A TUTORIAL INTRODUCTION TO NONLINEAR DYNAMICS AND CHAOS, PART II: MODELING AND CONTROL

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ABSTRACT - This is the second and final part of a series of two papers on nonlinear dynamics and chaos. In the first part some tools, developed for analysing nonlinear systems, were described in conjunction with a set of models commonly used as benchmarks in the literature. This paper investigates a number of issues concerning the modeling, signal processing and control of nonlinear dynamics. This is carried out using the tools and models described in the first paper. This investigation has thrown some new light on relevant problems such as model parametrization, model validation, data smoothing and control of nonliear systems. These issues are investigated using NARMAX polynomial models but it is believed that the conclusions are relevant to nonlinear representations in general. Some numerical examples are included.

1 INTRODUCTION

Chaotic systems have attracted a great deal of attention in the last three decades. Systems and models which undergo chaotic regimes for a rather wide range of operating conditions have been found in virtually every branch of science and engineering.

In the evolution of the study of chaotic systems, several distinct but sometimes co-existent phases can be distinguished. In the first phase, chaos was recognised as a deterministic dynamical regime which could be responsible for fluctuations that hitherto had been regarded as noise and therefore modeled as stochastic processes (Lorenz, 1963). In a subsequent phase, it was necessary both to develop criteria to detect chaotic dynamics and to establish dynamical invariants to quantify chaos (Guckenheimer, 1982; Eckmann and Ruelle, 1985; Denton and Diamond, 1991). Having succeeded in diagnosing chaos, the next step was to build models which would learn the dynamics from data on the strange attractor. In this respect a number of model structures have been investigated such as local linear mappings (Farmer and Sidorowich, 1987; Crutchfield and McNamara, 1987), radial basis functions (Broomhead and Lowe, 1988; Casdagli, 1989), neural networks (Elsner, 1992) and global nonlinear polynomials (Aguirre and Billings, 1995c; Kadtke *et alii*, 1993). This phase is being currently investigated with other equally important issues concerning nonlinear dynamics such as noise reduction and control. It is the objective of this paper to provide a brief introduction to these issues.

The outline of the paper is as follows. Section 2 discusses a number of issues concerning the modeling of nonlinea dynamics. Embedding techniques in general and NARMAX models in particular are briefly presented in that section. Section 3 is concerned with the noise reduction problem, which is a major limitation in the modeling of nonlinear dynamics. Section 4 provides a very superficial introduction to the subject of control and synchronization of chaos, nonetheless several references are provided for further reading. Some final remarks are made in section 5.

2 MODELING NONLINEAR DYNAMICS

This section gives a quick view at nonlinear dynamics modeling. This vital issue is discussed under several related headings. A helpful account of some of the main points on

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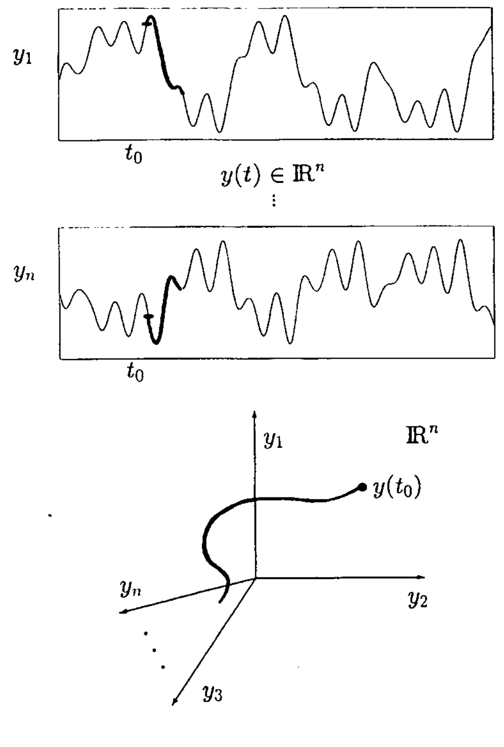


Figure 1 - The n time series defined by the state variables of an $n$tn-order dynamical system can be used to compose the trajectory in state space.

this subject can be found in the literature (Casdagli *et alii*, 1992).

2.1 Embedding Techniques

An nth-order dynamical system $\dot{y} = f(y)$ can be represented as a set of n first-order ordinary differential equations each one governed by a *state variable.* The global system would therefore have n time variables $\{y_1, y_2, \ldots, y_n\}$ and the solution of such a system could be thought of as n time series.

