A TUTORIAL INTRODUCTION TO NONLINEAR DYNAMICS AND CHAOS, PART I: TOOLS AND BENCHMARKS

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ABSTRACT - The relevance of nonlinear dynamics and chaos in science and engineering cannot be overemphasized. Unfortunately, the enormous wealth of techniques for the analysis of linear systems which is currently available is totally inadequate for handling nonlinear systems in a systematic, consistent and global way. This paper presents a brief introduction to some of the main concepts and tools used in the analysis of nonlinear dynamical system and chaos which have been currently used in the literature. The main objective is to present a readable introduction to the subject and provide several references for further reading. A number of well known and well documented nonlinear models are also included. Such models can be used as benchmarks not only for testing some of the tools described in this paper, but also for developing and troubleshooting other algorithms in the comprehensive fields of identification, analysis and control of nonlinear dynamical systems. Some of these aspects will be addressed in a companion paper which follows.

1 INTRODUCTION

An important step towards the analysis of real systems is to realize that virtually all systems are in a sense *dynamical.* This does not mean to say, of course, that the dynamics of every system are always necessarily relevant to the analysis. Thus although it is sometimes justifiable to regard certain systems as being *static*, in most cases it is worthwhile taking into account the dynamics inherent in the systems to be analyzed.

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Mathematically, dynamical systems are described by differential equations in continuous-time and by difference equations in discrete-time. On the other hand, most static systems are, of course, described by algebraic equations.

A second step in the analysis of real dynamical systems is to take into consideration the nonlinearities, which are often as important to the system as the dynamics. The use of linear models in science and engineering has always been common practice. A good linear model, however, describes the dynamics of the a system only in the neighborhood of the particular operating point for which such a model was derived. The need for a broader picture of the dynamics of real systems has prompted the development and use of dynamical models which included the nonlinear interactions observed in practice.

In this paper a few basic concepts related to nonlinear dynamical systems are briefly reviewed. The objective is twofold, namely to provide a brief introduction to nonlinear dynamics and chaos and to indicate a few basic references which can be used as a starting point for a more detailed study on this subject. In particular, this paper will describe a few mathematical tools such as Poincaré sections, bifurcation diagrams, Lyapunov exponents and correlation dimensions. Also, some of the most commonly used nonlinear models which display chaotic behavior will be presented thus providing the reader with both, a few basic tools and well established benchmarks for testing such tools.

The ambit of the techniques developed for nonlinear systems with chaotic dynamics can be appreciated by considering the wide range of examples in which chaos has been found. Different types of mathematical equations exhibit chaotic solutions, for instance ordinary differential equations, partial differential equations (Abhyankar *et alii*, 1993), continued fractions (Corless, 1992) and delay equations (Farmer, 1982).

Chaos is also quite common in many fields of control systems such as nonlinear feedback systems (Baillieul *et alii*, 1980; Genesio and Tesi, 1991), adaptive control (Mareels and Bitmead, 1986; Mareels and Bitmead, 1988; Golden and Ydstie, 1992) and digital control systems (Ushio and Hirai, 1983; Ushio and Hsu, 1987)

Chaos seems to be the rule rather than the exception in many nonlinear mechanical and electrical oscillators and pendula (Blackburn *et alii*, 1987; Hasler, 1987; Matsumoto, 1987; Ketema, 1991; Kleczka *et alii*, 1992).

Chaos, fractals and nonlinear dynamics are common in some aspects of human physiology (Mackey and Glass, 1977; Glass *et alii*, 1987; Goldberger *et alii*, 1990), population dynamics (May, 1987; Hassell *et alii*, 1991), ecology and epidemiology (May, 1980; Schaffer, 1985), and the solar system (Wisdom, 1987; Kern, 1992; Sussman and Wisdom, 1992).

Models of electrical systems have been found to exhibit chaotic dynamics. A few examples include DC-DC converters (Hamill *et alii*, 1992), digital filters (Lin and Chua, 1991; Ogorzalek, 1992), power electronic regulators (Tse, 1994), microelectronics (Van Buskirk and Jeffries, 1985), robotics (Varghese *et alii*, 1991) and power system models (Abed *et alii*, 1993).

There seems to be some evidence of low dimensional chaos in time series recorded from electroencephalogram (Babloyantz *et alii*, 1985; Babloyantz, 1986; Layne *et alii*, 1986) although such results are so far inconclusive. Other areas where there has been much debate concerning the possibility of chaotic dynamics are economics (Boldrin, 1992; Jaditz and Sayers, 1993) and the climate (Lorenz, 1963; Elgar and Kadtke, 1993).

Many other examples in which chaos has apparently been diagnosed include the models of a rotor blade lag (Flowers and Tongue, 1992), force impacting systems (Foale and Bishop, 1992), belt conveyors (Harrison, 1992), neural systems (Harth, 1983), biological networks (Lewis and Glass, 1991), spacecraft attitude control systems (Piper and Kwatny, 1991) and friction force (Wojewoda *et alii*, 1992), to mention a few.

An advantage of focusing on chaotic systems is that chaos is ubiquitous in nature, science and engineering. Thus simple systems which exhibit chaos commend themselves as valuable paradigms and benchmarks for developing and testing new concepts and algorithms which in principle would apply to a much wider class of problems. Therefore most of the tools and concepts reviewed in this paper are also very relevant to systems which display regular dynamics.

In the companion paper the tools and systems described here will be used in the identification, analysis and control of nonlinear dynamics and chaos.

2 NONLINEAR DYNAMICS: CONCEPTS AND TOOLS

This section provides some concepts and tools for the analysis of nonlinear dynamics. Some of the tools considered in this section currently constitute active fields of research *per se*. Although no attempt has been made to give a thorough treatment on such issues, a considerable number of references has been included for further reading.

2.1 Differential and difference equations

An nth-order continuous-time system can be described by the differential equation

$$\frac{dy}{dt} = \dot{y} = f(y,t) \ , \qquad (1)$$

where $y(t) \in \mathbb{R}^n$ is the *state* at time t and $f : \mathbb{R}^r \rightarrow \mathbb{R}^n$ is a smooth function called the *vector field*. f is said to generate a *flow* $\phi_t : \mathbb{R}^r \rightarrow \mathbb{R}^n$, where $\phi_t(y,t)$ is a smooth function which satisfies the group properties $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$, and $\phi(y,0) = y$.

Given an *initial condition*, $y_0 \in \mathbb{R}^n$ and a time t_0, a *trajectory*, *orbit* or *solution* of equation (1) passing through (or based at) y_0 at time t_0 is denoted as $\phi_t(y_0, t_0)$.

Because the time is explicit in equation (1), f is said to be *non-autonomous*. Conversely, systems in which the vector field does not contain time explicitly are called *autonomous*.

A system is said to be *time periodic* with period T if $f(y,t) = f(y,t+T)$, $\forall y,t$. An nth-order non-autonomous system with period T can be converted into an $(n+1)$th-order autonomous system by adding an extra state $\theta = 2/T$ in which case the state space will be transformed from the Euclidean space $\mathbb{R}^{n+1}$ to the *cylindrical space* $\mathbb{R}^n \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/T$ is the circle of length $T = 2\pi/\omega$. It is noted that all the non-autonomous systems considered in this work will be time periodic in most situations.

A *Fixed point* of f or *equilibrium*, $\bar{y}$, is defined as $f(\bar{y}) = 0$ for continuous-time systems and as $\bar{y} = f(\bar{y})$ for discrete-time systems. $\mathrm{D}f$ is the *Jacobian* matrix of the system, defined as the matrix of first partial derivatives. Evaluating the Jacobian at a particular point on a trajectory of the system, that is $\mathrm{D}f(y_i)$ gives a local approximation of the vector field f in the neighborhood of y_i, sometimes $\mathrm{D}f(y_i)$ is referred to as a *linearization* of f at y_i. If $\mathrm{D}f(\bar{y})$ has no zero or purely imaginary eigenvalues, then the eigenvalues of this matrix characterize the stability of the fixed point $\bar{y}$.

An nth-order discrete-time system can be described by a difference equation of the form

$$y(k+1) = f(y(k), t) \quad . \qquad (2)$$

A *trajectory* or *orbit* of a discrete system is a set of points $\{y(k+1)\}_{k=0}^{\infty}$. The definitions for discrete systems are analogous to the ones described for continuous-time systems and therefore will be omitted. For details see (Guckenheimer and Holmes, 1983; Parker and Chua, 1989; Wiggins, 1990).

2.2 Numerical simulation of dynamical systems

Generating time series for a system described by a difference equation is quite straightforward since $y(k)$, $k = n_y, n_y+1, n_y+2, \ldots$ can be computed by simply computing repeatedly an equation like (2) from a set of n_y initial conditions.

If the system is described by an ordinary differential equation, simulation cannot be performed as easily since an equation like (1) should be integrated. Fortunately, there are a number of well know algorithms available for performing this task such as Euler, trapezoidal, Runge-Kutta, Adams-Bashforth, Adams-Moulton and Gear's algorithm (Parker and Chua, 1989). The fourth-order Runge-Kutta is undoubtedly the most commonly used algorithm for integrating ordinary differential equations.

An important question when integrating differential equations on a digital computer is the choice of the integration interval. In the case of linear systems or nonlinear systems with relatively slow dynamics the choice of the integration interval is not usually critical. For some nonlinear systems, however, if such an interval is not sufficiently short spurious chaotic regimes may be induced when integrating the system using, for instance, a fourth-order Runge-Kutta algorithm, whilst second-order Runge-Kutta algorithms may induce spurious dynamics even for fairly short integration intervals (Grantham and Athalye, 1990). It has also been reported that in some cases the location of the bifurcation points depend on the integration interval if it exceeds a critical value (Aguirre and Billings, 1994a).

