chapellat, Dahleh, Bhattacharyya 91 [9])

**Problem:** Determine the minimum of the real part of $g(j\omega)$ over all $\omega \in \mathbb{R}$ and all elements $g(s)$ of a stable proper family $\mathcal{G}$ of real interval plants.

**Characterization of SPR:** Let $g(s) = \frac{n(s)}{d(s)}$ be a real proper transfer function with no poles in the closed RHP. we have the following theorem.

**Theorem** $g(s)$ is SPR iff the following three conditions are satisfied:

a) $\text{Re}[g(0)] > 0$,

b) $n(s)$ is Hurwitz stable,

c) $d(s) + j\alpha n(s)$ is Hurwitz stable for all $\alpha \in \mathbb{R}$.

**Problem:** Let $\gamma$ be any given real number, find necessary and sufficient conditions under which it is true that for all $g(s) \in \mathcal{G}$,

$$\mathrm{Re}[g(j\omega)] + \gamma > 0, \quad \text{for all } \omega \in \mathbb{R}.$$

i.e., under what conditions is $g(s) + \gamma$ SPR for all $g(s) \in \mathcal{G}$?

**Answer (lemma):** $g(s) + \gamma$ is SPR for every element in $\mathcal{G}$ iff it is SPR for the 16 K-systems in $\mathcal{G}_K$.

**Theorem** Given a proper stable family $\mathcal{G}$ of interval plant, the minimum of $\mathrm{Re}[g(j\omega)]$ over all $\omega$ and over all $g(s) \in \mathcal{G}$ is achieved at one of the 16 K-systems in $\mathcal{G}_K$.

**Theorem** Every plant $g(s)$ in $\mathcal{G}$ is SPR iff it is the case for the following 8 plants:

$$g_1(s) = \frac{K_A^2(s)}{K_B^1(s)} \quad g_2(s) = \frac{K_A^3(s)}{K_B^1(s)} \quad g_3(s) = \frac{K_A^1(s)}{K_B^2(s)} \quad g_4(s) = \frac{K_A^4(s)}{K_B^2(s)}$$

$$g_5(s) = \frac{K_A^1(s)}{K_B^3(s)} \quad g_6(s) = \frac{K_A^4(s)}{K_B^3(s)} \quad g_7(s) = \frac{K_A^2(s)}{K_B^4(s)} \quad g_8(s) = \frac{K_A^3(s)}{K_B^4(s)}$$

**Remark:** When the family is proper but not strictly proper, then the overall minimum is achieved at one of the 16 K-systems even though one only has to check 8 plants to verify the SPR property for the entire family.

## 5.3. Example

Consider the following stable family $\mathbf{G}(s)$ of interval systems whose generic element is given by

$$G(s) = \frac{1 + \alpha s + \beta s^2 + s^3}{\gamma + \delta s + \epsilon s^2 + s^3}$$

where

$$\alpha \in [1, 2],\ \beta \in [3, 4],\ \gamma \in [1, 2],\ \delta \in [5, 6],\ \epsilon \in [3, 4].$$

$\mathbf{G}_K(s)$ consists of the following 16 rational functions

$$r_1(s) = \frac{1 + s + 3s^2 + s^3}{1 + 5s + 4s^2 + s^3}, \quad r_2(s) = \frac{1 + s + 3s^2 + s^3}{1 + 6s + 4s^2 + s^3},$$

$$r_3(s) = \frac{1 + s + 3s^2 + s^3}{2 + 5s + 3s^2 + s^3}, \quad r_4(s) = \frac{1 + s + 3s^2 + s^3}{2 + 6s + 3s^2 + s^3},$$

$$r_5(s) = \frac{1 + s + 4s^2 + s^3}{1 + 5s + 4s^2 + s^3}, \quad r_6(s) = \frac{1 + s + 4s^2 + s^3}{1 + 6s + 4s^2 + s^3},$$

$$r_7(s) = \frac{1 + s + 4s^2 + s^3}{2 + 5s + 3s^2 + s^3}, \quad r_8(s) = \frac{1 + s + 4s^2 + s^3}{2 + 6s + 3s^2 + s^3},$$

$$r_9(s) = \frac{1 + 2s + 3s^2 + s^3}{1 + 5s + 4s^2 + s^3}, \quad r_{10}(s) = \frac{1 + 2s + 3s^2 + s^3}{1 + 6s + 4s^2 + s^3},$$

$$r_{11}(s) = \frac{1 + 2s + 3s^2 + s^3}{2 + 5s + 3s^2 + s^3}, \quad r_{12}(s) = \frac{1 + 2s + 3s^2 + s^3}{2 + 6s + 3s^2 + s^3},$$

$$r_{13}(s) = \frac{1 + 2s + 4s^2 + s^3}{1 + 5s + 4s^2 + s^3}, \quad r_{14}(s) = \frac{1 + 2s + 4s^2 + s^3}{1 + 6s + 4s^2 + s^3},$$

$$r_{15}(s) = \frac{1 + 2s + 4s^2 + s^3}{2 + 5s + 3s^2 + s^3}, \quad r_{16}(s) = \frac{1 + 2s + 4s^2 + s^3}{2 + 6s + 3s^2 + s^3}.$$

The corresponding minima of their respective real parts along the imaginary axis and over the whole family is achieved at $r_8$:

$$\inf_{\omega \in R} Re\ r_8(j\omega) = 0.0563546.$$

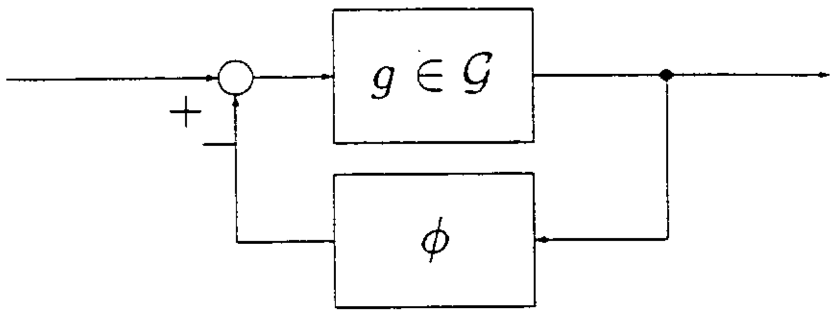
Therefore, the entire family is SPR and the minimum is achieved at $r_8(s)$.

**Remark:** However $r_8(s)$ corresponds to $\frac{K_A^1(s)}{K_B^4(s)}$ which is not among the 8 rational functions of the Theorem.

## 5.4. The Robust Lur'e Problem

(Chapellat, Dahleh and Bhattacharyya 91[9])

Given an interval plant $\mathcal{G}$, consider the following family, $\mathcal{G}^{\phi}$, of feedback systems.




By performing 16 computations we can determine the size of a sector for which the family $\mathcal{G}^{\phi}$ is absolutely stable.

**Remark:** Real part of $g$ at $s = j\omega$: $\mathrm{Re}[g(j\omega)]$

$$\mathrm{Re}[g(j\omega)] + \frac{1}{k} > 0, \quad \forall\, \omega.$$

Thus,

$$\min_{g \in \mathcal{G}} \min_{\omega} \mathrm{Re}[g(j\omega)]$$

determines the size of the sector.