In a sense, the n time series mentioned above are obtained from the original nth-order system by decomposition. Also, given th n times series it is possible to recover the original n-dimensional solution by taking each state variable to be a coordinate of a 'reconstruction space' and to represent each time series in such a space. Thus n time series can be used to *compose* or *reconstruct* the system solution or trajectory. This is illustrated in figure 1.

A difficulty encountered in practice with this approach is that the order of the system n is seldom known and even when an accurate estimate of this variable exists the number of measurements will not be as large as n. Take for instance the atmosphere which is usually thought of as a high-order system, nonetheless monitoring and weather forecasting stations only measure a very limited number of variables of this system in order to make predictions.

This can be described in a more mathematical way by considering the action of a *measuring* function $h(y) : \mathbb{R}^n \rightarrow \mathbb{R}$ which operates on the entire state or phase space but which yields just a scalar which is called the *measured variable.* The question which naturally arises at this stage is the following: given $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $h(y) : \mathbb{R}^n \rightarrow \mathbb{R}$, is it possible to reconstruct a trajectory or sulution of f from the scalar measurement $h(y)$?

Fortunately, it turns out that this question has an affirmative answer if certain requirements are met (Takens, 1980; Packard *et alii*, 1980; Sauer *et alii*, 1991). Thus *embedology* is concerned with how to reconstruct the phase space of a dynamical system of order n from a limited set of measurements q where $q < n$, and more often than not $q = 1$. In other words, the objective is to reconstruct the phase space of a system from a single time series. The resulting phase space is usually referred to as *embedded phase space*, *embedding space* or just *embedding*.

Another question which should be addressed is: why should we be concerned in reconstructing the trajectories of a dynamical system? In the companion paper it was shown that in state (or phase) space the steady state dynamics of a system are represented by geometrical figures which are called attractors. A stable autonomous linear system only has one kind of attractor, a point attractor. However, nonlinear systems may have more complicated attractors such as limit cycles, tori and the so-called strange attractors.

Therefore if time series are used to reconstruct the phase space of dynamical systems via embedding techniques, it is possible to use results from differential geometry and topology to analyse the resulting attractors which are *geometrical objects* in the reconstructed space. Moreover, if the embedding is successful, both the reconstructed and the original attractors are equivalent from a topological point of view, or in other words, they are said to be diffeomorphic.

The practical consequences of this are obvious. No matter how complex a dynamical system might be, even if only one variable of such a system is measured, it is possible to reconstruct the original phase space via embedding techniques. It is also possible to estimate qalitative and quantitative invariants of the original attractor, such as Poincaré maps, fractal dimension and Lyapunov exponents, directly from the reconstructed attractor which is topologically equivalent to the original one. These ideas are illustrated in figure 2.

A convenient but by no means unique way of reconstructing phase spaces from scalar measurements is achieved by using *delay coordinates* (Packard *et alii*, 1980; Takens, 1980; Sauer *et alii*, 1991). Other coordinates include the *singular value* (Broomhead and King, 1986; Albano *et alii*, 1988) and *derivatives* (Baake *et alii*, 1992; Gouesbet and Maquet, 1992). A framework for the comparison of several reconstructions has been developed in (Casdagli *et alii*, 1991) and three of the most common methods have been studied in (Gibson *et alii*, 1992).

A *delay vector* has the following form

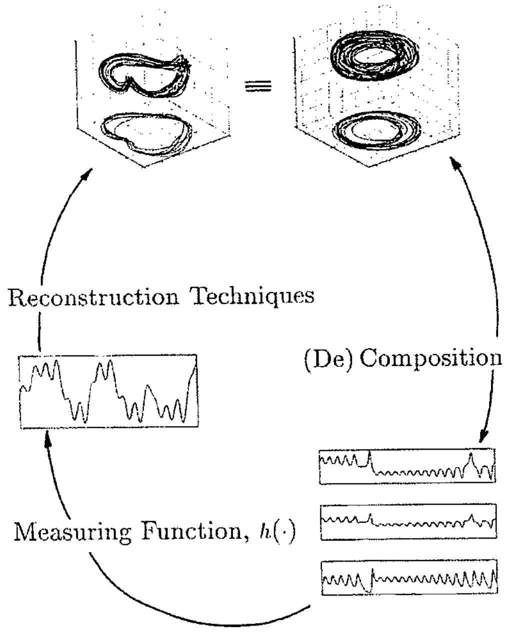


Figure 2 - In many practical situations the number of measured variables is limited. Embedding techniques enable the reconstruction of the state space even from a single measurement. The reconstructed (or embedded) and the original state spaces are equivalent.