Irrespective of the type of the dynamical equations or the algorithm used to solve such equations, an important question which should be answered is whether the simulated results are representative of the 'real solution'. This is a nontrivial matter, and to address it would involve a detailed look into the *shadowing lemma* (Guckenheimer and Holmes, 1983). For the purposes of this tutorial, it suffices to mention that there is abundant evidence that computer simulations are generally reliable as numerical tools for the analysis of dynamical systems (Sauer and Yorke, 1991). However, it should also be borne in mind that pitfalls exist (Troparevsky, 1992), some of them as a consequence of the extreme sensitivity to initial conditions exhibited by some systems. This characteristic is one of the most peculiar features of a chaotic system and will be briefly illustrated in section 2.9. Extreme sensitivity to initial conditions does not invalidate numerical computations but certainly calls for caution in analyzing the results.

2.3 Spectral methods

One of the first tools used in diagnosing chaos was the power spectrum (Mees and Sparrow, 1981). The appearance of a broad spectrum of frequencies of highly structured humps near the low-order resonances is usually credited to chaos in low-order systems (Blacher and Perdang, 1981). However, broad-band noise and the existence of phase coherence can make it difficult to discriminate experimentally between chaotic and periodic behavior by means of power spectrum (Farmer *et alii*, 1980). More recently the *raw spectrum* (sum of the absolute values of the real and imaginary components) and the *log spectrum* (log of the raw spectrum) have been compared with more classical techniques in the context of chaotic time series analysis (Denton and Diamond, 1991).

Recently, the application of spectral techniques to the analysis of chaotic systems has concentrated on the bispectrum and trispectrum (Pezeshki *et alii*, 1990; Subba Rao, 1992; Chandran *et alii*, 1993; Elgar and Chandran, 1993; Elgar and Kennedy, 1993). See (Nikias and Mendel, 1993; Nikias and Petropulu, 1993) for an introduction on higher-order spectral analysis. Such techniques have been used to detect and, to a certain extent, to quantify the energy transfer among frequency modes in chaotic systems.

2.4 Embedded trajectories

One technique used in the analysis of nonlinear dynamical systems is to plot a steady-state trajectory of a system in the phase-space. Thus if $y(t)$ is a trajectory of a given system this can be achieved by plotting $\dot{y}(t)$ against $y(t)$. For low-order systems this procedure can be used to distinguish between different dynamical regimes.

In many practical situations, however, only one variable is measured. In these cases an alternative procedure is to plot $y(t-T_{\rm p})$ against $y(t)$ where $T_{\rm p}$ is a time lag. These variables can be used in the reconstruction of attractors (Packard *et alii*, 1980; Takens, 1980) and such variables also define the so-called pseudo-phase plane. This is motivated by the fact that $y(t-T_{\rm p})$ is, in a way, related to $\dot{y}(t)$ and consequently the embedded trajectories represented in the pseudo-phase plane should have properties similar to those of the original attractor represented in the phase plane (Moon, 1987).

A further advantage of this technique is that it enables the comparison of trajectories computed from continuous systems where $\dot{y}(t)$ is usually available, and from discrete models where $\dot{y}(t)$ is often not available and would have to be estimated.

The choice of $T_{\rm p}$ for graphical representation purposes is not critical and plotting a trajectory onto the pseudo-phase plane for varying values of $T_{\rm p}$ may give some insight regarding the information flow on the attractor (Fraser and Swinney, 1986).

Phase portraits and plots of trajectory embeddings can be used not only as a means of distinguishing different dynamical regimes, but also to demonstrate qualitative relationships between original and reconstructed attractors.

2.5 Dynamical attractors

If a deterministic and stable system is simulated for a sufficiently long time it reaches *steady-state*. In state space this corresponds to the trajectories of the system falling on a particular 'object' which is called the *attractor*. Asymptotically stable linear systems excited by constant inputs have point attractors which have dimension zero and correspond to a constant time series.

Nonlinear systems, on the other hand, usually display a wealth of possible attractors. To which attractor the system will finally settle depends on the system itself and also on the initial conditions.

An advantage of considering attractors in state space as alternative representations of time series is that a number of geometrical and topological results can be used. For the purposes of this tutorial, it will suffice to point out that the *shape* and *dimension* of the attractors in state space are directly linked to the complexity of the dynamics of the respective time series. Thus simple low dimensional attractors correspond to simple time series dynamics.

The most common attractors are the *point attractor* (dimension zero), *limit cycles* (dimension one) and *tori* (dimension two). Another type of attractor which has recently attracted a great deal of attention are the so-called *strange* or *chaotic attractors* which are *fractal* objects. The determination of the dimension of such attractors will be briefly addressed in section 2.11.

2.6 Bifurcation diagrams

Another useful tool for assessing the characteristics of the steady-state solutions of a system over a range of parameter values is the bifurcation diagram which reveals how the system bifurcates as a certain parameter, called the bifurcation parameter, is varied. Roughly, a system is said to undergo a bifurcation when there is a qualitative change in the trajectory of the system as the bifurcation parameter is varied. At the bifurcation point, the Jacobian of the system has at least one eigenvalue with the real part equal to zero for continuous-time systems or on the unit circle for discrete-time systems.

There are a number of known bifurcations. The most common co-dimension one bifurcations are the *pitchfork*, the *saddle-node*, the *transcritical*, the *Hopf* bifurcation, and the *flip* or *period doubling*, which only occur in discrete maps or periodically driven systems. For an introduction to bifurcation and a description of the aforementioned types see (Guckenheimer and Holmes, 1983; Mees, 1983; Thompson and Stewart, 1986) .

Approaches to calculate bifurcation diagrams include the *brute force*, *path following* (Parker and Chua, 1989), the *cell-to-cell mapping* technique (Hsu, 1987) and frequency domain methods (Moiola and Chen, 1993). For reasons of simplicity, the brute force approach is described in what follows. This approach is simple and robust but in general it is computationally intensive.

Thus a point r of a bifurcation diagram of a nonautonomous systems driven by $A\ \cos(\omega\, t)$ with A as the bifurcation parameter is defined as

$$r = \{\ (y, A)\ \in\ \mathbb{R}\times I\ |\ y = y(t_i),\ A = A_0;\quad t_i = t_\circ + K_{\rm ss}\times 2\pi/\omega\}\ \ , \tag{3}$$

where I is the interval $I=[A_{\rm i}\ A_{\rm f}]\subset\mathbb{R}$, $0\leq t_\circ\leq 2\pi/\omega$ and $K_{\rm ss}$ is a constant. This means that the point r is chosen by simulating the system for a sufficiently long time $K_{\rm ss}\times 2\pi/\omega$ with $A=A_0$ to ensure that transients have died out before plotting $y(K_{\rm ss}\times 2\pi/\omega)$ against A_0. In practice for each value of the parameter A, $n_{\rm b}$ points are taken at the instants

$$t_i = t_\circ + (K_{\rm ss} +\ i\ )\times 2\pi/\omega, \qquad i = 0, 1, ..., n_{\rm b}-1\ . \tag{4}$$

Clearly, the input frequency ω can also be used as a bifurcation parameter. For autonomous systems a bifurcation diagram can be obtained in an analogous way by choosing

$$t_i = t_\circ + (K_{\rm ss} +\ i), \qquad i = 0, 1, ..., n_{\rm b}-1\ . \tag{5}$$

A bifurcation diagram will therefore reveal at which values of the parameter $A\in I$ the solution of the system bifurcates and how it bifurcates. When studying chaos such diagrams are also useful in detecting parameter ranges for which the system behavior is chaotic.

As an example of a bifurcation diagram consider figure 1. This diagram and the respective system will be described in some detain in section 4.5. Throughout this tutorial, bifurcation parameters are denoted by A. Thus, figure 1 shows some of the different types of attractors displayed by the system as A is varied. In particular, for $A = 4.5$, 9 and 11 the system displays period-one, period-three and chaotic dynamics, respectively. For clarity the respective attractors represented in the cylindrical state-space (see section 2.1) are also shown.

2.7 Poincaré sections

A bifurcation diagram shows the different types of attractors to which the system settles to as the bifurcation parameter is varied. However, a bifurcation diagram provides very little information concerning the shape of the attractors in state-space. In order to gain further insight into the geometry of attractors one may use the so-called Poincaré maps. Such a map is a cross section of the attractor and can be obtained by defining a plane which should be transversal to the flow in state space as shown in figure 2.

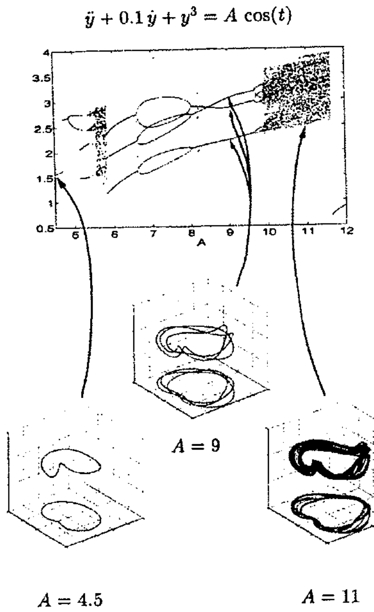


Figure 1 - Bifurcation diagram for the Duffing-Ueda oscillator, see section 4.5. A, the amplitude of the input, is the bifurcation parameter.