**Theorem** Given the interval family $\mathcal{G}$ of stable proper systems. The nonlinear family $\mathcal{G}^\circ$ is absolutely stable if the nonlinearity $\phi$ belongs to the sector $[0, k]$, where $k > 0$ is any number such that

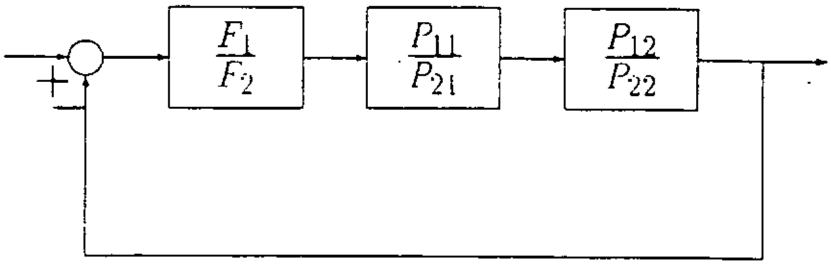
$$k < \infty, \qquad \text{if} \quad \inf_{\mathcal{G}_K} \inf_{\omega \in \mathbb{R}} \text{Re}[g(j\omega)] \geq 0$$

otherwise,

$$k < -\frac{1}{\inf_{\mathcal{G}_K} \inf_{\omega \in \mathbb{R}} \text{Re}[g(j\omega)]}$$

where $\mathcal{G}_K$ is the set of 16 K-systems corresponding to $\mathcal{G}$.

# 6. Multilinear Parametric Perturbations




**Characteristic Polynomial:**

$$\delta(s) = F_1(s)P_{11}(s)P_{12}(s) + F_2(s)P_{21}(s)P_{22}(s)$$

with $P_{ij}(s) \in \mathcal{P}_{ij}(s)$ being interval polynomials $i, j = 1, 2$.

**Uncertainty Set:**

$$\mathbf{\Pi} := \mathcal{P}_{11} \times \mathcal{P}_{12}(s) \times \mathcal{P}_{21}(s) \times \mathcal{P}_{22}(s)$$

**Problem:** Determine conditions under which $\delta(s)$ is Hurwitz over $\mathbf{\Pi}$ (equivalently $\underline{F}$ stabilizes $\mathbf{\Pi}$). Note that $\delta(s)$ is a multilinear function of the uncertain parameters.

## 6.1. Notation

Consider the multilinearly parametrized family of polynomials

$$\delta(s) = F_1(s)P_{11}(s)\cdots P_{1r_1}(s) + \cdots + F_m P_{m1}(s)\cdots P_{mr_m}(s).$$

In the space of the coefficients of the polynomials $P_{ij}(s)$ we have the box $\mathbf{\Pi}$ of uncertain parameters:

$$\mathbf{\Pi} = \{\mathbf{p} \ : \ p_{ij}^l \in [\alpha_{ij}, \beta_{ij}^l], i \in \underline{m}, j \in \underline{r_i}, l = 0, \cdots, d^o(P_{ij})\}.$$

Then we write

$$\begin{aligned}\delta(s,\mathbf{p}) &= F_1(s)P_{11}(s,\mathbf{p})\cdots P_{1r_1}(s,\mathbf{p}) + \cdots \\ &\quad + F_m P_{m1}(s,\mathbf{p})\cdots P_{mr_m}(s,\mathbf{p}).\end{aligned}$$

Let $\mathbf{\Delta}$ denote the family of polynomials generated by the map $\mathbf{p} \longrightarrow \delta(s)$ and obtained by letting the parameter vector $\mathbf{p}$ range over the box $\mathbf{\Pi}$:

$$\mathbf{\Delta} = \{\delta(s,\mathbf{p}) \mid \mathbf{p} \in \mathbf{\Pi}\}.$$

- $K_{ij}^k(s), k = 1,2,3,4$ : K-polynomials associated with $P_{ij}(s)$.
- Kharitonov segments for $P_{ij}(s)$:

$$S_{ij}^1 = [K_{ij}^1(s), K_{ij}^2(s)] \qquad S_{ij}^2 = [K_{ij}^1(s), K_{ij}^3(s)]$$
$$S_{ij}^3 = [K_{ij}^2(s), K_{ij}^4(s)] \qquad S_{ij}^4 = [K_{ij}^3(s), K_{ij}^4(s)]$$

- $\mathbf{\Delta}_l$ is the polynomial manifold which contains all polynomials of the form

$$\begin{aligned}\delta(s) &= F_1(s)K_{11}^{i(i,1)}(s)K_{12}^{i(1,2)}(s)\cdots K_{1r_1}^{i(1,r_1)}(s)+\cdots \\ &\quad +F_l(s)S_{l1}^{i(l,1)}(\lambda_1,s)\cdots S_{lr_1}^{i(l,r_1)}(\lambda_{r_l},s)+\cdots \\ &\quad +F_m(s)K_{m1}^{i(m,1)}(s)K_{m2}^{i(m,2)}(s)\cdots K_{mr_m}^{i(m,r_m)}(s).\end{aligned}$$

  Equivalently,

$$\mathbf{\Delta}_l := \{\delta(s,\mathbf{p}) \mid \mathbf{p} \in \mathbf{\Pi}_l\}.$$

- $\mathbf{\Pi}_{\text{CB}} := \cup_{l=1}^{m}\mathbf{\Pi}_l$.
- $\mathbf{\Delta}_{\text{CB}} := \cup_{l=1}^{m}\mathbf{\Delta}_l = \{\delta(s,\mathbf{p}) \mid \mathbf{p} \in \mathbf{\Pi}_{\text{CB}}\}$.

**Remark** It is not difficult to see that the number of distinct manifolds in $\mathbf{\Pi}_{\text{CB}}$ or $\mathbf{\Delta}_{\text{CB}}$ is

$$m4^R, \qquad R = r_1 + r_2 + \cdots + r_m.$$

## 6.2. Multlinear CB Theorem

(Chapellat, Keel and Bhattacharyya 91[10])
For a given fixed set of polynomials $F_i(s), i = 1, m$, let

$$\underline{F} := \left[ F_1(s) \;\; F_2(s) \;\; \cdots \;\; F_m(s) \right].$$

We shall say that $\underline{F}$ stabilizes the family $\mathbf{\Pi}$ iff each polynomial of the family $\mathbf{\Delta}$ is Hurwitz stable. Similarly, we shall say that $\underline{F}$ stabilizes $\mathbf{\Pi}_{\mathrm{CB}}$ iff every polynomial in $\mathbf{\Delta}_{\mathrm{CB}}$ is Hurwitz stable.

**Theorem** $\underline{F}$ stabilizes $\mathbf{\Pi}$ iff $\underline{F}$ stabilizes $\underline{\mathbf{\Pi}}_{\mathrm{CB}}$.

**Remark** The theorem is stated assuming that the polynomials $P_{ij}(s)$ perturb independently. In an interconnected multiloop control system it can happen that some of the polynomials $P_{ij}(s)$ are in fact identical. It is however easy to take this dependence into account.

