EXACT POLE RETENTION IN LEAST SQUARES PADÉ REDUCTION OF SISO SYSTEMS

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ABSTRACT - One of the advantages of least squares Padé reduction is that an extra number of constraints can be taken into account to determine a reduced-order model. Because the coefficients of such a model are determined by satisfying the constraints only *approximately*, in a least squares sense, the exact retention of poles cannot be guaranteed in spite of the fact that in some situations the *exact* retention of poles might be of paramount importance. This work investigates the use of a model matching algorithm to achieve *exact* pole retention while computing part of the reduced-order model by least squares Padé techniques. Numerical examples are included to illustrate the new method.

1 INTRODUCTION

The standard model order reduction problem, as it is known nowadays, can be traced back to the sixties and has attracted much attention ever since. The existing methods can be roughly divided into two groups. The first such group includes techniques developed in the time domain (Davison, 1966; Chidambara, 1967; Fossard, 1969). A feature common to these methods is the that the reduced-order model is constrained to retain specified modes of the original system. This has two main advantages, namely i) the simplified model retains fundamental physical properties of the original system such as time constants, and ii) the stability of the final model is guaranteed. These characteristics confer a greater physical meaning upon the models produced by such methods which is usually welcome by most control engineers. The second group includes methods developed in the frequency domain. The most popular methods in this group are those which constrain the reduced-order model to match a finite number of coefficients computed directly from the original model. Such coefficients are often purely mathematical and have no obvious physical interpretation. A few examples include *partial quotients* (Chen and Shieh, 1968) and *time moments* (Gibilaro and Lees, 1969). The main advantage of these techniques is that the algorithms are simple and the simplified models are usually more accurate than those produced by methods in the first group. The main disadvantage, however, is that the reduced-order models may turn out to be unstable. It seems appropriate to conjecture that the appearance of unstable models can be partly explained by the lack of *physical constraints* which is characteristic of most frequency domain methods for order reduction.

In subsequent years, a number of methods were developed in order to overcome the aforementioned stability problem. One of the simplest stability preserving methods consists of retaining dominant poles of the original model (Shamash, 1975). This can be interpreted as an attempt to impose some physical constraints on the reduced-order models derived by frequency domain methods. Other stability preserving methods developed in the frequency domain include the Routh method (Hutton and Friedland, 1975), the stability equation method (Chen *et alii*, 1980) and mixed methods (Pal, 1980). See (Bultheel and van Barel, 1986) for a more complete list.

Two other techniques for model simplification which have received much attention in the literature are the balanced truncation approximation (Moore, 1980; Pernebo and Silverman, 1982) and the Hankel-norm approximation (Glover, 1984). The main advantages of these methods is

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that stability of the reduced-order models is guaranteed and that bounds on the approximation error can be obtained *a priori*. These properties commend these methods for controller design purposes (McFarlane and Glover, 1989). Some difficulties related to these techniques are i) they assume that the original model is asymptotically stable, and ii) if the smallest Hankel singular values of a model are relatively large, then the approximation is not very accurate. Some attempts have been made to minimize such difficulties. For instance, the simplification of unstable systems has been considered in (Yang *et alii*, 1993) where a translation transformation was used in connection with the balancing technique. Enhanced accuracy can be attained over certain frequency ranges by choosing appropriate weighting functions (Latham and Anderson, 1985) which might be important in many situations [chap. 10](Aguirre, 1993b),(Skelton, 1988).

Alternative methods for solving the stability problem in the frequency domain exploit the concept of least-squares Padé approximation. This approach was originally proposed as a solution to the stability problem (Shoji *et alii*, 1985) but it was shown that stability is not mathematically guaranteed by this method (Lucas and Beat, 1990; Lucas and Munro, 1991). However, one of the main advantages is that additional Padé coefficients and Markov parameters are employed in the simplification and consequently better accuracy is usually achieved (Aguirre, 1992a). Another advantage is that the number of constraints used to determine the reduced-order model may be varied, thus a family of simplified models can be easily computed. The best model is then selected using appropriate criteria which should also reflect the needs of the application at hand. Moreover, the concept of least-squares Padé approximation has been recently exploited to overcome problems related to the appearance of purely imaginary poles in squared-magnitude transfer functions (Aguirre, 1994b).