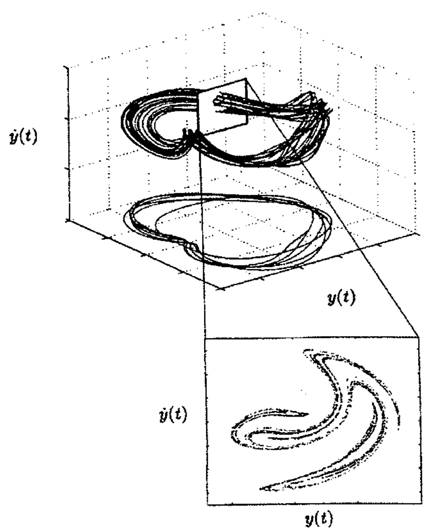


Figure 2 - A Poincaré section is obtained by defining a plane in state space which is transversal to the flow. The image formed on such a plane is the Poincaré section of the attractor and will display fractal structure if such an attractor is chaotic.

More precisely, consider a periodic orbit γ of some flow ϕ_t in $\mathbb{R}^n$ arising from a nonlinear vector field. Let $\Sigma \subset \mathbb{R}^n$ be a hypersurface of dimension $n-1$ which is transverse to the flow ϕ_t. Thus the first return or Poincaré map $\mathbf{P} = \Sigma \rightarrow \Sigma$ is defined for a point $q \in \Sigma$ by

$$\mathbf{P}(q) = \phi_{\tau_p}(q) \ , \qquad (6)$$

where τ_p is the time taken for the orbit $\phi_t(q)$ based at q to first return to Σ.

This map is very useful in the analysis of nonlinear systems since it takes place in a space which is of lower dimension than the actual system. It is therefore easy to see that a fixed point of $\mathbf{P}$ corresponds to a periodic orbit of period $2\pi/\omega$ for the flow. Similarly, a subharmonic of period $K \times 2\pi/\omega$ will appear as K fixed points of $\mathbf{P}$. Quasiperiodic and chaotic regimes can also be readily recognized using Poincaré maps. For instance, the first-return map of a chaotic solution is formed by a well-defined and finely-structured set of points for noise-free dissipative systems. Such maps ca be used in the validation of identified models and reconstructed attractors (Aguirre and Billings, 1994b).

From the above definition it is clear that if a system has $n > 3$, the Poincaré map would require more than two dimensions for a graphical presentation. In order to restrict the plots to two-dimensional figures, $y(t - T_p)$ is plotted against $y(t)$ at a constant period. For periodically driven systems the input period is a natural choice and the resulting plot is called a Poincaré section.

This procedure amounts to defining the Poincaré plane Σ_p in the pseudo-phase-space and then sampling the orbit represented in such a space. The choice of T_p is not critical but it should not be chosen to be too small nor too large compared to the correlation time of the trajectory. Otherwise the geometry and fine structure of the attractor would not be well represented. The qualitative information conveyed by both Poincaré maps and sections are equivalent as demonstrated by the theory of embeddings (Takens, 1980; Sauer *et alii*, 1991).

Although the Poincaré sections are usually obtained by means of numerical simulation, it is possible, although not always feasible, to determine Poincaré maps analytically (Guckenheimer and Holmes, 1983; Brown and Chua, 1993).

2.8 Routes to chaos

In the study of chaotic systems it is somewhat instructive to consider the different routes to chaos in order to gain further insight about the dynamics of the system under investigation. As pointed out "the benefit in identifying a particular prechaos pattern of motion with one of these now *classic* models is that a body of mathematical work on each exists which may offer better understanding of the chaotic phenomenon under study" (Moon, 1987,page 62).

Because a thorough study of the routes to chaos is beyond the immediate scope of this work, some of the most well-known patterns will be listed with some references for further reading. Some of the routes to chaos reported in the literature include *period doubling cascade* (Feigenbaum, 1983; Wiesenfeld, 1989), *quasi-periodic route to chaos* (Moon, 1987), *itermittency* (Manneville and Pomeau, 1980; Kadanoff, 1983), *frequency locking* (Swinney, 1983). For other routes to chaos see (Robinson, 1982) and references therein.

2.9 Sensitivity to initial conditions

Probably the most fundamental property of chaotic systems is the sensitive dependence on initial conditions. This feature arises due to the local divergence of trajectories in state space in at least one 'direction'. This will be also addressed in the next section.

In order to illustrate sensitivity to initial conditions and one of its main consequences, it will be helpful to consider the map

$$y(k) = A\,[1 - y(k-1)]\,y(k-1)\ . \qquad (7)$$

In order to iterate equation (7) on a digital computer, an initial condition $y(0)$ is required. Using this value, the right hand side of equation (7) can be evaluated for any value of A. This produces $y(1)$ which should be 'fedback' and used as the initial condition in the following iteration. This procedure can be then repeated as many times as necessary to generate a time series $y(0), y(1), y(2), \ldots$.

A graphical way of seeing this is illustrated in figure 3. It should be noted that the right hand side of equation (7) is a parabola, as shown in figure 3a. Thus to evaluate equation (7) is equivalent to find the value on the parabola which corresponds to the initial condition. This is represented in figure 3a by the first vertical line. The feeding back of the new value is then represented by projecting the value found on the parabola on the bisector. This completes one iteration.

Choosing the initial condition $y(0) = 0.22$ and $A = 2.6$, figure 3a shows the iterative procedure and reveals that after a few iterations the equation settles to a point attractor. The respective time series is shown in figure 3b. The same procedure was followed for the same initial condition and $A = 3.9$. The results are shown in figures 3c-d. Clearly, the equation does not settle onto any fixed point and not even onto a limit cycle. In fact, it is known that equation (7) displays chaos for $A = 3.9$.

What happens if instead of a single initial condition an interval of initial conditions is iterated? This is shown in figures 3e-f. For $A = 2.6$, the map will eventually settle to the same point attractor as before. This is a typical result for regular stable systems and it illustrates how all the trajectories based on the initial conditions taken from the original interval converge to the same attractor.

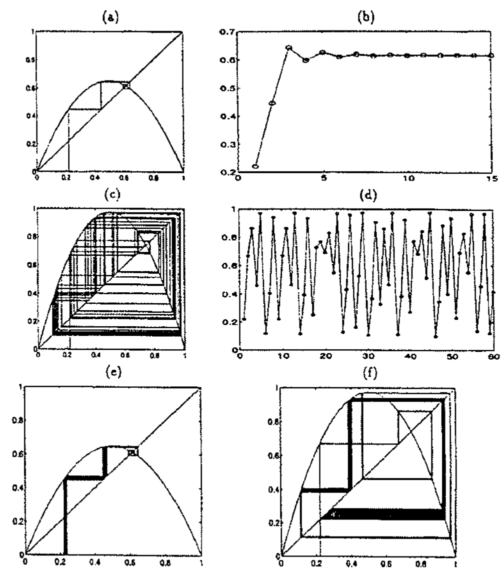


Figure 3 - Graphical iteration of the logistic equation (7) (a) regular motion ($A = 2.6$) and (b) respective time series. (c) chaotic motion ($A = 3.9$), and (d) respective time series. In these figures the same initial condition has been used, namely $y(0) = 0.22$. In figures (e) and (f) an interval of initial conditions has been iterated for the same values of A as above. The intervals used were $y(0) \in [0.22\ 0.24]$ and $y(0) \in [0.220\ 0.221]$, respectively. Note how such an interval is amplified when the system is chaotic, (f). This is due to the sensitive dependence on initial conditions.

Considering a much narrower interval of initial conditions and proceeding as before yielded the results shown in figure 3f for which $A = 3.9$. Clearly, the interval of initial conditions was widened at each iteration. Such an interval can be interpreted as an error in the original initial condition, $y(0) = 0.22$. In practice errors in initial conditions will be always present due to a number of factors such as noise, digitalization effects, round-off errors, finite wordlength precision, etc. It is this effect of amplifying errors in initial conditions which is known as the sensitive dependence on initial conditions and an immediate consequence of this feature is the impossibility of making long-term prediction for chaotic systems. The next section describes indices which quantify the sensitivity to initial conditions.

2.10 Lyapunov exponents

Lyapunov exponents measure the average divergence of nearby trajectories along certain 'directions' in state space. A chaotic attracting set has at least one positive Lyapunov exponent and no Lyapunov exponent of a non-chaotic attracting set can be positive. Consequently such exponents have been used as a criterion to determine if a given attracting set is or is not chaotic (Wolf, 1986). Recently the concept of *local* Lyapunov exponents has been investigated (Abarbanel, 1992). The local exponents describe orbit instabilities a fixed number of steps ahead rather than an infinite number. The (global) Lyapunov exponents of an attracting set of length N can be defined as [1]

[1] Many authors use $\log_2$ in this definition

$$\lambda_i = \lim_{N\to\infty} \frac{1}{N} \log_e j_i(N), \qquad i = 1, 2, ..., n \ , \qquad (8)$$

where $\log_e = \ln$ and the $\{j_i(N)\}_{i=1}^{n}$ are the absolute values of the eigenvalues of

$$[\mathrm{D}f(y_N)][\mathrm{D}f(y_{N-1})]...[\mathrm{D}f(y_1)] \ , \qquad (9)$$

where $\mathrm{D}f(y_i) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of the n-dimensional differential equation (or discrete map) evaluated at y_i, and $\{y_k\}_{k=1}^{N}$ is a trajectory on the attractor. Note that n is the dynamical order of the system.