## 6.3. Extremal Properties

(Bhattacharyya and Keel 92[11])

The CB segments and manifolds enjoy many extremal properties. In other words the points closest to instability over the entire uncertainty set in fact occurs on the CB subsets. Since these segments and manifolds form a highly reduced dimensional subset of the original uncertainty set the CB subsets provide efficient means of computing stability and performance margins for interval control systems. We briefly explore these here.

### *6.3.1. Parametric Stability Margin*

**Problem:** Given a family of polynomials $\mathbf{\Pi}$ which is stable, we wish to know the "distance" to the closest unstable polynomial as the point $\mathbf{p}$ varies over the box $\mathbf{\Pi}$.

- Let

$$\mathbf{p} := \begin{bmatrix} \underline{p}_{11} & \underline{p}_{12} & \cdots & \underline{p}_{nr_m} \end{bmatrix},$$

denote the $n$ dimensional parameter vector consisting of the ordered set of coefficients of the polynomials $P_{ij}(s)$ and let $\mathbf{p} \in \mathbb{R}^n$ vary in the prescribed box $\mathbf{\Pi}$ specified by the given upper and lower bounds:

$$p_{ij}^l \in [\alpha_{ij}^l, \beta_{ij}^l], \quad l = 0, \cdots d^o(P_{ij}), i \in \underline{m}, j \in \underline{r}_i.$$

- $\mathcal{P}_u$ - the set of points $\mathbf{u} \in \mathbb{R}^n$ for which $\delta(s, \mathbf{u})$ is unstable or loses degree.

- Radius of the stability ball around $\mathbf{p}$.

$$\rho(\mathbf{p}) = \inf_{\mathbf{u} \in \mathcal{P}_u} ||\mathbf{p} - \mathbf{u}||.$$

**Remark** The number $\rho(\mathbf{p})$ serves as the stability margin associated with the point $\mathbf{p}$. If the box $\mathbf{\Pi}$ is stable we can associate a stability margin with each point in $\mathbf{\Pi}$.

**Question:** Is there a point in $\mathbf{\Pi}$ which is closest to instability in the norm $||\cdot||$ and where is it?

Defining a mapping from $\mathbf{\Pi}$ to the set of all positive real numbers:

$$\begin{array}{ccc} \mathbf{\Pi} & \stackrel{\rho}{\Longrightarrow} & \mathcal{R}^+ \backslash \{0\} \\ \mathbf{p} & \Longrightarrow & \rho(\mathbf{p}) \end{array}$$

**Theorem** The function

$$\begin{array}{ccc} \mathbf{\Pi} & \stackrel{\rho}{\Longrightarrow} & \mathcal{R}^+ \backslash \{0\} \\ \mathbf{p} & \Longrightarrow & \rho(\mathbf{p}) \end{array}$$

has a minimum which is reached at a point on the CB manifolds $\mathbf{\Pi}_{CB}$.

## 6.3.2. *Parametric and Unstructured Perturbaitons*

Let

$$g(s) = \frac{\gamma(s)}{\delta(s)}$$

where

$$\begin{aligned}\gamma(s) &= H_1(s)L_{11}(s)L_{12}(s)\cdots L_{1r_1}(s) + \cdots\cdots \\ &\quad + H_m(s)L_{m1}(s)L_{m2}(s)\cdots L_{mr_m}(s)\end{aligned}$$

where the polynomials $H_i(s)$ are fixed and the polynomials $L_{ij}(s)$ are interval polynomials, that is their coefficients vary in a prescribed box $\mathbf{\Lambda}$; the corresponding family of polynomials $\gamma(s)$ is denoted by $\mathbf{\Gamma}$. We suppose as before that

$$\begin{aligned}\delta(s) &:= F_1(s)P_{11}(s)P_{12}(s)\cdots P_{1r_1}(s) + \cdots\cdots \\ &\quad + F_m(s)P_{m1}(s)P_{m2}(s)\cdots P_{mr_m}(s)\end{aligned}$$

where the polynomials $F_i(s)$ are fixed, the polynomials $P_{ij}(s)$ are interval polynomials, with coefficients that vary in the prescribed box $\mathbf{\Pi}$ and the resulting family of polynomials $\delta(s)$ is denoted $\mathbf{\Delta}$.

¿From these polynomial families we form the parametrized family of transfer functions

$$\mathcal{G} = \{\frac{\gamma(s,\mathbf{l})}{\delta(s,\mathbf{p})} | \mathbf{p} \in \mathbf{\Pi}, \text{ and } \mathbf{l} \in \mathbf{\Lambda}\}. \tag{1}$$

To display the dependence of a typical element $g(s)$ of $\mathcal{G}$ on $\mathbf{l}$ and $\mathbf{p}$ we write $g(s,\mathbf{p},\mathbf{l})$.

Define

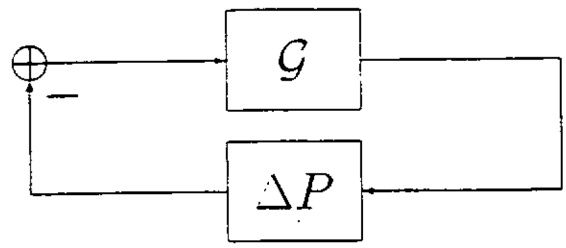
$$\Gamma_{CB} = \{\gamma(s, \mathbf{l}) | \mathbf{l} \in \Lambda_{CB}\}, \Gamma_K = \{\gamma(s, \mathbf{l}) | \mathbf{l} \in \mathbf{K}(\Lambda)\}$$

$$\Delta_{CB} = \{\delta(s, \mathbf{p}) | \mathbf{p} \in \Pi_{CB}\}, \Delta_K = \{\delta(s, \mathbf{p}) | \mathbf{p} \in \mathbf{K}(\Pi)\}.$$

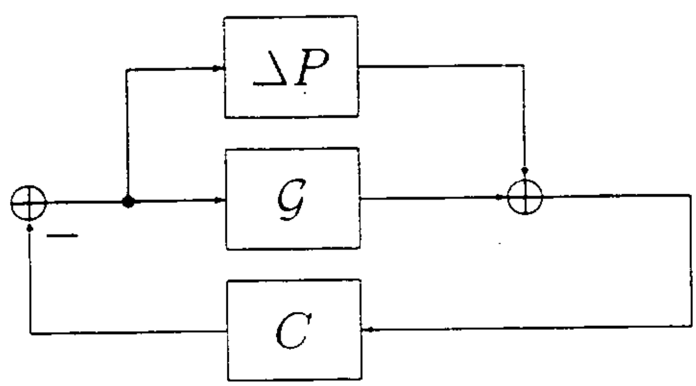
**Problem:** Calculation of the $H_\infty$ stability margin for systems containing parameter uncertainty as defined above.

To determine the unstructured stability margin of the family $\mathcal{G}$ we need to determine the supremum of the $H_\infty$ norm of certain transfer functions over $\mathcal{G}$. Specifically we formulate the following problems: Let $W(s)$ be a scalar stable weight, with a stable inverse, and write $W(s) = \frac{n_w(s)}{d_w(s)}$.

A) *Multiplicative Perturbations:* Consider the feedback configuration shown below. $\mathcal{G}$ is a stable family, and $\Delta P$ is any $H_\infty$ perturbation that satisfies $\|\Delta P\| < \alpha$.