In this paper a procedure is proposed to achieve *exact* pole retention while determining the rest of the coefficients in the reduced-order model by least squares matching of Padé coefficients and/or Markov parameters. This combines three of the aforementioned advantages, namely i) exact pole retention, ii) stability preservation if the original model is stable, and iii) the approximate matching of a relatively large (compared to the classical Padé and mixed methods) number of Padé and Markov parameters. The extension of this method to the multivariable case has been considered elsewhere (Aguirre and Mendes, 1994).

The outline of the paper is as follows. The necessity of developing special algorithms for least-squares Padé approximation with *exact* pole retention is discussed in the following section. The new procedure is based on an algorithm previously suggested for open-loop model matching problems and is reviewed in § 3. The new procedure is described in § 4. The problem of selecting the dominant poles of a given model is briefly addressed in § 5. Numerical examples are provided in § 6. Final remarks and conclusions are given in § 7.

2 STATEMENT OF PROBLEM

In many cases of model simplification via moment matching, the retention of only one or two poles is sufficient to guarantee stability. Consequently, not only the numerator polynomial but also some of the denominator coefficients are 'free variables' which can be used to improve accuracy by matching some more moments. Reduced-order models obtained in this way are in general more accurate than those obtained by mixed methods (where usually only the numerator is determined by moment matching) because a larger number of time-moments is matched. The appeal of this property is reflected by a number of papers in which the principal aim is to allow the matching of one or at most two *extra* moments in order to increase accuracy (Singh, 1981; Lepschy and Viaro, 1982; Alexandro, 1984).

The chief claim of the least squares Padé method of model reduction is that better accuracy is sometimes possible if an even greater number of constraints is used. In such cases, since the number of constraints exceeds the number of free parameters, the resulting overdetermined set of equations has to be solved by standard least squares techniques. Consequently, all the constraints used, that is moment matching and pole retention constraints, are taken into account but are satisfied only approximately despite the fact that in many applications it is desirable that poles of the original system be retained exactly in the reduced-order model.

Clearly, it is desirable to develop an algorithm which would retain such poles (the fixed part of the simplified model) exactly whilst using the moment matching constraints to determine the rest of the coefficients (the variable part of the reduced-order model). When the number of constraints equals that of coefficients, the solution to this problem is trivial and well known.

The main point of this paper is to show that by stating the aforementioned problem as a controller design problem instead of a model simplification problem, an existing algorithm, originally developed for performing open-loop model matching, can be satisfactorily used. Conversely, the model matching problem can also be stated as a partial model reduction problem.

Therefore, the model reduction problem stated above will be solved by using constraints which will guarantee exact pole locations even when the resulting set of equations is overdetermined. Such constraints will be taken from the algorithm described in the next section.

The two main differences between this paper and the reference (Aguirre, 1992b) are as follows i) in this paper, model reduction is the main focus whilst controller design was the main issue in the latter reference, and ii) overdetermined systems are solved in this study to gain further flexibility and enhance accuracy whilst in the aforementioned reference the number of constraints always equaled the number of coefficients thus yielding exact moment matching. An additional difference is that in problems of model reduction with pole retention, the issue of which poles to retain arises. This important matter, which is absent in controller design

problems, is dealt with in some detail in section 5.

3 THE MODEL MATCHING ALGORITHM

Consider figure 1 where the transfer functions are

$$G(s) = \frac{g_0 + g_1 s + \ldots + g_q s^q}{h_0 + h_1 s + \ldots + h_n s^n} \quad , \tag{1}$$

$$D(s) = \frac{a_0 + a_1 s + \ldots + a_p s^p}{b_0 + b_1 s + \ldots + b_m s^m} \quad . \tag{2}$$

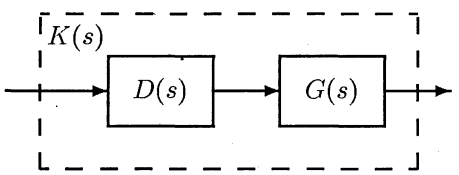

Figure 1 - Open-loop model matching schematic diagram.

The open-loop model matching problem consists in finding a transfer function $D(s)$ such that $D(j\omega)G(j\omega) = K(j\omega)$ when the plant $G(s)$ and a reference model $K(s)$ are given. This can be achieved by moment matching in which case the solution is, in a sense, approximate, that is $D(j\omega)G(j\omega) \approx K(j\omega)$. Before presenting an algorithm to solve this approximate model matching problem, a few definitions are required.