In many situations the reconstructed or identified models may have a dimension which is larger than that of the original systems and therefore such models have more Lyapunov exponents. These 'extra' exponents are called *spurious Lyapunov exponents.* The estimation of Lyapunov exponents is known to be a nontrivial task. The simplest algorithms (Wolf *et alii*, 1985; Moon, 1987) can only reliably estimate the largest Lyapunov exponent (Vastano and Kostelich, 1986). Estimating the entire spectrum is a typically ill-conditioned problem and requires more sophisticated algorithms (Parker and Chua, 1989). Further problems arise when it comes to deciding which of the estimated exponents are *true* and which are *spurious* (Stoop and Parisi, 1991; Parlitz, 1992; Abarbanel, 1992). The estimation of Lyapunov exponents is currently an active field of research as can be verified from the following references (Sano and Sawada, 1985; Eckmann *et alii*, 1986; Bryant *et alii*, 1990; Brown *et alii*, 1991; Parlitz, 1992; Kadtke *et alii*, 1993; Nicolis and Nicolis, 1993; Chialina *et alii*, 1994). For application of Lyapunov exponents in the quantification of real data see (Brandstäter *et alii*, 1983; Wolf and Bessoir, 1991; Vastano and Kostelich, 1986).

In view of such difficulties and the fact that the largest Lyapunov exponent, λ_1, is in many cases the only positive exponent[2] and that this gives an indication of how far into the future accurate predictions can be made, it seems appropriate to use λ_1 to characterize a chaotic attracting set (Rosenstein *et alii*, 1993). Indeed, the largest Lyapunov exponent has been used in this way and to compare several identified models (Abarbanel *et alii*, 1989; Abarbanel *et alii*, 1990; Principe *et alii*, 1992).

The algorithm suggested in (Moon, 1987) for estimating λ_1 is described below. A similar algorithm which simultaneously estimates the correlation dimension to be defined in section 2.11 has been recently investigated in (Rosenstein *et alii*, 1993).

Consider a point x_0 on the trajectory $x(k)$ (for the moment it is assumed that such a trajectory is available *a priori*), say $x_0 = x(0)$, and a nearby point $x_0+\delta_0$. For simplicity it is assumed that $x(k) \in \mathbb{R}$, but in general higher-dimensional systems will be the case. The largest Lyapunov exponent of an attracting set of length N can be defined as (see also equation (8))

$$\lambda_1 \doteq \frac{1}{N} \lim_{N\to\infty} \sum_{k=1}^{N} \log_e \frac{\| \delta_{k+1} \|}{\| \delta_k \|} \ , \qquad (10)$$

where δ_k is the distance between two points on nearby trajectories at time k. The estimation of λ_1 is a simulation-based calculation (Moon, 1987; Parker and Chua, 1989). The main idea is to be able to determine the ratio

$$\frac{\| x_1 - (x_1 + \delta_1) \|}{\| x_0 - (x_0 + \delta_0) \|} = \frac{\| \delta_1 \|}{\| \delta_0 \|} \ , \qquad (11)$$

where x_1 is another point on the trajectory $x(k)$, namely $x(\Delta L)$, $x_1+\delta_1$ is a point obtained by following the evolution of the randomly chosen initial condition $x_0 + \delta_0$ over the interval ΔL where ΔL will be referred to as the Lyapunov interval.

From the last equation it is clear that one only needs to follow the evolution of perturbations δ_i along the reference trajectory $x(k)$. It is well known that the Jacobian matrix $\mathrm{D}f(x_i)$ describes the dynamics of the system for small perturbations in the neighborhood of x_i. Thus the computation of the largest Lyapunov exponent, λ_1, consists in solving the variational equations

$$\dot{\delta} = \mathrm{D}f(x_i)\,\delta \ , \qquad (12)$$

where $\mathrm{D}f(x_i)$ is the Jacobian matrix of $f(\cdot)$ evaluated at x_i, and also of simulating the system

$$\dot{x} = f(x) \qquad (13)$$

if the trajectory $x(k) = \{x_i\}_{i=0}^{N}$ is not available in advance. Equations (12) and (13) are simulated and the ratio $\| \delta_{k+1} \|$ / $\| \delta_k \|$ is calculated once at each ΔL interval. Therefore estimating λ_1 consists in successively predicting the systems governed by $\mathrm{D}f(\cdot)$ and $f(\cdot)$ ΔL seconds into the future and assessing the expansion of the perturbations δ_i.

Some of the ideas described above are illustrated in figures 4a-b. The former figure is the bifurcation diagram of the logistic equation (7). Figure 4b shows the largest Lyapunov exponent of such an equation for a range of values of A. The largest Lyapunov exponent was calculated as described above. Note that $\lambda_1 = 0$ at bifurcation points and that $\lambda_1 > 0$ for chaotic regimes as predicted by the theory. These figures also reveal the narrow windows of regular dynamics which are surrounded by chaos.

2.11 Correlation dimension

Another quantitative measure of an attracting set is the fractal dimension. In theory, the fractal dimension of a

[2] In this case $\lambda_1 \geq h$, where h is the Kolmogorov-Sinai or metric entropy. Note that for dissipative systems (chaotic and non-chaotic) $\sum_{i=1}^{n} \lambda_i < 0$ (Eckmann and Ruelle, 1985; Wolf, 1986).

chaotic (non-chaotic) attracting set is non-integer (integer). An exception to this rule are *fat fractals* which have integer fractal dimension which is consequently inadequate to describe the properties of such fractals (Farmer, 1986). Nonetheless, like the largest Lyapunov exponent, the fractal dimension can be, in principle, used not only to diagnose chaos but also to provide some further dynamical information in most cases (Grassberger *et alii*, 1991). A deeper treatment can be found in (Russell *et alii*, 1980; Farmer *et alii*, 1983; Grassberger and Procaccia, 1983a; Atten *et alii*, 1984; Caputo *et alii*, 1986) for raw data and in (Badii and Politi, 1986; Badii *et alii*, 1988; Mitschke, 1990; Brown *et alii*, 1992; Sauer and Yorke, 1993) for filtered time series.

The fractal dimension is related to the amount of information required to characterize a certain trajectory. If the fractal dimension of an attracting set is $D+\delta$, $D \in \mathbb{Z}^+$, where $0<\delta<1$, then the smallest number of first-order differential equations required to describe the data is $D+1$.

There are several types of fractal dimension such as the *pointwise dimension*, *correlation dimension*, *information dimension*, *Hausdorff dimension*, *Lyapunov dimension*, for a comparison of some of these dimensions see (Farmer, 1982; Hentschel and Procaccia, 1983; Moon, 1987). For many strange attractors, however, such measures give roughly the same value (Moon, 1987; Parker and Chua, 1989). The correlation dimension[3] (Grassberger and Procaccia, 1983b), however, is clearly the most widely used measure of fractal dimension employed in the literature.

A time series $\{y_i\}_{i=1}^N$ can be embedded in the phase space where it is represented as a sequence of d_e-dimensional points $\mathbf{y}_j = [y_j \; y_{j-1} \; \dots \; y_{j-d_e+1}]$. Suppose the distance between two such points is[4] $S_{ij} = \mid \mathbf{y}_i - \mathbf{y}_j \mid$ then a correlation function is defined as (Grassberger and Procaccia, 1983b)

$$C(\epsilon) = \lim_{N\to\infty} \frac{1}{N} (\text{number of pairs } (i,j) \text{ with } S_{ij} < \epsilon) \quad . \tag{14}$$

The correlation dimension is then defined as

$$D_c = \lim_{\epsilon\to\infty} \frac{\log_e C(\epsilon)}{\log_e \epsilon} \quad . \tag{15}$$

For many attractors D_c will be (roughly) constant for values of ϵ within a certain range. In theory, the choice of d_e does not influence the final value of D_c if d_e is greater than a certain value. In particular, it has been shown that provided there are sufficient noise-free data, $d_e = \text{Ceil}(D_c)$, where Ceil($\cdot$) is the smallest integer greater than or equal to D_c (Ding *et alii*, 1993) and that this result remains true in the case the data have been filtered using *finite impulse*

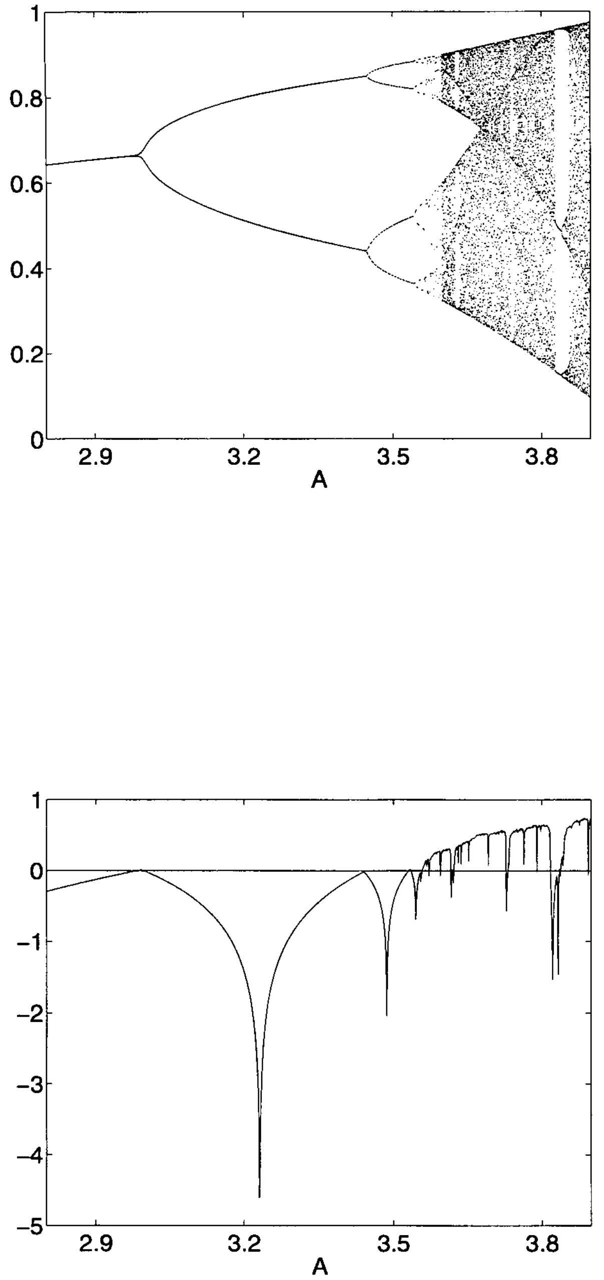


Figure 4 - (a) Bifurcation diagram of the logistic map, and (b) respective largest Lyapunov exponent, λ_1. Note that $\lambda_1 = 0$ at bifurcation points and that $\lambda_1 > 0$ for chaotic regimes.