B) *Additive Perturbations:* Consider the feedback configuration shown below. $\Delta P$ is any $H_\infty$ perturbation that satisfies $\|\Delta P\| < \alpha$, and $C$ is a controller that simultaneously stabilizes every element in the set $\mathcal{G}$.



Define

$$\mathcal{G}_{CB} := \{\frac{\gamma(s,\mathbf{l})}{\delta(s,\mathbf{p})} | (\mathbf{l} \in \mathbf{K}(\mathbf{\Lambda}), \mathbf{p} \in \mathbf{\Pi}_{CB}) \text{ or } (\mathbf{l} \in \mathbf{\Lambda}_{CB}, \mathbf{p} \in \mathbf{K}(\mathbf{\Pi})\}.$$

**Theorem (Extremal properties)**

A) $$\sup_{g\in\mathcal{G}} \|Wg\|_\infty = \sup_{g\in\mathcal{G}_{CB}} \|Wg\|_\infty,$$

B) $$\sup_{g\in\mathcal{G}} \|WC(1+gC)^{-1}\|_\infty = \sup_{g\in\mathcal{G}_{CB}} \|WC(1+gC)^{-1}\|_\infty.$$

**Corollary (Unstructured Margins)**

1) The configuration (A) will be stable if and only if $\alpha$ satisfies

$$\alpha \leq \frac{1}{\sup_{g\in\mathcal{G}_{CB}} \|g\|_\infty} := \alpha_o^*.$$

2) The configuration (B) will be stable if and only if $\alpha$ satisfies

$$\alpha \leq \frac{1}{\sup_{g\in\mathcal{G}_{CB}} \|C(1+gC)^{-1}\|_\infty} := \alpha_c^*.$$

**Lemma** Let $h(s) = n(s)/d(s)$ be a proper (real or complex) rational function in $H_\infty(\mathbf{C}_+)$, with $deg(d(s)) = q$. then $\|h\|_\infty < 1$ if and only if

a1) $|n_q| < |d_q|$,

b1) $d(s) + e^{j\theta}n(s)$ is Hurwitz for all $\theta$ in $[0, 2\pi)$.

**Remark** The quantities $\alpha_o^*$ and $\alpha_c^*$ serve as unstructured $H_\infty$ stability margins for the respective open and closed loop parametrized systems treated in Problems A and B.

**Theorem (Extremal properties)**

1) Let $\mathcal{G}$ be the multilinear family defined above, and assume that $\mathcal{G}$ is stable then

$$\inf_{g\in\mathcal{G}} \inf_{\omega\in R} Re(W(j\omega)g(j\omega)) = \inf_{g\in\mathcal{G}_{CB}} \inf_{\omega\in R} Re(W(j\omega)g(j\omega)).$$

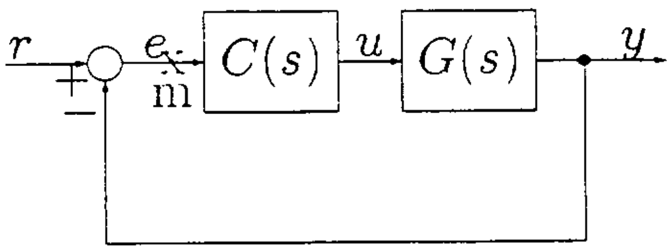
2) If $C$ is a controller that stabilizes the entire family $\mathcal{G}$, then

$$\inf_{g\in\mathcal{G}} \inf_{\omega\in R} Re(WC(1+gC)^{-1}(j\omega)) =$$

$$\inf_{g\in\mathcal{G}_{CB}} \inf_{\omega\in R} Re(WC(1+gC)^{-1}(j\omega)).$$

# 7. Frequency Response of Interval Control Systems

(Keel and Bhattacharyya 91[12]) In this section we deal with the frequency domain properties of interval control systems. Our motivation is that these properties allow us to apply standard results of classical control to the analysis and design of such systems.




## 7.1. Preliminaries

### Interval System

$$\mathcal{G}(s) := \left\{ \tfrac{n(s)}{d(s)} \mid n(s) \in \mathcal{N}(s), d(s) \in \mathcal{D}(s) \right\}$$

where $\mathcal{N}(s)$ and $\mathcal{D}(s)$ are sets of interval polynomials

$$\begin{aligned}
\mathcal{N}(s) &:= \{n(s) \mid n_0 + n_1 s + n_2 s^2 + \cdots + n_q s^q, \\
&\qquad n_i \in [n_i^-, n_i^+], \quad \forall i\} \\
\mathcal{D}(s) &:= \{d(s) \mid d_0 + d_1 s + d_2 s^2 + \cdots + d_q s^q, \\
&\qquad d_i \in [d_i^-, d_i^+], \quad \forall i\}
\end{aligned}$$

$$n(\omega) := n(s)|_{s=j\omega}, \quad d(\omega) := d(s)|_{s=j\omega}, \quad G(\omega) := G(s)|_{s=j\omega}$$

## 7.2. Bode Envelopes

## Magnitude Envelopes

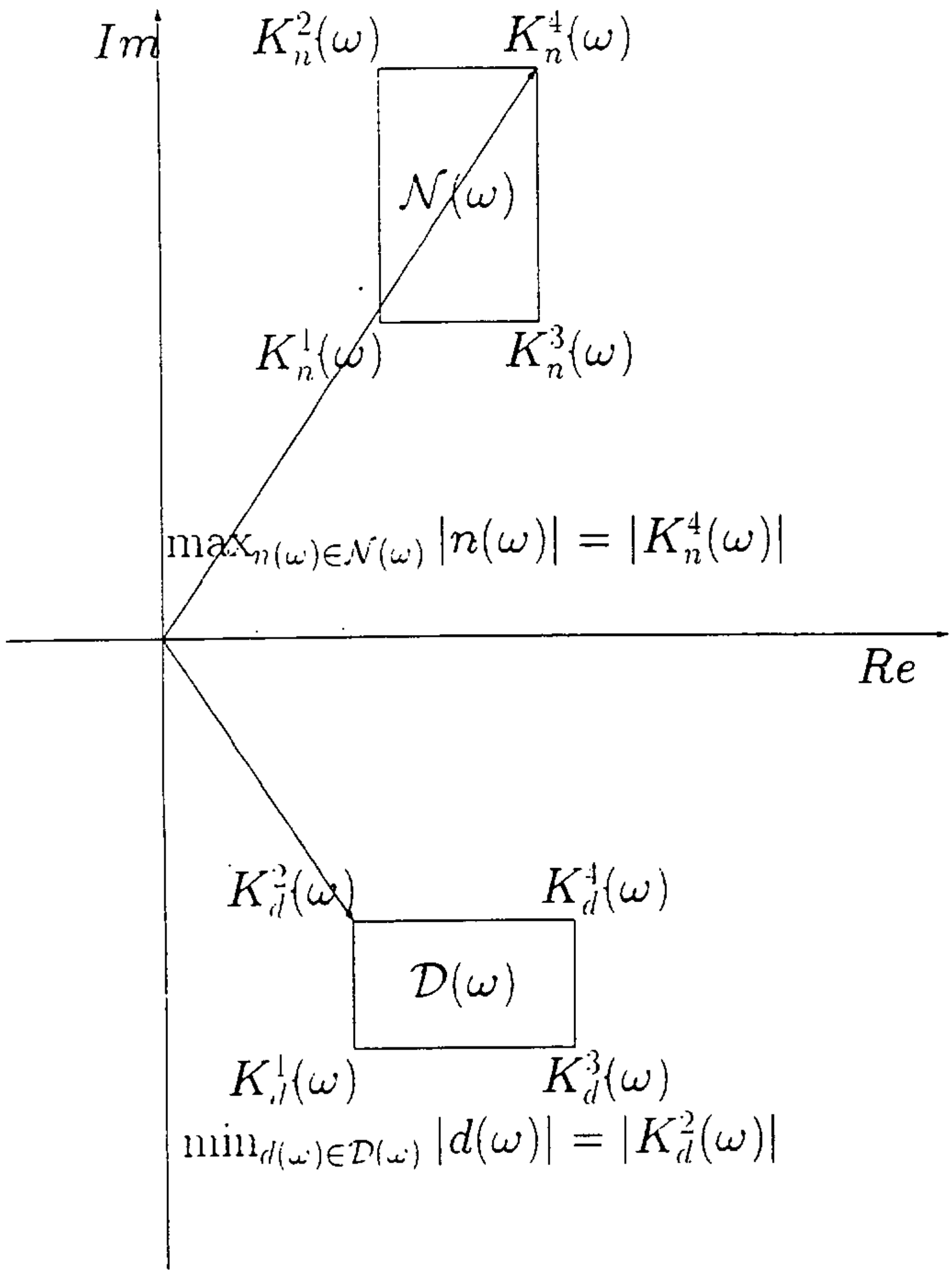
For a transfer function $T(s) \in \mathcal{T}(s)$. Define