The Padé coefficients of a transfer function, say $G(s)$, are defined as

$$\begin{cases} c_0 = g_0 / h_0, \\ c_k = \left[g_k - \sum_{j=1}^{k} h_j \; c_{k-j} \right] / h_0, \quad k > 0 \; , \end{cases} \tag{3}$$

where $g_i = 0$ for $i > q$, $h_i = 0$ for $i > n$. The numbers $\{c_i\}_{i=0}^{\infty} \in \mathbb{R}$, known as Padé coefficients, are the coefficients of the Taylor expansion of $G(s)$ around the point $s = 0$ and are proportional to the time-moments. Conversely, the Markov parameters are defined as

$$\begin{cases} m_0 = g_n / h_n, \\ m_k = \left[g_{n-k} - \sum_{j=1}^{k} h_{n-j} \; m_{k-j} \right] / h_n, \quad k > 0 \; , \end{cases} \tag{4}$$

where the numbers $\{m_i\}_{i=0}^{\infty} \in \mathbb{R}$ are the coefficients of the Taylor expansion of $G(s)$ around the point $s = \infty$.

Lemma 3.1
(Open-loop approximate model matching). The transfer function $D(s)$ which will make $D(s)G(s)$ match the prespecified set of P Padé coefficients and M Markov parameters $\{c_0, c_1, \ldots, c_{P-1}, m_\nu, m_{\nu+1}, \ldots, m_{\nu+M-1}\}$, where $P+M = p+m+1$, is given by the solution of the following set of linear equations

$$\begin{cases} y_0 c_0 = x_0, \\ y_0 c_k = x_k - \sum_{j=1}^{k} y_j c_{k-j}, \quad k = 1, 2, \ldots, P-1, \\ 1 = y_{n+m}, \\ m_k = x_{n+m-k} - \sum_{j=1}^{k} y_{n+m-j} m_{k-j}, \quad k = \nu, \nu+1, \ldots, \\ \qquad\qquad\qquad\qquad\qquad\qquad \nu+M-1, \end{cases} \tag{5}$$

where, c_i is the ith Padé coefficient of a given reference model $K(s)$, m_i is the ith Markov parameter of $K(s)$, m_ν is the first non-zero Markov parameter of $K(s)$ and

$$x_i = \sum_{j=0}^{i} a_j \; g_{i-j}, \qquad i = 0, 1, ..., q+p, \tag{6}$$

$$y_i = \sum_{j=0}^{i} b_j \; h_{i-j}, \qquad i = 0, 1, ..., n+m \; . \tag{7}$$

Proof.

See (Aguirre, 1992b). □

4 ORDER-REDUCTION PROCEDURE

The model matching problem, as stated above, is different from the conventional model reduction problem (where the entire simplified model is to be computed) because only one part, namely $D(s)$, of the open-loop transfer function $D(s)G(s)$ is to be determined and $G(s)$ is known *a priori*.

On the other hand, the model reduction problem of concern in this paper is similar to the controller design problem because only one part of the reduced model is to be determined and the other part is assumed to have been previously chosen, namely the poles to be retained. In other words, the desired reduced model is $R(s) = D(s)G(s)$, where $G(s) = 1/P_o(s)$ and the zeros of the polynomial $P_o(s)$ are the poles of the original system, $K(s)$, which should be retained in $R(s)$. Thus least squares Padé reduction with *exact* pole retention can be achieved using the following procedure

Step 1: Choose the poles to be retained and form $P_o(s)$

Step 2: Compute Padé and Markov parameters from the original model $K(s)$ using equations (3) and (4)

Step 3: Form a set of $P+M+1 \geq p+m+2$ linear equations from (5)

Step 4: Determine the coefficients of $D(s)$ by solving the resulting set of equations by the least squares method

Step 5: The desired simplified model is
$R(s)=k \times D(s)/P_o(s)$,
where k is such that $k \times D(0)/P_o(0)=K(0)$

Remark 4.1 The last step in the procedure above guarantees that the reduced-order model matches the steady-state gain of the original model.

Remark 4.2 If $P+M+1=p+m+2$ in step 3, this procedure is equivalent to classical techniques which match Padé and Markov parameters exactly. Note that, by choosing different values of P and M for which the inequality in step 3 holds, a family of reduced-order models is obtained. The best model can then be chosen by using quality or robustness criteria. It should be emphasized that having to choose a performance criterion in order to select the best candidate of a family of models need not be seen as a limitation of the method. On the contrary, in many practical situations this may confer greater flexibility to the procedure and may also provide the means by which the best model for a certain application can be selected via practical and meaningful engineering constraints (Lastman and Sinha, 1985; Aguirre, 1994a). In the example provided, the H_∞-norm of the error has been used to select the best simplified model.