[3] This measure can be seen as a *generalised dimension* and is considered to be the easiest to estimate reliably (Grassberger, 1986b) and thus remains the most popular procedure so far.

[4] Several norms can be used here such as Euclidean, ℓ_1, etc.

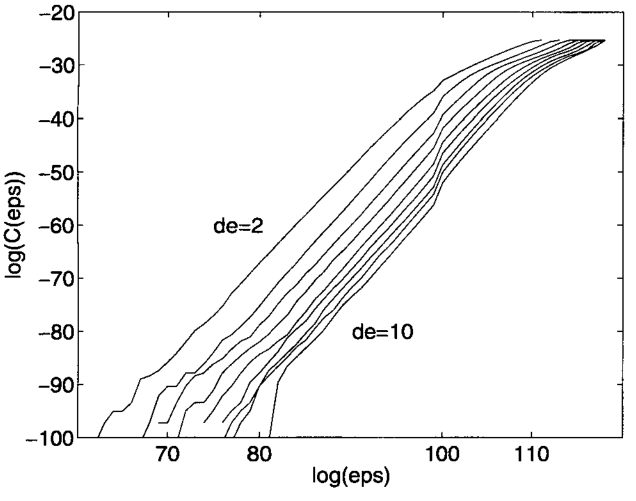


Figure 5 - Logarithm of the correlation function $C(\epsilon)$ plotted against $\log(\epsilon)$ for embedding dimensions $d_e = 2$ to $d_e = 10$. The correct value, $D_c \approx 2.0$ is attained for $d_e \geq 5$.

response (FIR) filters (Sauer and Yorke, 1993). In practice, due to the lack of data and to the presence of noise, $d_e > \text{Ceil}(D_c)$, thus several estimates of the correlation dimension are obtained for increasing values of d_e. If the data were produced by a low-dimensional system, such estimates would eventually converge. Of course, these results depend largely on both the amount and quality of the data available. For a brief account of data requirements, see section 3.1 below.

In order to illustrate the estimation of D_c a time series with $N = 15000$ data points was obtained by simulating Chua's circuit (see section 4.3) operating on the double scroll attractor. The correlation function $C(\epsilon)$ was then calculated for $2 \leq d_e \leq 10$ and plotted in figure 5. For small embedding dimensions ($d_e = 2$) the correlation dimension is $D_c \approx 1.8$ but as d_e is increased the scaling region converges to the correct value $D_c \approx 2.0$ for $d_e \geq 5$.

One of the properties of some fractals is *self-similarity*. This is illustrated in figure 6 which shows the well known Hénon attractor (see also section 4.2) and an amplification of a small section of one of its legs. It should be observed that what appears to be a single 'line' in the attractor turns out to be two lines (see zoom in figure 6). However, if each of these lines were zoomed again it would become apparent that they are composed of other two lines each and this continues *ad infinitum*. This particular fractal structure is sometimes referred to as having a Cantor set structure.

Probably the greatest application of the correlation dimension is to diagnose if the underlying dynamics of a time series have been produced by a low-order system (Grassberger, 1986a; Lorenz, 1991). Because this is an important problem, the estimation of correlation dimension has attracted much attention in the last years. Many papers have focused on determining the causes of bad estimates (Theiler, 1986), estimating error-bounds (Holzfuss and Mayer-Kress, 1986; Judd and Mees, 1991) and suggesting improvements on the original algorithm described in (Grassberger and Procaccia, 1983b).

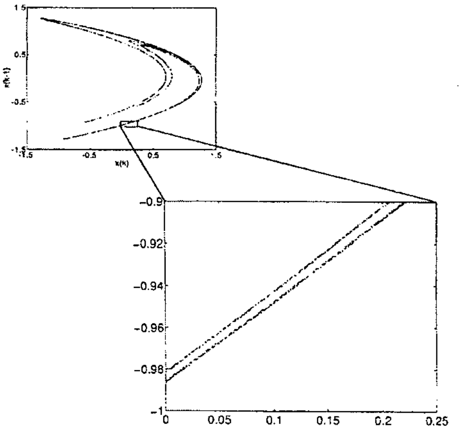


Figure 6 - Fractal structure of the Hénon attractor.

2.12 Other invariants

There are a number of less used invariants of strange attractors reported in the literature such as the Kolmogorov or metric entropy, topological entropy, generalised entropies and dimensions, partial dimensions, mutual information, etc. (Grassberger and Procaccia, 1984; Eckmann and Ruelle, 1985; Fraser, 1986; Grassberger, 1986b).

With few exceptions (Hsu and Kim, 1985), statistics have received little attention as invariant measures of strange attractors. Apparently, the most useful such measure is the *probability density function* (Packard *et alii*, 1980; Moon, 1987; Vallée *et alii*, 1984; Kapitaniak, 1988)

The estimation of unstable limit cycles has also been put forward as a way of characterizing strange attractors. The motivation behind this approach is that because a strange attractor can be viewed as a bundle of infinite unstable limit cycles, the number of the periodic orbits, the respective distribution and properties should be representative of the attractor dynamics. Indeed, from such information other invariants such as entropies and dimensions can be estimated (Auerbach *et alii*, 1987). For more information on this subject, see (Grebogi *et alii*, 1987; Cvitanović, 1988; Lathorp and Kostelich, 1989; Lathorp and Kostelich, 1992)

3 DIAGNOSING CHAOS

In general, the problem of diagnosing chaos can be reduced to estimating invariants which would suggest that the data are chaotic. For instance, positive Lyapunov exponents, non-integer dimensions and fractal structures in Poincaré sections would suggest the presence of chaos. The main question is how to confidently estimate such properties from the data, especially when the available records are relatively short and possibly noisy. The techniques that have been suggested in the literature can be divided in two major groups.

Non-parametric methods. These include the use of tools which take the data and estimate dynamical invariants which, in turn, will give an indication of the presence of chaos. Such tools include power spectra, the largest Lyapunov exponent, the correlation dimension, reconstructed trajectories, Poincaré sections, relative rotation rates etc. Detailed description and application of these techniques can be found in the literature (Moon, 1987; Tufillaro *et alii*, 1990; Denton and Diamond, 1991). For a recent comment of the practical difficulties in using Lyapunov exponents and dimensions for diagnosing chaos see (Mitschke and Dämmig, 1993).

Two practical difficulties common to most of these approaches are the number of data points available and the noise present in the data. These aspects are briefly discussed in the following section.

Poincaré sections are very popular for detecting chaos because for a chaotic system the Poincaré section reveals the fractal structure of the attractor. However, in order to be able to distinguish between a fractal object and a fuzzy cloud of points a certain amount of data is necessary. Moon (1987) has suggested that a Poincaré section should consist of at least 4000 points before declaring a system chaotic. For non-autonomous systems this means 4×10^3 forcing periods which could amount to 4×10^5 data points.

Prediction-based techniques. Some methods try to diagnose chaos in a data set based upon prediction errors (Sugihara and May, 1990; Casdagli, 1991; Elsner, 1992; Kennel and Isabelle, 1992). Thus predictors are estimated from, say, the first half of the data records and used to predict over the last half. Chaos can, in principle, be diagnosed based on how the prediction errors behave as the prediction time is increased (Sugihara and May, 1990), or based on how the prediction errors related to the true data compare to the prediction errors obtained from 'faked' data which are random but have the same length and spectral magnitude as the original data (Kennel and Isabelle, 1992). A related approach has been termed *the method of surrogate data* (Theiler *et alii*, 1992a; Theiler *et alii*, 1992aa).

Regardless of which criterion is used to decide if the data are chaotic or not, predictions have to be made. Clearly, the viability of these approaches depends on how easily predictors can be estimated and on the convenience of making predictions. Once a predictor is estimated criteria and statistics such as the ones presented in (Sugihara and May, 1990; Kennel and Isabelle, 1992) can be used to diagnose chaos.

3.1 Data requirements

The length and quality of the data records are crucial in the problem of characterization of strange attractors. At present, there seems to be no general rule which determines the amount of data required to learn the dynamics, to estimate Lyapunov exponents and the correlation dimension of attractors. However it is known that "in general the detailed diagnosis of chaotic dynamical systems requires long time series of high quality" (Ruelle, 1987).