$$\mathbf{y}(k) = [y(k) \;\; y(k-\tau) \;\; \ldots \;\; y(k-(d_e-1)\tau)]^T \;, \qquad (1)$$

where d_e is the *embedding dimension* and τ is the *delay time*. Clearly, $\mathbf{y}(k)$ can be represented as a point in the d_e-dimensional *embedding space*. Takens (1980) has shown that embeddings with $d_e > 2n$ will be faithful generically so that there is a smooth map $f_T : \mathbb{R}^{d_e} \rightarrow \mathbb{R}$ such that

$$y(k+T) = f_T(\mathbf{y}(k)) \qquad (2)$$

for all integers k, and where the *forecasting time* T and τ are also assumed to be integers. A consequence of Taken's theorem is that the attractor reconstructed in $\mathbb{R}_e^d$ is diffeomorphic to the original attractor in state space and therefore the former retains dynamical and topological characteristics of the latter.

In the case of delay reconstructions, the choice of the reconstruction parameters, that is, the embedding dimension d_e and the delay time τ is of the greatest importance since such parameters strongly affect the quality of the embedded space. The selection of d_e has been investigated in (Cenys and Pyragas, 1988; Aleksić, 1991; Cheng and Tong, 1992; Kennel *et alii*, 1992). The choice of the delay time has been discussed in (Albano *et alii*, 1991; Buzug *et alii*, 1990; Fraser, 1989; Kember and Fowler, 1993; Liebert and Schuster, 1989; Billings and Aguirre, 1995; Aguirre, 1995a). Many authors have suggested that in some applications it is more meaningful to estimate these parameters simultaneously, this is tantamount to estimating the embedding window defined as $(d_e-1)\tau$ (Albano *et alii*, 1988; Buzug and Pfister, 1992; Martinerie *et alii*, 1992). Some of these methods have recently been compared in (Rosenstein *et alii*, 1994). There is some evidence that the 'optimum' value of τ in system identification problems is shorter than for phas-space reconstructions (Aguirre, 1994). Dynamical reconstructions from nonuniformly sampled data has been addressed in (Breedon and Packard, 1992) and phase space reconstruction of symmetric attractors has been considered in (King and Stewart, 1992).

Taken's theorem gives sufficient conditions for equation (2 to hold, that is, in order to be able to infer dynamical invariants of the original system from the time series of a single variable, however no indication is given as to how to estimate the map f_T. A number of papers have been devoted to this goal and such methods can be separated into two major groups, namely *local* and *global* approximation techniques.

The local approaches usually begin by partitioning the embedding space into neighbourhoods $\{\mathcal{U}_i\}_{i=1}^{N_n}$ within which the dynamics can be appropriately described by a linear map $g_T : \mathbb{R}_e^d \rightarrow \mathbb{R}$ such that

$$y(k+T) \approx g_{T\,i}(\mathbf{y}(k)) \quad \text{for } \mathbf{y}(k) \in \mathcal{U}_i, \; i = 1, \ldots, N_n \;. \; (3)$$

Several choices for g_T have been suggested in the literature such as linear polynomials (Farmer and Sidorowich, 1987; Casdagli, 1991) which can be interpolated to obtain an approximation of the map f_T (Abarbanel *et alii*, 1990). Simpler choices include *zeroth-order approximations*, also known as *local constant predictors* (Farmer and Sidorowich, 1987; Kennel and Isabelle, 1992; Wayland *et alii*, 1993) and a *weighted predictor* (Linsay, 1991).

Global approximators overcome some of the difficulties faced by local maps. Although global models have problems of their own, some attention has been devoted to the investigation of such models (Cremers and Hübler, 1987; Crutchfield and McNamara, 1987; Kadtke *et alii*, 1993; Aguirre and Billings, 1995e).

2.2 Representation of Nonlinear Systems

The Volterra series and other related functional representations were among the first models to be used in nonlinear approximation. A well known difficulty with such representations is the enormous amount of parameters required in order to approximate simple nonlinearities (Billings, 1980). Related techniques seem to suffer from the same problem and, in addition, tend to require very large data sets (Giona *et alii*, 1991).