Remark 4.3 Since lemma 3.1 holds for rational models, the above procedure can be also used to exactly retain zeros of the original model by taking $G(s) = Z_o(s)/P_o(s)$, where the zeros of the polynomial $Z_o(s)$ are the zeros of the original model which should be retained in the reduced-order model. The retention of zeros is important in the reduction of non-minimum phase systems and in many other situations (Marshall, 1980).

5 SELECTING THE DOMINANT POLES

The retention of poles in reduced-order models is a standard procedure (Shamash, 1975; Kimura, 1983; Krajewski *et alii*, 1990). In the introduction it has been conjectured that imposing constraints for the retention of dominant poles in simplified models usually yields improved results because such constraints are in a sense *physically motivated* and not a mere mathematical exercise.

So far it has been assumed that the poles to be retained in the reduced-order model are known in advance. In practice, however, this will hardly be the case and the identification of the truly dominant poles of a system may not be easy. The common, but rather naïve, approach to this problem is to choose the slowest poles to be the most dominant. It has been pointed out that a slow mode might have a small 'weight' at the output and therefore might not be dominant (Moore, 1980; Aguirre, 1993a).

The choice of which poles to retain in a reduced-order model is crucial because the accuracy of such a model will depend greatly on the *true dominance* of the retained poles. Furthermore, a reduced-order model intended for analysis and design would be of great help if, besides being accurate, it retained those poles which characterized the most important modes of the original system because it is usually required that such poles be confined to a 'safe' region of the s-plane in some design procedures (Liaw and Chao, 1993).

There are some methods which quantify modal dominance and therefore permit the selection of the most dominant poles (Zhao *et alii*, 1981; Ouyang *et alii*, 1987; Aguirre, 1993a). Among these methods, the easiest to implement is the one based on the *modal dominance indices* (MDI) defined as (Aguirre, 1993a)

$$\begin{cases} \gamma_i \doteq -\frac{J_i}{\lambda_i}, & i = 1,2,\ldots,\alpha, \\ \gamma_i \doteq \frac{-[J_{\alpha+l}\lambda^*_{\alpha+l}+J^*_{\alpha+l}\lambda_{\alpha+l}]}{2\lambda_{\alpha+l}\lambda^*_{\alpha+l}}, & l = 1,2,\ldots,\beta, \\ & i = \alpha+2l-1, \alpha+2l \end{cases} , \tag{8}$$

where J_i is the ith residue corresponding to the pole at λ_i, α is the number of real poles and β is the number of conjugate pairs and asterisks denote complex conjugates. It is often useful to express the γ_i as a percentage of the total contribution, that is $\mid \gamma_i \mid \% = \mid \gamma_i \mid \times 100/\Sigma \mid \gamma \mid$, where the summation is taken over all the poles. The MDI can also be readily determined directly from state-space realizations (Aguirre, 1993a), and are invariant with respect to coordinate transformations.

It is interesting to note that the MDI are, roughly speaking, analogous to the Hankel singular values in the sense that both quantities try to quantify aspects of input-output dominance of a system. There is a crucial difference, however. Whilst the MDI essentially quantify the importance of the *time constants* of a system, the Hankel singular values quantify the input-output contribution of *state variables* of a state-space realization. In most cases, the state variables of a linear model are mathematical abstractions and have no obvious physical meaning, this is likely to be true after representing the original model in balanced coordinates. On the other hand, although poles in a transfer function might appear scaled in time, there is often a clear correspondence between some poles in a transfer function and real-life time constants. These characteristics are not, *per se*, necessarily good nor bad as far as model reduction is concerned but have been described to point out fundamental differences between the two approaches. A more rigorous justification of the MDI was given in (Aguirre, 1993a) where, for instance, is was pointed out that the impulse response energy of unrepeated real poles are proportional to the squared value of the respective MDI.

6 NUMERICAL RESULTS

Two numerical examples are discussed in order to illustrate some of the main points of the paper.