Typical values of data length for learning the dynamics are 2×10^4 (Farmer and Sidorowich, 1987; Abarbanel *et alii*, 1990) for systems of dimension 2 to 3, $1.2 \times 10^4 - 4 \times 10^4$ (Casdagli, 1991).

It has been argued that to estimate the Lyapunov exponents $10^3 - 10^4$ forcing periods should be used (Denton and Diamond, 1991). Other estimates are $N > 10^D$ (quoted in (Rosenstein *et alii*, 1993) and $N > 30^D$ where D is the dimension of the system (Wolf *et alii*, 1985) but in some cases at least 2×30^D was required (Abarbanel *et alii*, 1990). Typical examples in the literature use $4 \times 10^4 - 6.4 \times 10^4$ (Eckmann *et alii*, 1986) 1.6×10^4 (Wolf and Bessoir, 1991) and 2×10^4 data points (Ellner *et alii*, 1991).

Fairly long time series are also required for estimating the correlation dimension. In fact, it has been pointed out that dimension calculations generally require larger data records (Wolf and Bessoir, 1991). For a strange attractor, if insufficient data is used the results would indicate the dimension of certain parts of the attractor rather than the dimension of the entire attractor (Denton and Diamond, 1991). However, results have been reported which suggest that consistent estimates of the correlation dimension can be obtained from data sequences with less than 1000 points (Abraham *et alii*, 1986). On the other hand, there seems to be evidence that "spuriously small dimension estimates can be obtained from using too few, too finely sampled and too highly smoothed data" (Grassberger, 1986a). Moreover, the use of short and noisy data sets may cause the correct scaling regions to become increasingly shorter and may cause the estimate of the correlation dimension to converge to the correct result for relatively large values of the embedding dimension (Ding *et alii*, 1993). Thus typical examples use $1.5 \times 10^4 - 2.5 \times 10^4$ (Grassberger and Procaccia, 1983b) and $0.8 \times 10^4 - 30 \times 10^4$ data points (Atten *et alii*, 1984). Thus there seems to be no agreed upon rule to determine the amount of data required to estimate dimensions with confidence but it appears that at least a few thousand points for low dimensional attractors are needed (Theiler, 1986; Havstad and Ehlers, 1989; Ruelle, 1990; Essex and Nerenberg, 1991). In particular, $N \geq 10^{D_c/2}$ has been quoted in (Ding *et alii*, 1993).

It should be realized that the difficulties in obtaining long time series goes beyond problems such as storage and computation time. Indeed, it has been pointed out that for some real systems, stationarity cannot always be guaranteed even over relatively short periods of time. Examples of this include biological systems (May, 1987; Denton and Diamond, 1991), ecological and epidemiological data (Schaffer, 1985; Sugihara and May, 1990). A test for stationarity has been recently suggested in (Isliker and Kurths, 1993).

4 SOME NONLINEAR MODELS

The objective of this section is to provide the reader will a collection of well documented simple nonlinear models which display a wide variety of dynamical regimes which include self-oscillations, limit cycles, period-doubling cascades and chaos.

4.1 THE LOGISTIC EQUATION

In many fields of science the dynamics of a particular system are better represented by discrete maps, also called difference equations, of the form

$$y(k+T) = F[y(k)] \ . \tag{16}$$

This equation indicates that the value of the variable y at time $k+T$ is a function of the same variable at time k. The basic equation (16) applies to a number of situations which include switching power circuits (Tse, 1994), population dynamics, genetics, demography, economics and social sciences, see (May, 1975; May, 1976) and references therein.

The most well known, and certainly one of the simplest, examples of (16) is equation (7) known as the *logistic equation* which was originally suggested as a population dynamics model (May, 1976) and displays a variety of dynamical regimes as the bifurcation parameter, A, is varied in the interval $2.8 \leq A \leq 3.9$. The bifurcation diagram of this map, which is shown in figure 4a, is a classical example of the *period-doubling route to chaos.*

Due to its simplicity and also because some dynamical invariants can be derived analytically for this model, the logistic map is frequently used as a bench test in the study of dynamical systems. For a review on first-order maps see (May, 1980; May, 1987) .

4.2 The Hénon map

The map (Hénon, 1976)

$$\begin{cases} x(k) = 1 - a\,x(k-1)^2 + y(k-1), \\ y(k) = b\,x(k-1) \end{cases} \tag{17}$$

was proposed by the French astronomer Michel Hénon as an approximation of a Poincaré mapping of the Lorenz system (see section 4.6). In the literature the parameter values usually used are $a = 1.4$ and $b = 0.3$. Thus these values lead to the so-called Hénon attractor shown in figure 6. For a more detailed description of the dynamical properties of this map see (Moon, 1987; Peitgen *et alii*, 1992) .

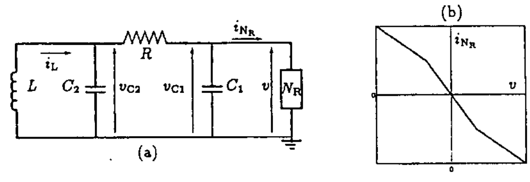


Figure 7 - (a) The Chua circuit and (b) voltage-current characteristic of the piecewise linear component, N_R, known as Chua's diode.

As the logistic equation, the Hénon map is a popular benchmark used to test a variety of algorithms concerning the analysis and signal processing of nonlinear dynamical systems.

4.3 The Chua circuit

The normalized equations of Chua's circuit can be written as (Chua *et alii*, 1986; Chua, 1992; Madan, 1993; Chua and Hasler, 1993)

$$\begin{cases} \dot{x} = \alpha(y - h(x)) \\ \dot{y} = x - y + z \quad , \\ \dot{z} = -\beta y \end{cases}$$
$$h(x) = \begin{cases} m_1 x + (m_0 - m_1) & x \geq 1 \\ m_0 x & |\, x \,| \leq 1 \\ m_1 x - (m_0 - m_1) & x \leq -1 \ , \end{cases} \tag{18}$$

where $m_0 = -1/7$ and $m_1 = 2/7$. Varying the parameters α and β the circuit displays several regular and chaotic regimes. This system is a particular case of the more general *unfolded Chua's circuit* (Chua, 1993). The well known double scroll attractor, for instance, is obtained for $\alpha = 9$ and $\beta = 100/7$, see figure. 8. For this attractor $\lambda_1 = 0.23$ (Matsumoto *et alii*, 1985; Chialina *et alii*, 1994). The estimated value of the correlation dimension for this attractor was $D_c = 1.99 \pm 0.023$.

Chua's circuit is certainly one of the most well studied nonlinear circuits and a great number of papers ensure that the dynamics of this circuit are also well documented, see for instance (Chua and Hasler, 1993; Madan, 1993; Matsumoto *et alii*, 1993).

The only nonlinear element in Chua's circuit is a two-terminal piecewise-linear resistor, called Chua's diode, which has also been implemented as an integrated circuit (Cruz and Chua, 1992), however the entire circuit can be implemented with 'off-the-shelf' components (Kennedy, 1992).

4.4 The Duffing-Holmes oscillator

One of the classical bench tests in mechanics is the Duffing oscillator (Duffing, 1918). Two different versions of this oscillator have been investigated in connection with chaos

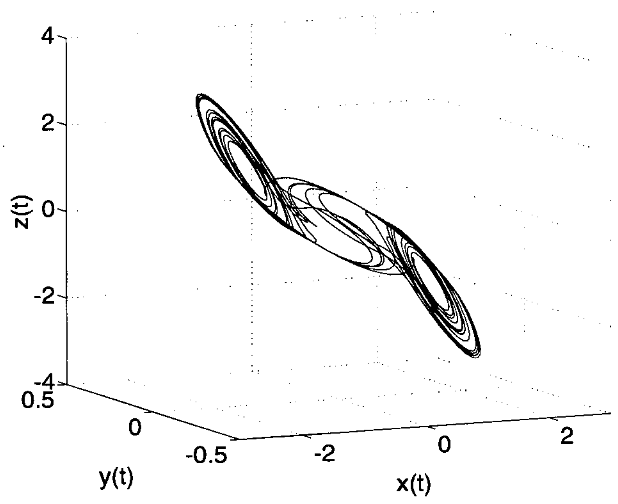


Figure 8 - The double scroll Chua's attractor.

by Philip Holmes and Yoshishuke Ueda. The equations which describe the dynamics of such oscillators are briefly described in this and the following section, respectively.

The well known Duffing-Holmes equation is commonly used to model mechanical oscillations arising in two-well potential problems (Moon, 1987) because this system is characteristic of many of the structural nonlinearities encountered in practice (Hunter, 1992). The equation which models this system is (Holmes, 1979; Moon and Holmes, 1979)

$$\ddot{y} + \delta\,\dot{y} - \beta\,y + y^3 = A\cos(\omega\,t) \ . \qquad (19)$$

The bifurcation diagram for this system for $\omega = 1$ rad/s and $0.22 \leq A \leq 0.35$ is shown in figure 9a. For $\delta = 0.15$, $\beta = 1$, $A = 0.3$ and $\omega = 1$ rad/s, this system settles to a strange attractor which is shown in figure 9b. The largest Lyapunov exponent of this attractor is $\lambda_1 = 0.20$ and the correlation dimension equals $D_c = 2.40 \pm 0.019$.