$$\begin{aligned} \mu_T(\omega) &:= |T(j\omega)| \\ \bar{\mu}_{\mathcal{T}}(\omega) &:= \sup_{\mathcal{T}(j\omega)} |T(j\omega)| \\ \underline{\mu}_{\mathcal{T}}(\omega) &:= \inf_{\mathcal{T}(j\omega)} |T(j\omega)| \end{aligned}$$

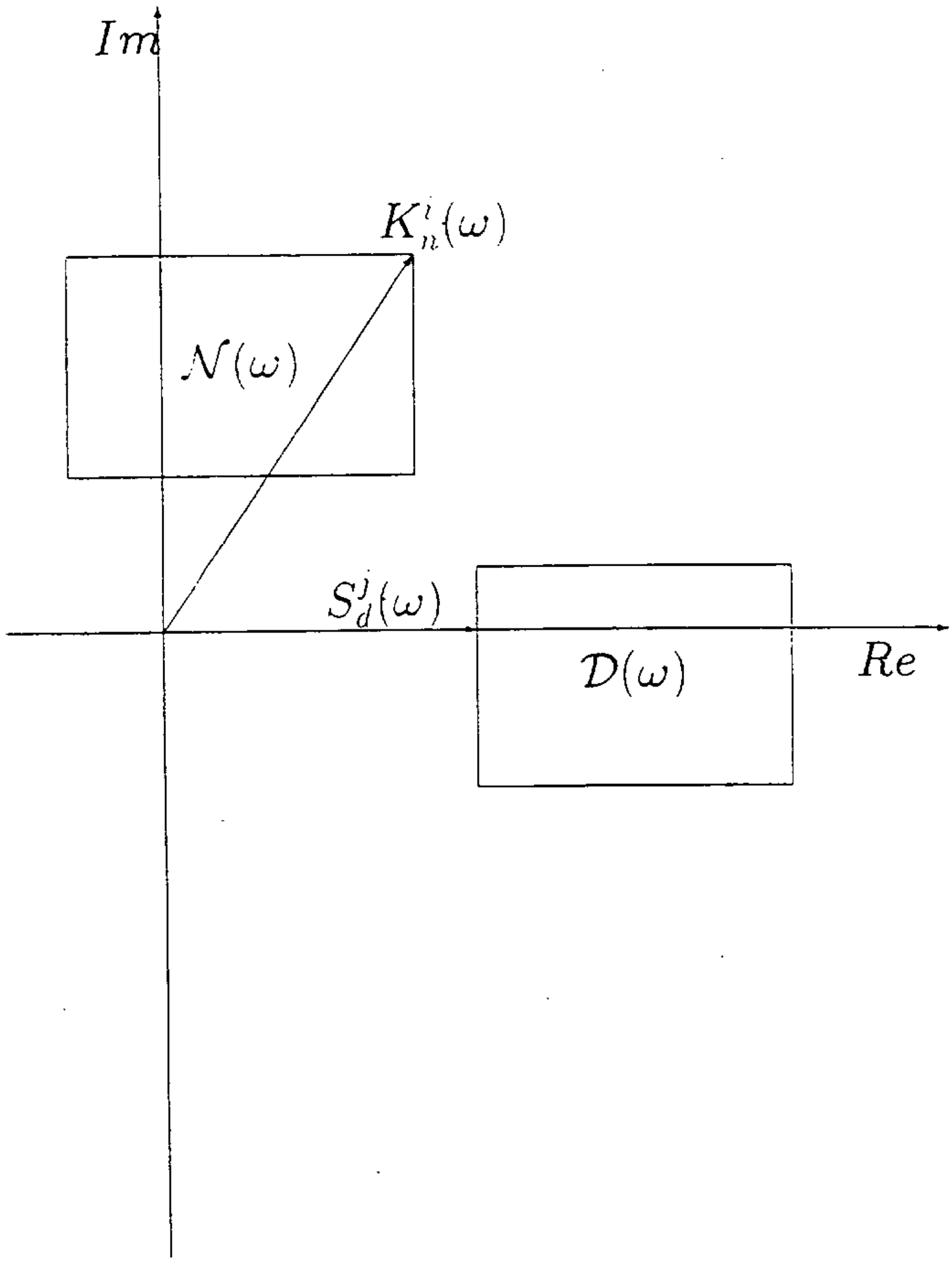
Consequently, for the given family of systems

$$\mathcal{P}(s) := \{P(s) = C(s)G(s) \mid G(s) \in \mathcal{G}(s)\}$$

$$\begin{aligned} \mu_P(\omega) &:= |C(j\omega)G(j\omega)| \\ \bar{\mu}_{\mathcal{P}}(\omega) &:= \sup_{G \in \mathcal{G}} |C(j\omega)|\,|G(j\omega)| \\ \underline{\mu}_{\mathcal{P}}(\omega) &:= \inf_{G \in \mathcal{G}} |C(j\omega)|\,|G(j\omega)| \end{aligned}$$




$$\max_{G(\omega)\in\mathcal{G}(\omega)} |G(\omega)| = \frac{\max_{n(\omega)\in\mathcal{N}(\omega)} |n(\omega)|}{\min_{d(\omega)\in\mathcal{D}(\omega)} |d(\omega)|} = \frac{|K_n^4(\omega)|}{|K_d^2(\omega)|}$$




$$\max_{G(\omega)\in\mathcal{G}(\omega)} |G(\omega)| = \frac{\max_{n(\omega)\in\mathcal{N}(\omega)} |n(\omega)|}{\min_{d(\omega)\in\mathcal{D}(\omega)} |d(\omega)|} = \frac{|K_n^i(\omega)|}{|S_d^j(\omega)|}$$