Example 6.1

Consider the transfer function of a real combustion control system (Pena *et alii*, 1990)

$$K_1(s) = \frac{7.526s^3 + 10.357s^2 + 0.92051s + 0.63827}{20.0141s^4 + 11.749s^3 + 16.987s^2 + 1.1602s + 1} \quad . \tag{9}$$

This transfer function is rather difficult to simplify because it has two pairs of complex poles and the fastest poles are more dominant according to the MDI. In particular, the poles at $s = -0.2795 \pm j0.8306$ have percentage MDI of $|\ \gamma_1\ |\ \% = 2 \times 48.86\%$ while the MDI for the poles at $s = -0.0141 \pm j0.2547$ are just $|\ \gamma_2\ |\ \% = 2 \times 1.14\%$. Unsurprisingly, the classical Padé method yields an unstable model.

Therefore it is desired that the complex pair of poles at $s = -0.2795 \pm j0.8306$ be exactly retained in the reduced model. In addition, a third-order reduced model with a second-order numerator is desired. Retaining such poles and matching Padé coefficients exactly produces

$$R_{11}(s) = \frac{0.5171s^2 + 0.5433s + 0.0526}{s^3 + 0.6663s^2 + 0.8280s + 0.0825} \quad , \tag{10}$$

with $\| K(j\omega) - R_{11}(j\omega) \|_\infty = 0.42$. The least squares Padé with *approximate* pole retention for $P = 5$ and $M = 1$ yields

$$R_{12}(s) = \frac{0.3525s^2 + 0.0563s + 0.0207}{s^3 + 0.5285s^2 + 0.0790s + 0.0325} \quad , \tag{11}$$

with $\| K(j\omega) - R_{12}(j\omega) \|_\infty = 2.65$.

It is interesting to note that although constraints for the retention of the fastest poles were used, the slowest poles were nearly retained by the least squares Padé method (Aguirre, 1992a). This illustrates the need to retain poles *exactly*. On the other hand, the importance of being able to discern which poles should be retained in a model can be appreciated by noticing that $R_{12}(s)$ has a pair of poles which almost coincides with the slowest poles in the original model. However, because such poles are not dominant, the performance of such a transfer function is inadequate. Indeed, Shamash's method fails to produce a stable model when the slowest poles are retained. This clearly illustrates that the slowest poles are not necessarily dominant.

Finally, the new algorithm was used to obtain a reduced-order model which retained the fastest poles. Thus $p = 2$ and $m = 1$ since $P_o(s)$ is a second-order polynomial. The number of Padé and Markov constraints taken into account, P and M, were varied in the ranges $1 \leq P \leq 10$ and $p+m+1-P \leq M \leq 10$. The best simplified model, chosen according to the H_∞-norm of the error, was

$$R_{13}(s) = \frac{0.45914s^2 + 0.55711s + 0.04947}{0.97056s^3 + 0.64337s^2 + 0.80177s + 0.077507} \tag{12}$$

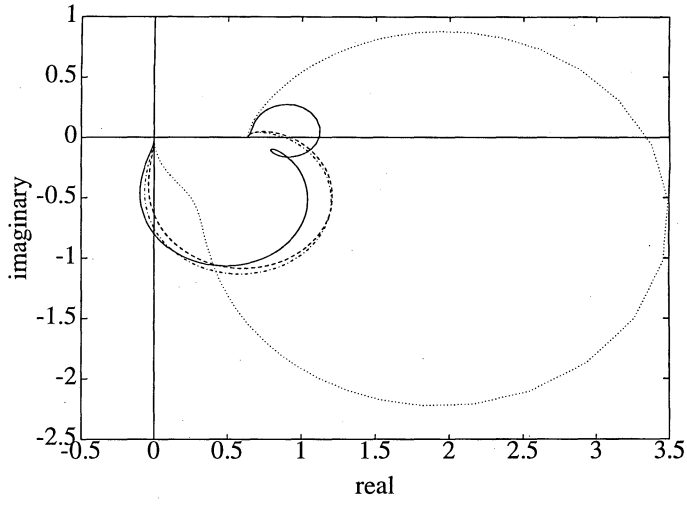

Figure 2 - Nyquist diagrams (—) $K_1(j\omega)$, (- -) $R_{11}(j\omega)$, $(\cdots)$ $R_{12}(j\omega)$ and (-·-) $R_{13}(j\omega)$

for $P = 4$, $M = 3$ and $\| K(j\omega) - R_{13}(j\omega) \|_\infty = 0.38$.