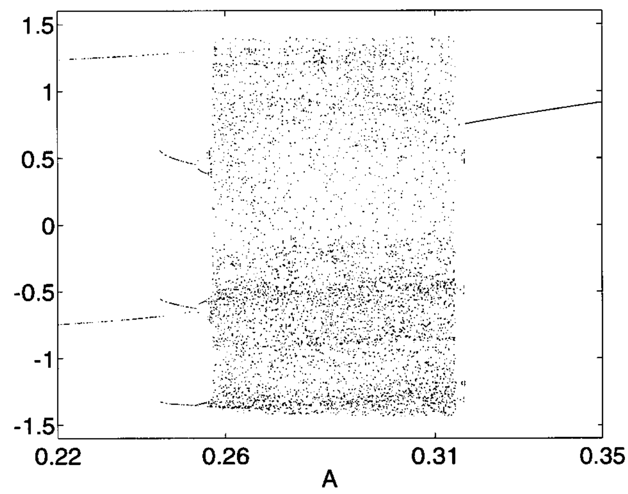
Equation (19) has been used to model a wide range of systems in science and engineering, a few examples include (McCallum and Gilmore, 1993; Parlitz, 1993).

4.5 The Duffing-Ueda oscillator

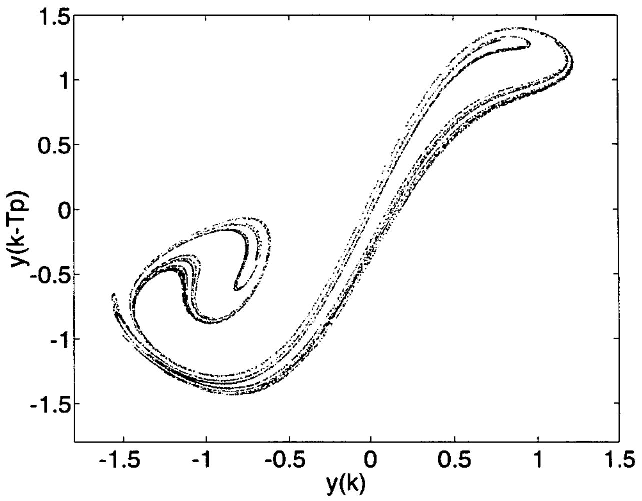
The Duffing-Ueda equation (Ueda, 1985)

$$\ddot{y} + k\,\dot{y} + y^3 = u(t) \qquad (20)$$

was originally proposed as a model for nonlinear oscillators and has become a bench test for the study of non-



a



b

Figure 9 - (a) bifurcation diagram, and (b) Poincaré section of the Duffing-Holmes oscillator, for $A = 0.3$ and $T_p = 5$.

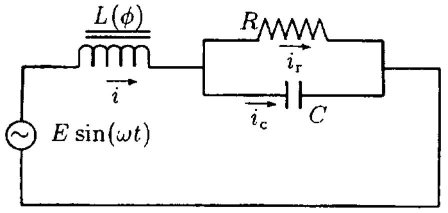
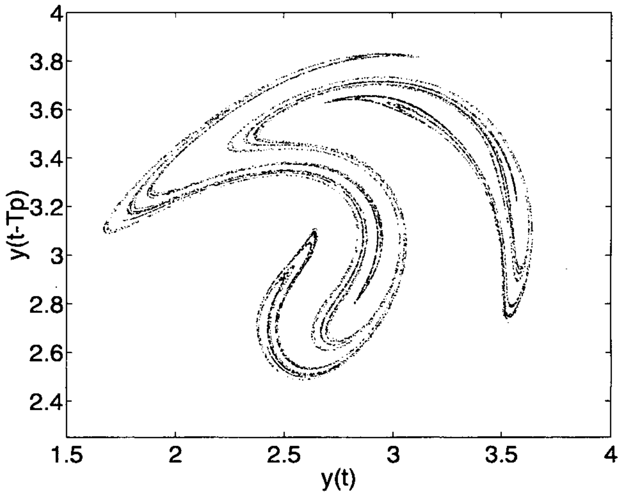


Figure 10 - The Duffing-Ueda oscillator.

linear dynamics. It has also been considered as a simple paradigm for chaotic dynamics in electrical science (Moon, 1987). Consequently this system has attracted some attention and has been used to investigate nonlinear dynamics in different situations (Feigenbaum, 1983; Kapitaniak, 1988). One of the main reasons for this is that in spite of being simple this model can produce a variety of dynamical regimes, from period-one motions to chaos (Ueda, 1980; Ueda, 1985; Kawakami, 1986).

The Poincaré sections and bifurcation diagrams were obtained as indicated respectively in sections 2.7 and 2.6 for the input $u(t) = A\cos(\omega t)$. A was used as the bifurcation parameter in the bifurcation diagrams. The bifurcation shown in figure 1 was obtained by taking $k=0.1$, $\omega=1$ rad/s and simulating equation (20) digitally using a fourth-order Runge-Kutta algorithm with an integration interval equal to $\pi/3000$. Figures 11a and 11b shows the Poincaré section of the attractors at $A=5.7$ and at $A=11$, respectively. The largest Lyapunov exponent of these attractors are respectively $\lambda_1 = 0.099$ and $\lambda_1 = 0.11$ and the correlation dimensions equal $D_c = 2.10 \pm 0.050$ and $D_c = 2.19 \pm 0.020$.

The bifurcation diagram in figure 1 reveals a number of dynamical regimes displayed by this oscillator in the range of values of A considered. In particular, at $A \approx 4.86$ the system undergoes a period doubling (flip) bifurcation. This happens again at $A \approx 5.41$ and characterizes the well known period doubling route to chaos (Feigenbaum, 1983). Another similar cascade begins at $A \approx 9.67$ preceding a different chaotic regime. Two chaotic windows can be distinguished at approximately $5.55 \leq A \leq 5.82$ and $9.94 \leq A \leq 11.64$. At $A \approx 6.61$ the system undergoes a supercritical pitchfork bifurcation and at $A \approx 9.67$ it undergoes a subcritical pitchfork bifurcation. The bifurcation diagram begins and ends with period-1 regimes and displays period-3 dynamics for $5.82 \leq A \leq 9.67$. It should be noted that in the range $4.5 \leq A \leq 5.5$ there are two co-existent attractors undergoing a sequence of period-doublings, this becomes apparent by using a different set of initial conditions and explains the broken lines.



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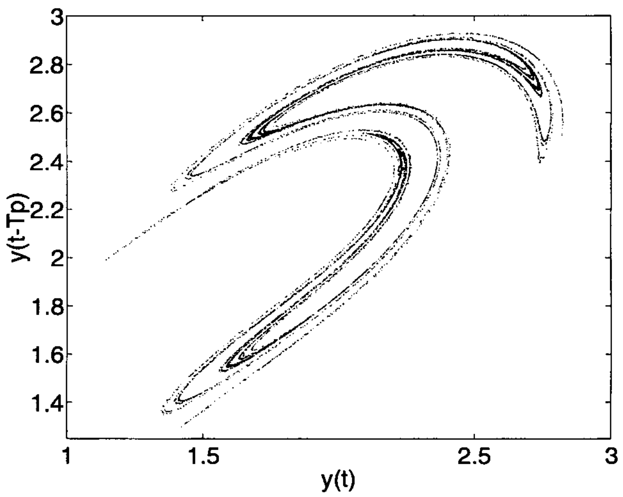


b

Figure 11 - Poincaré sections for the Duffing-Ueda oscillator, for $T_p = 200 \times \pi/3000$ and (a) $A=5.7$, (b) $A=11$.

4.6 The Lorenz equations

The well known Lorenz equations are (Lorenz, 1963)

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = xy - \beta z \; . \end{cases} \qquad (21)$$

Choosing $\sigma=10$, $\beta=8/3$ and $\rho=28$, the system described

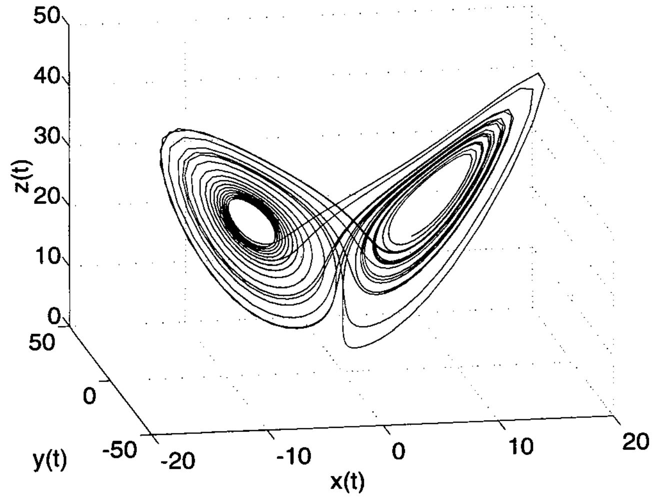


Figure 12 - The Lorenz attractor.

by equations (21) settles to the well known Lorenz attractor, shown in figure 12.

Since the Lorenz equations were first published in 1963 they have become a standard for studying complex dynamics. Fairly detailed descriptions of the dynamics of this system can be found in (Sparrow, 1982; Thompson and Stewart, 1986; Peitgen *et alii*, 1992). The popularity of the Lorenz equations is not only due to the fact that it was one of the first systems published in connection with chaos but also because it is a physically motivated model. Further, although the Lorenz equations model some aspects of fluid dynamics, it has been shown that the instabilities of such a model are identical with that of the single mode laser and applicable to underdamped laser spikes (Haken, 1975) and that the open-loop dynamics of smooth-air-gap brushless DC motors can also be modeled by such equations (Hemati, 1994).