## Phase Envelopes

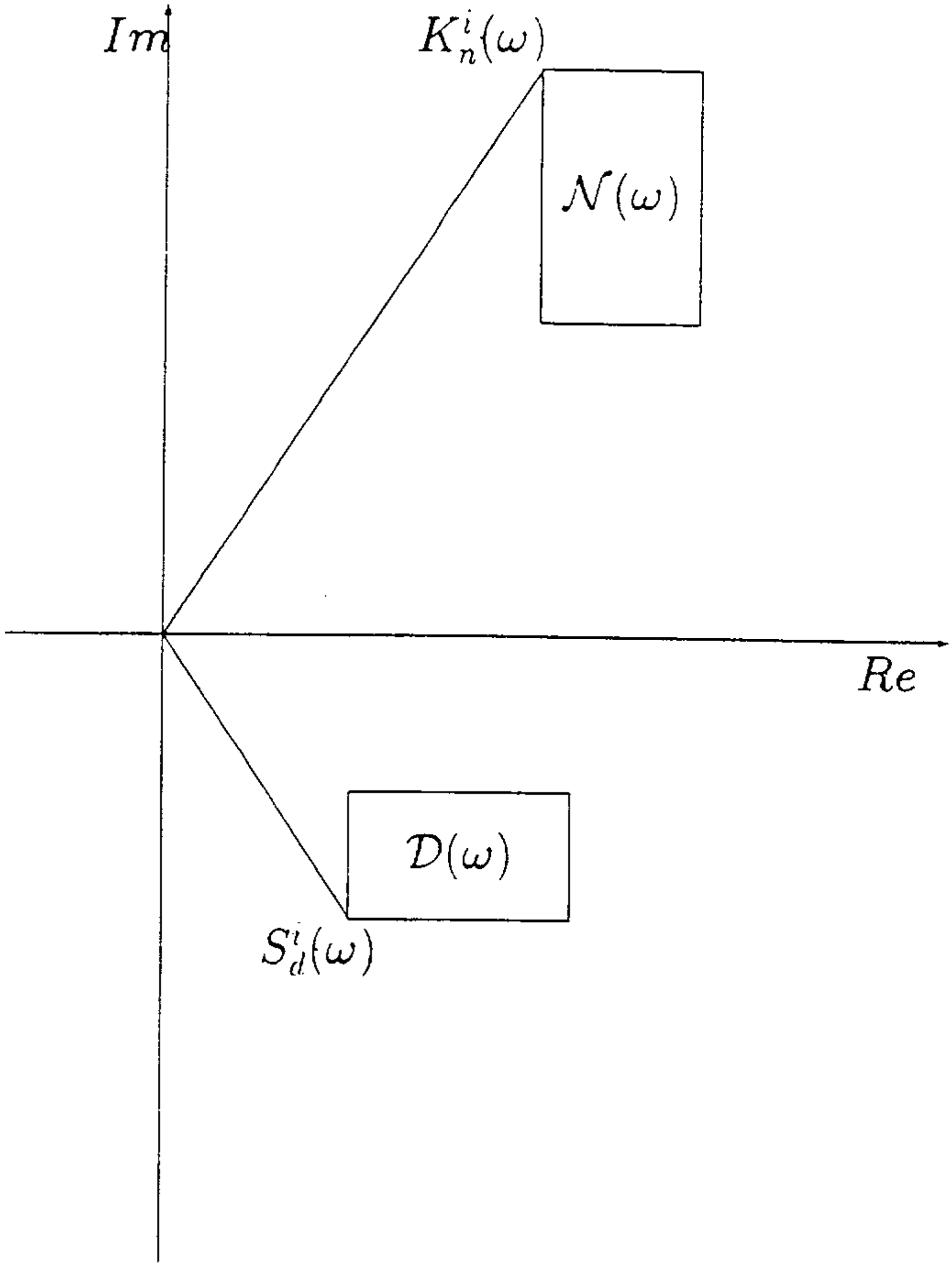
For a transfer function $T(s) \in \mathcal{T}(s)$. Define

$$\begin{aligned} \phi_T(\omega) &:= \angle T(j\omega) \\ \bar{\phi}_{\mathcal{T}}(\omega) &:= \sup_{\mathcal{T}(j\omega)} \angle T(j\omega) \\ \underline{\phi}_{\mathcal{T}}(\omega) &:= \inf_{\mathcal{T}(j\omega)} \angle T(j\omega) \end{aligned}$$

Consequently, for the given family of systems

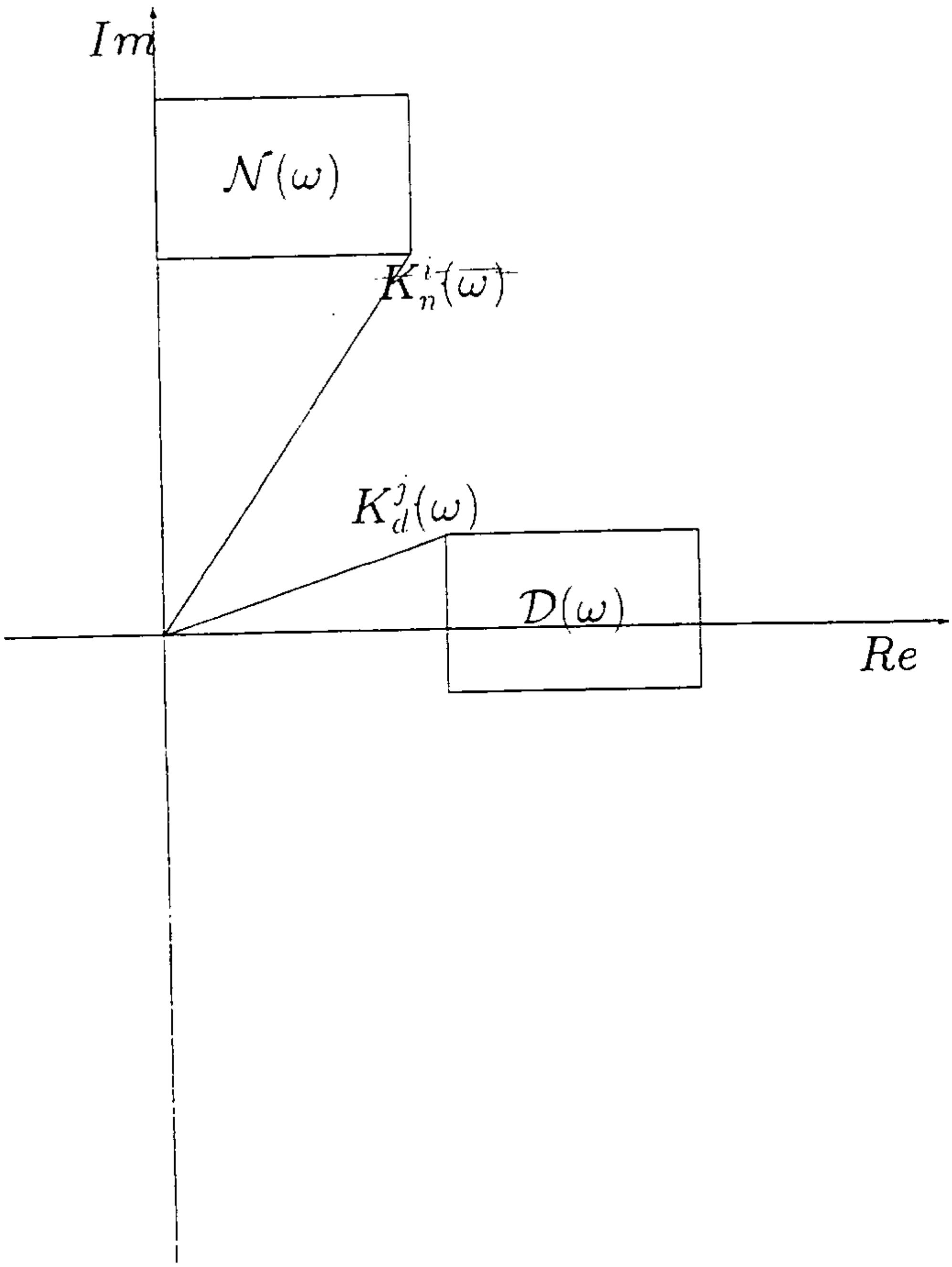
$$\mathcal{P}(s) := \{P(s) = C(s)G(s) \mid G(s) \in \mathcal{G}(s)\}$$