Figure 2 shows the Nyquist diagram of the transfer functions above. It is interesting to note that the plot of $K_1(j\omega)$ has two loops. The smallest one corresponds to the slowest pair of complex poles. Clearly, the model which nearly retains these poles, namely $R_2(s)$, is the best approximant only at very low frequencies, however its overall accuracy is totally inadequate. □

Example 6.2

Consider the non-minimum phase transfer function (El-Attar and Vidyasagar, 1978)

$$K_2(s) = \frac{35.8223s^3 - 120.9286s^2 + 2327.8s - 2863}{s^5 + 9.8s^4 + 162.9s^3 + 872.3s^2 + 4284.3s + 5751.6} \quad . \tag{13}$$

This model has been recently considered in (Al-Saggaf and Bettayeb, 1993) where the following reduced-order models have been given

$$R_{oH}(s) = \frac{-0.1433s^4 + 3.5368s^3 - 19.5681s^2 + 217.4474s - 246.3792}{s^4 + 5.0870s^3 + 94.8419s^2 + 270.1647s + 694.665} \quad , \tag{14}$$

$$R_{EV}(s) = \frac{33.8059s^2 - 7.1888s + 2330.9}{s^4 + 10.974s^3 + 227.7036s^2 + 1018.5s + 10154.5} \quad . \tag{15}$$

The former reduced-order model, $R_{oH}(s)$, was obtained using the optimal Hankel approximation whereas $R_{EV}(s)$ was derived using uniform approximation for all L_1- and L_∞-inputs as described in (El-Attar and Vidyasagar, 1978). The fourth- and third-order models obtained via the balancing technique described in (Moore, 1980) are

$$R_{b4}(s) = \frac{2.0576s^3 - 5.2754s^2 + 132.9445s - 127.3847}{s^4 + 2.6190s^3 + 89.2455s^2 + 143.9681s + 603.0204} \quad (16)$$

and

$$R_{b3}(s) = \frac{0.7928s^2 + 0.8533s - 1.6371}{s^3 + 0.9969s^2 + 15.0708s + 1.6521}, \quad (17)$$

respectively. Retaining the pole with largest MDI, namely $s = -1.8$ and using the same criterion as in the previous example to choose the best candidate, the following model was obtained using the method described in this paper for $P=6$ and $M=0$

$$R_{21}(s) = \frac{17.9781s^2 + 14.5190s - 48.2358}{s^4 + 9.5961s^3 + 51.7925s^2 + 121.8022s + 96.9029} . \quad (18)$$

The approximation can be improved further by *exactly* retaining the three non-minimum phase zeros of the original model $K_2(s)$ in addition to the dominant pole at $s=-1.8$. In so doing the following model was obtained for $P=5$ and $M=0$

$$R_{22}(s) = \frac{4.9250s^3 - 16.6256s^2 + 320.0331s - 393.6140}{0.9954s^4 + 22.1367s^3 + 119.7580s^2 + 588.9464s + 790.7477} . \quad (19)$$

Finally, the following third-order model was obtained for $P=4$ and $M=0$ and which also retains the dominant pole at $s=-1.8$

$$R_{23}(s) = \frac{18.5465s - 23.7864}{s^3 + 7.7119s^2 + 37.1887s + 47.7856} . \quad (20)$$

The step responses of these models are shown in figure 3. It should be noted that because the responses of $R_{21}(s)$ and $R_{23}(s)$ are very similar, only the latter was included in the figure.

Based on this figure, the following remarks can be made. The reduced-order model $R_{EV}(s)$ seems the most accurate at high frequencies but presents unacceptable mismatch at low frequencies. The most accurate model in the range of medium frequencies is clearly $R_{oH}(s)$ but this model also presents mismatches at both high frequencies (because it is proper transfer function while the original model is strictly proper) and low frequencies. The only models which match the dynamics at low frequencies are $R_{22}(s)$ and $R_{23}(s)$, see remark 4.1, which also seem to tradeoff satisfactorily the approximation at high and medium frequencies. The most inaccurate models have been produced by the 'plain' balancing approach.