The largest Lyapunov exponent of the attractor shown in figure 12 is $\lambda_1 = 0.90$ (Peitgen *et alii*, 1992) and the correlation dimension equals $D_c = 2.01 \pm 0.017$.

4.7 The modified van der Pol oscillator

During the twenties van der Pol investigated an electrical circuit in which the nonlinearity was introduced by a vacuum valve. One of the principal characteristics of such a circuit was that it exhibited self-sustained oscillations also called relaxation oscillations, thus lending itself as a model for a number of real oscillating systems such as the heart (van der Pol and van der Mark, 1928).

The normalized equations of a modified version of the van der Pol oscillator, which has negative resistance, are (Ueda

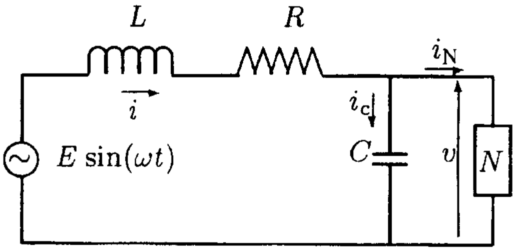


Figure 13 - The modified van der Pol oscillator.

and Akamatsu, 1981)

$$\ddot{y} + \mu(y^2 - 1)\dot{y} + y^3 = A\cos(\omega t) \ . \tag{22}$$

This is known as the *modified van der Pol oscillator* (Moon, 1987). Alike the diverse versions of the Duffing system, this equation also has a cubic term which is absent in the original van der Pol equation. On the other hand, alike the van der Pol oscillator, the circuit governed by equation (22) also exhibits self-sustained oscillations, that is, oscillations for $A = 0$ with $\omega = 1.62\,\text{rad/s}$.

The bifurcation diagram of this system is shown in figure 14a. This bifurcation diagram presents a number of coexistent attractors and dynamical regimes interwoven in a very complicated manner. It is worth pointing out that for $0 < A \leq 3$ the oscillator presents quasi-periodic motions which cannot be distinguished from the chaotic dynamics based upon bifurcation diagrams only.

Taking $\mu = 0.2$, $A = 17$ and $\omega = 4\,\text{rad/s}$, this system settles to the strange attractor shown in figure 14b. The largest Lyapunov exponent of this attractor is $\lambda_1 = 0.33$ and the correlation dimension equals $D_c = 2.18 \pm 0.028$.

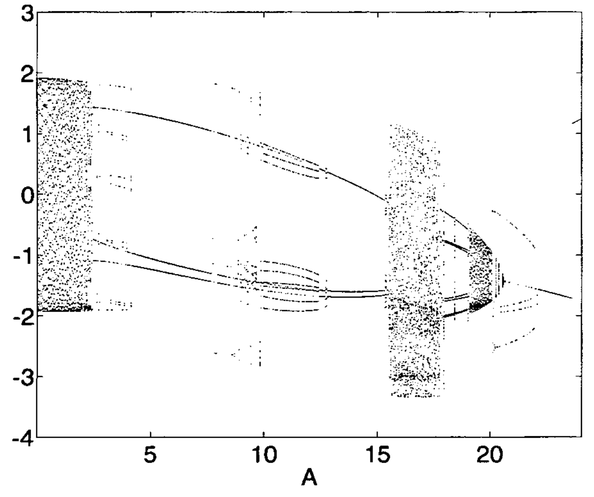
4.8 The Rössler equations

The Hénon map described in section 4.2 was originally suggested as a model for the Poicaré map of the Lorenz equations. Similarly, Rössler proposed the following set of equations (Rössler, 1976)

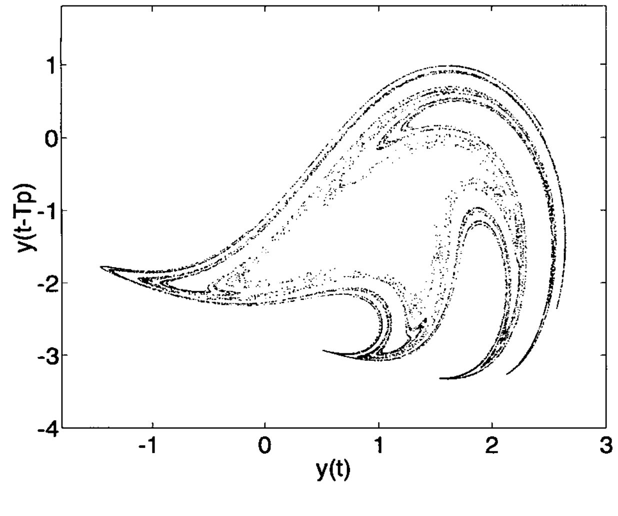
$$\begin{cases} \dot{x} = -(y + z) \\ \dot{y} = x + \alpha y \\ \dot{z} = \alpha + z(x - \mu) \ , \end{cases} \tag{23}$$

as a simplified version of the Lorenz system, or in Rössler's words a "model of a model". The simplification attained by Rössler can be appreciated by noticing that the attractor exhibited by (23), with $\alpha = 0.2$ and $\mu = 5.7$ see figure 15., is composed of a single spiral which resembles a Möbius band instead of the two spirals which can be clearly distinguished in the Lorenz attractor.

The largest Lyapunov exponent of the attractor shown in figure 15 is $\lambda_1 = 0.074$, the Lyapunov dimension and the correlation dimension of this attractor are $D_L = 2.01$ and $D_c = 1.91 \pm 0.002$, respectively.



a



b

Figure 14 - (a) bifurcation diagram, and (b) Poincaré section of the modified van der Pol oscillator, $A=17$, $\omega=4\,\mathrm{rad/s}$ and $T_{\mathrm{p}}=16$.

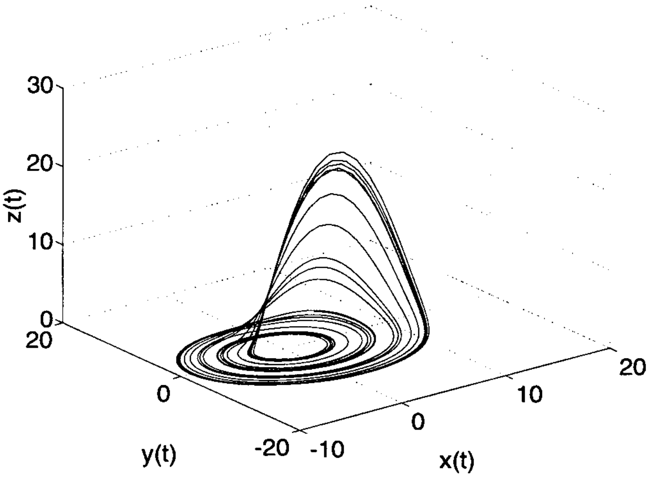


Figure 15 - The Rössler attractor.

The field of possible applications of equations of the type of (23) ranges from astrophysics, via chemistry and biology, to economics (Rössler, 1976). In fact, the Rössler and Lorenz attractors are regarded as paradigms for chaos thus the chaotic dynamics of some systems are sometimes classified as "Rössler-like" or "Lorenz-like" chaos (Gilmore, 1993).

5 DISCUSSION AND FURTHER READING

The analysis and quantification of chaotic dynamics is a relatively recent area. Nevertheless there is an immense collection of scientific papers and books devoted to this subject and any attempt to produce a survey on nonlinear dynamics and chaos, no matter how thorough, would be, in all certainty, just a rough sketch on this fascinating subject.

The main objective of this paper has been to review in a very pragmatic way a few concepts which are believed to be basic. Since it would be inappropriate to produce an in-depth review, a rather generous number of references has been cited for further reading. Needless to say, the reference list does not exhaust the wealth of papers and books currently available.

The following references seem to be a good starting point. The books (Gleick, 1987) and (Stewart, 1989) are a good introduction for the average reader. A more formal coverage is given by (Thompson and Stewart, 1986) and (Moon, 1987). For a mathematical exposition on the subject see (Guckenheimer and Holmes, 1983) and (Wiggins, 1990). Some practical aspects of bifurcation and chaos are discussed in (Matsumoto *et alii*, 1993) and a good account on computer algorithms for nonlinear systems applications can be found in (Parker and Chua, 1989). See also (Abraham and Shaw, 1992) for a beautifully illustrated introduction to nonlinear dynamics and bifurcations. The following papers are also good introductions to nonlinear dynamics and chaos (Mees and Sparrow, 1981; Shaw, 1981; Mees, 1983; Eckmann and Ruelle, 1985; Crutchfield *et alii*, 1986; Mees and Sparrow, 1987; Parker and Chua, 1987; Argyris *et alii*, 1991; Thompson and Stewart, 1993). Good surveys on modeling and analysis of chaotic signals can be found in (Grassberger *et alii*, 1991; Abarbanel *et alii*, 1993). See also (Hayashi, 1964; Atherton and Dorrah, 1980) for a rather 'classical' approach to the analysis of nonlinear oscillations. Finally, it is worth pointing out that some software packages are available for analysis of nonlinear and chaotic systems. In particular, the program *kaos* (Guckenheimer and Kim, 1990; Guckenheimer, 1991) which runs on Sun workstations and can be obtained by ftp on macomb.tn.cornell.edu 128.84.237.12. The programs MTR-CHAOS and MTRLYAP (Rosenstein, 1993) can be sued for analysing chaotic signals and estimating correlation dimension and largest Lyapunov exponents. These programs are available under request to MTR1a@aol.com.

Some of the concepts and mathematical tools discussed in this paper will be used in a companion paper which will address some aspects of the identification, analysis and control of nonlinear dynamics and chaos.

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