$$\begin{aligned} \phi_P(\omega) &:= \angle C(j\omega)G(j\omega) \\ \bar{\phi}_{\mathcal{P}}(\omega) &:= \sup_{G \in \mathcal{G}} \angle C(j\omega)G(j\omega) \\ \underline{\phi}_{\mathcal{P}}(\omega) &:= \inf_{G \in \mathcal{G}} \angle C(j\omega)G(j\omega) \end{aligned}$$




$$\max_{n(\omega)\in\mathcal{N}(\omega),d(\omega)\in\mathcal{D}(\omega)} \angle G(\omega) = \max_{n(\omega)\in\mathcal{N}(\omega)} \angle n(\omega) - \min_{d(\omega)\in\mathcal{D}(\omega)} \angle d(\omega$$

$$= \angle K_n^i(\omega) - \angle K_d^j(\omega)$$




$$\min_{n(\omega)\in\mathcal{N}(\omega),d(\omega)\in\mathcal{D}(\omega)} \angle G(\omega) = \min_{n(\omega)\in\mathcal{N}(\omega)} \angle n(\omega) - \max_{d(\omega)\in\mathcal{D}(\omega)} \angle d(\omega)$$

$$= \angle K_n^i(\omega) - \angle K_d^j(\omega)$$

Define

$$\mathcal{G}(\omega) := \left\{ G(j\omega) = \tfrac{n(j\omega)}{d(j\omega)} \mid n(j\omega) \in \mathcal{N}(j\omega), \; d(j\omega) \in \mathcal{D}(j\omega) \right\}$$

$$\mathcal{G}_K(\omega) := \left\{ G(j\omega) = \tfrac{n(j\omega)}{d(j\omega)} \mid n(j\omega) \in \mathcal{K}_{\mathcal{N}}(j\omega), \; d(j\omega) \in \mathcal{K}_{\mathcal{D}}(j\omega) \right\}$$

$$\bar{\mathcal{G}}_{CB}(\omega) := \left\{ G(j\omega) = \tfrac{n(j\omega)}{d(j\omega)} \mid n(j\omega) \in \mathcal{K}_{\mathcal{N}}(j\omega), \; d(j\omega) \in \mathcal{S}_{\mathcal{D}}(j\omega) \right\}$$

$$\underline{\mathcal{G}}_{CB}(\omega) := \left\{ G(j\omega) = \tfrac{n(j\omega)}{d(j\omega)} \mid n(j\omega) \in \mathcal{S}_{\mathcal{N}}(j\omega), \; d(j\omega) \in \mathcal{K}_{\mathcal{D}}(j\omega) \right\}$$

For every $\omega > 0$

$$\bar{\mu}_{\mathcal{G}}(\omega) = \bar{\mu}_{\mathcal{G}_{CB}}(\omega)$$

$$\underline{\mu}_{\mathcal{G}}(\omega) = \underline{\mu}_{\mathcal{G}_{CB}}(\omega)$$

$$\bar{\phi}_{\mathcal{G}}(\omega) = \bar{\phi}_{\mathcal{G}_K}(\omega)$$

$$\underline{\phi}_{\mathcal{G}}(\omega) = \underline{\phi}_{\mathcal{G}_K}(\omega)$$

## 7.3. Nyquist Envelopes

Nyquist plots of the family

$$\mathcal{P} = \{C(s)G(s) = \frac{n_c(s)n(s)}{d_c(s)d(s)} \mid n(s) \in \mathcal{N}(s), d(s) \in \mathcal{D}(s)\}$$

are bounded by the Nyquist plots of the following subsets.

$$\{C(s)G(s) \mid G(s) \in \bar{\mathcal{G}}_{CB}(s) \cup \underline{\mathcal{G}}_{CB}(s)\}$$

where

$$\begin{aligned} \bar{\mathcal{G}}_{CB}(s) &= \{G(s) \mid n(s) \in \mathcal{K}_{\mathcal{N}}(s), d(s) \in \mathcal{S}_{\mathcal{D}}(s)\} \\ \underline{\mathcal{G}}_{CB}(s) &= \{G(s) \mid n(s) \in \mathcal{S}_{\mathcal{N}}(s), d(s) \in \mathcal{K}_{\mathcal{D}}(s)\} \end{aligned}$$

### The CB Systems generate the Nyquist plot boundary

For every fixed frequency $\omega \geq 0$, the CB systems

$$\frac{K_n^{i1}(j\omega)}{S_d^j(j\omega)} \qquad \frac{K_n^{i2}(j\omega)}{S_d^j(j\omega)}$$

$$\frac{S_n^i(j\omega)}{K_d^{j1}(j\omega)} \qquad \frac{S_n^i(j\omega)}{K_d^{j2}(j\omega)}$$

$$\Downarrow$$

create a region bounded by two arcs and two segments. By sweeping these with respect to frequency we obtain the Nyquist envelope of the family.

# 8. Design Example: Lead - Lag Compensation

$$G(s) = \frac{a_0}{b_3 s^3 + b_2 s^2 + b_1 s}$$

where its coefficients are bounded by the given intervals as follows:

$$a_0 \in [4, 6] \qquad \begin{aligned} b_3 &\in [.4, .6] \\ b_2 &\in [1.4., 1.6] \\ b_1 &\in [.8, 1.2] \end{aligned}$$

The objective of the design is to achieve that the entire family of systems has the phase margin at least 30° and gain margin at least 20dB.

With the lag compensator

$$C_1(s) = \frac{21.2766s + 1}{280.505s + 1}$$

we have achieved approximately 35° of guaranteed phase margin and 10dB of guaranteed gain margin.