It is noted that the accuracy of $R_{oH}(s)$ over the mid-frequency range is a direct result of a rather elaborate procedure which involves choosing a frequency weighting function $W(s)$, obtaining unstable and stable projections

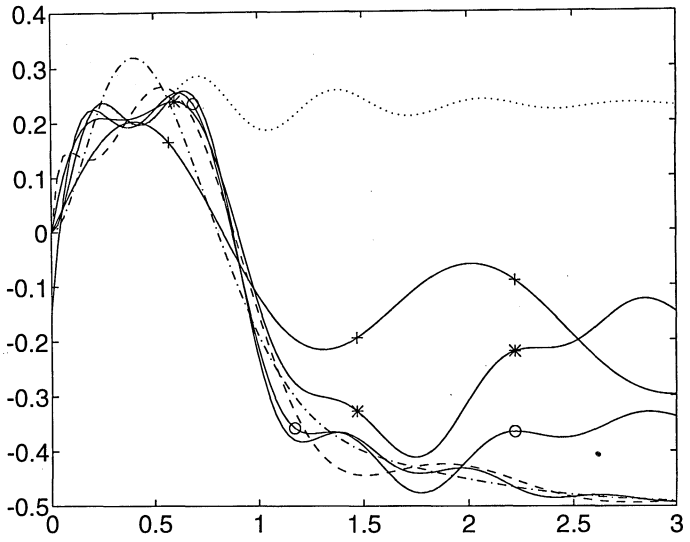

Figure 3 - Step response of models (—) $K_2(s)$, (-o-) $R_{oH}(s)$, (···) $R_{EV}(s)$, (-*-) $R_{b4}(s)$, (-+-) $R_{b3}(s)$, (- -) $R_{22}(s)$ and (-·-) $R_{23}(s)$.

of $W(s)$ and of $K_2(s)$, respectively, performing an optimal Hankel-norm approximation and finally obtaining the reduced-order model by performing a decomposition involving four rational functions.

It is worth pointing out that $R_{21}(s)$, $R_{23}(s)$ and $R_{EV}(s)$ are the only models which retained the pole-zero difference of the original model. This is usually considered relevant in some practical situations (Marshall, 1980). The other reduced-order models have roll-off rates which are different from the original model. This would introduce difficulties in situations where the noise amplification is important and also in closed-loop applications with high loop gain (Marshall, 1980; Aguirre, 1993b). □

7 FINAL REMARKS AND CONCLUSIONS

It should be noted that for all combinations for which $P+M=p+m+1$ holds, *exact* pole *and* coefficient matching is achieved. When $P+M > p+m+1$ however, the Padé and Markov parameters are *approximately* matched but the selected poles (the zeros of $P_o(s)$) are still *exactly* retained. This confers greater meaning to the reduced-order model since the retained poles provide a physical link with the original system. Moreover, since all the poles of the reduced-order model can be exactly placed stability is ultimately guaranteed by specifying the entire reduced denominator. In most practical situations, however, the retention of one or two truly dominant poles suffices to yield a stable model, but this cannot be, of course, mathematically guaranteed unless all the poles in the simplified model are placed in the left half plane.

It is noted that four Padé constraints, three Markov constraints and two pole retention constraints were used in the determination of six coefficients in example 6.1 and that five Padé, one pole and three zero retention constraints were

used to determine eight coefficients of $R_{22}(s)$. This illustrates that further improvement may be achieved by taking into account some extra Padé coefficients and Markov parameters and matching them in a least squares sense.

If the sequences of Padé coefficients and Markov parameters display *explosive* behavior, and if too many extra coefficients are taken into account in a certain application, the problem may become numerically ill-conditioned. A solution to this problem has been suggested in (Lucas and Beat, 1990) but this was unnecessary in the many examples tried.

Step 5 of the procedure in §3 is needed to match the steady-state gain of the original model. Thus, in reality, the procedure guarantees exact pole retention *and* exact steady-state agreement.

A procedure has been suggested for the simplification of transfer functions. The main features of the new method are i) it takes into account additional information via extra Padé and Markov parameters which are *approximately* matched using least squares techniques, ii) poles of the original system are matched *exactly* and therefore iii) stability is guaranteed, iv) the methods in (Shamash, 1975; Shoji *et alii*, 1985; Lucas and Beat, 1990; Lucas and Munro, 1991; Aguirre, 1992a) are special cases of the procedure presented in this paper which v) uses a simple and computer-oriented algorithm which can also be used in open-loop controller design (Aguirre, 1992a) problems, and vi) a family of simplified models (all of which retain the same specified poles) may be obtained by varying two parameters within limited ranges and the best model can be selected using a cost function.

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