With the additional lead compensator

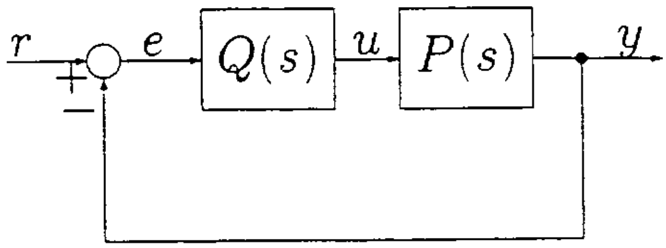
$$C_2(s) = \frac{s. + .4}{s + 2.5}$$

we have achieved approximately 55° of guaranteed phase margin and 22dB of guaranteed gain margin. Therefore, the controller is

$$C(s) = \frac{21.2766s + 1}{280.505s + 1} \frac{s + .4}{s + 2.5}.$$

## 9. Robust Performance Problem

(Keel & Bhattacharyya 91[12, 13])




$$Q(s) := \frac{Q_1(s)}{Q_2(s)} \qquad P(s) := \frac{N(s)}{D(s)}$$

**System Transfer Functions:**

$$\begin{aligned}
T^o(s) &:= \frac{y}{e} = P(s)Q(s) \\
T^y(s) &:= \frac{y}{r} = \frac{P(s)Q(s)}{1+P(s)Q(s)} \\
T^e(s) &:= \frac{e}{r} = \frac{1}{1+P(s)Q(s)} \\
T^u(s) &:= \frac{u}{r} = \frac{Q(s)}{1+P(s)Q(s)}
\end{aligned}$$

**Characteristic Polynomial:**

$$\Pi(s) = Q_2(s)D(s) + Q_1(s)N(s)$$

**Uncertainty Set:**

$$P(s) = \{\frac{N(s)}{D(s)} \; : \; (N(s), D(s)) \in \mathcal{N} \times \mathcal{D}\}$$

**Uncertain Transfer Function Set:** As $P(s)$ ranges over $\mathcal{P}(s)$, equivalently $(N(s), D(s))$ ranges over $\mathcal{N}(s) \times \mathcal{D}(s)$ we have the corresponding uncertain transfer function sets:

$$\begin{aligned}
\mathcal{T}^{o}(s) &:= \{P(s)Q(s) \ : \ P(s) \in \mathcal{P}(s)\} \\
\mathcal{T}^{y}(s) &:= \{\frac{P(s)Q(s)}{1+P(s)Q(s)} \ : \ P(s) \in \mathcal{P}(s)\} \\
\mathcal{T}^{\epsilon}(s) &:= \{\frac{1}{1+P(s)Q(s)} \ : \ P(s) \in \mathcal{P}(s)\} \\
\mathcal{T}^{u}(s) &:= \{\frac{Q(s)}{1+P(s)Q(s)} \ : \ P(s) \in \mathcal{P}(s)\}
\end{aligned}$$

$$\mathbf{\Pi}(s) := \{Q_2(s)D(s) + Q_1(s)N(s) \ : \ (N, D) \in \mathcal{N} \times \mathcal{D}\}$$

**Remark** We are interested in the complex plane image at $s = j\omega$ of the above transfer functions $\mathcal{T}$ and polynomial $\mathbf{\Pi}$. In other words in frequency response. Nyquist plot etc..

- $\mathcal{T}^o(\omega)$, $\mathcal{T}^y(\omega)$, $\mathcal{T}^u(\omega)$, $\mathcal{T}^e(\omega)$ - The complex plane image of each of the above sets. For example.

$$\mathcal{T}^o(\omega) = \{\mathcal{T}^o(s)|_{s=j\omega}\}.$$

- Nyquist plot of a transfer function $T(s)$ is

$$T = \cup_{0\leq\omega<\infty} T(j\omega).$$

- Similarly, for the set of transfer function $\mathcal{T}(s)$ the Nyquist plot is the complex plane set

$$\mathcal{T} = \cup_{0\leq\omega<\infty} \mathcal{T}(j\omega).$$

- The boundary of a set $\mathcal{S}$ - $\partial\mathcal{S}$.

**Remark** We are interested in finding the envelopes of frequency response of $\mathcal{T}^o(s)$. $\mathcal{T}^y(s)$, $\mathcal{T}^u(s)$, $\mathcal{T}^e(s)$.

Define subsets of the uncertain transfer functions:

$$\begin{aligned}
\mathcal{T}^{o}_{\mathrm{CB}}(s) &:= \{P(s)Q(s) \ : \ P(s) \in \mathcal{P}_{\mathrm{CB}}(s)\} \\
\mathcal{T}^{y}_{\mathrm{CB}}(s) &:= \{\frac{P(s)Q(s)}{1+P(s)Q(s)} \ : \ P(s) \in \mathcal{P}_{\mathrm{CB}}(s)\} \\
\mathcal{T}^{e}_{\mathrm{CB}}(s) &:= \{\frac{1}{1+P(s)Q(s)} \ : \ P(s) \in \mathcal{P}_{\mathrm{CB}}(s)\} \\
\mathcal{T}^{u}_{\mathrm{CB}}(s) &:= \{\frac{Q(s)}{1+P(s)Q(s)} \ : \ P(s) \in \mathcal{P}_{\mathrm{CB}}(s)\}
\end{aligned}$$

**Theorem** For every $\omega \geq 0$.

$$\begin{aligned}
\partial\mathcal{P}(\omega) &\subset \mathcal{P}_{\mathrm{CB}}(\omega) \\
\partial\mathcal{T}^{o}(\omega) &\subset \mathcal{T}^{o}_{\mathrm{CB}}(\omega) \\
\partial\mathcal{T}^{y}(\omega) &\subset \mathcal{T}^{y}_{\mathrm{CB}}(\omega) \\
\partial\mathcal{T}^{u}(\omega) &\subset \mathcal{T}^{u}_{\mathrm{CB}}(\omega) \\
\partial\mathcal{T}^{e}(\omega) &\subset \mathcal{T}^{e}_{\mathrm{CB}}(\omega)
\end{aligned}$$

**Theorem**

$$\begin{aligned}
\partial\mathcal{T}^{o} &\subset \mathcal{T}^{o}_{\mathrm{CB}} \\
\partial\mathcal{T}^{y} &\subset \mathcal{T}^{y}_{\mathrm{CB}} \\
\partial\mathcal{T}^{u} &\subset \mathcal{T}^{u}_{\mathrm{CB}} \\
\partial\mathcal{T}^{e} &\subset \mathcal{T}^{e}_{\mathrm{CB}}
\end{aligned}$$

## Remark

- Most of the closed loop transfer functions are no longer interval plants, yet extremality of CB systems still hold.
- All the results extend to the form of

$$P(s) = \frac{A_1(s)N_1(s) + \cdots + A_l(s)N_l(s)}{B_1(s)D_1(s) + \cdots + B_m(s)D_m(s)}$$

where

$$(N_1, \cdots, N_l, D_1, \cdots, D_m) \in \mathcal{N}_1 \times \cdots \times \mathcal{N}_l \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_m$$

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