One of the most popular representations of dynamical models is the polynomial form. Apart from being easy to interpret, simulate and operate, algorithms for the estimation of the parameters of polynomial models are currently widely available. One of the disadvantages of global polynomials, however, is that even for polynomial models of moderate order, the number of terms can become impractically large (Farmer and Sidorowich, 1988a; Casdagli, 1989). Using polynomials to forecast chaotic time series, Casdagli (1989) has reported that such predictors blow up in the iterative procedure and suggests that this is because polynomial predictors give bad approximants to the true dynamics except very close to the attractor. On the other hand, some of the problems related to global polynomials are believed to be connected to the structure of the models (Aguirre and Billings, 1995b) and promising results have been reported for some systems using nonlinear global polynomials with simplified structure (Kadtke *et alii*, 1993; Aguirre and Billings, 1994c).

Rational models share with polynomials the advantage of being linear in the parameters. This feature makes it possible to use well known and numerically robust algorithms to estimate the parameters of such models. Moreover, rational models seem to extrapolate better than polynomials (Farmer and Sidorowich, 1988a).

The *radial basis function* (RBF) approach is a global interpolation technique with good localization properties and it is easy to implement as the algorithm is essentially independent of the dimension (Broomhead and Lowe, 1988; Casdagli, 1989; Whaba, 1992). However performance of radial basis functions depends critically upon the centres (Chen *et alii*, 1990). For a few hundred data points the choice of the centres is a difficult task and the solution of the problem could become infeasible (Casdagli, 1989; Billings and Chen, 1992).

Local approximants are concerned with the mapping of a set of neighbouring points in a reconstructed state space into their future values. A major problem here is to select the neighbourhoods because such a choice is critical and there could be hundreds or even thousands of these (Farmer and Sidorowich, 1988ab). The size of the neighbourhoods depends on the noise level and the complexity of the dynamics (Farmer and Sidorowich, 1991).

A simple alternative to nonlinear modeling is the use of piecewise-linear representations (Billings and Voon, 1987). The discontinuities among the several linear models which compose a piecewise-linear model, can provide effects similar to those observed in nonlinear models such as chaos (Mahfouz and Badrakhan, 1990a; Mahfouz and Badrakhan, 1990b). However, as other local representations, the final model is piecewise-linear and therefore discontinuous. Piecewise-linear models have been found to be unreliable indicators of the underlying dynamics in some cases (Billings and Voon, 1987), and a possible explanation for this is that such models violate the physically motivated hypothesis of smooth dynamical systems (Crutchfield and McNamara, 1987). Thus local predictors may not always be suitable for predicting invariant measures (Brown *et alii*, 1991).

Smooth interpolation functions have been suggested as a way of alleviating the problem caused by discontinuities in piecewise-linear models (Johansen and Foss, 1993). Such functions have localised properties which confer to the final models composed in this way some similarities with radial basis functions. As would be expected, the quality of the approximation depends on the choice of several operating regimes where the system dynamics are approximately linear. This information has to be available *a priori* and is somewhat critical. The problem of selecting the operating points is similar to the choice of neighborhoods and of centres in other approaches.

Other representations for modelling nonlinear systems include Legendre polynomials (Cremers and Hübler, 1987), neural networks (Elsner, 1992; Principe *et alii*, 1992) and weighted maps (Stokbro and Umberger, 1992).

At present no particular representation can be regarded as the best for any application and "finding a good representation is largely a matter of trial and error" (Farmer and Sidorowich, 1988a). On the other hand, it seems that global polynomial models are in many respects simpler and therefore more convenient (Kadtke *et alii*, 1993).

The remainder of this section investigates the use of global polynomials with simplified structure to estimate dynamical invariants of strange attractors.

2.3 NARMAX Models

Consider the nonlinear autoregressive moving average model with exogenous inputs (NARMAX) (Leontaritis and Billings, 1985a; Leontaritis and Billings, 1985b)

$$\begin{aligned} y(k) = F^{\ell}[\, y(k-1), & \ \ldots, \ y(k-n_y), \\ u(k-d), & \ \ldots, \ u(k-d-n_u+1), \\ e(k), & \ \ldots \ , e(k-n_e)\,] \ , \end{aligned} \tag{4}$$

where n_y, n_u and n_e are the maximum lags considered for the output, input and noise terms, respectively and d is the delay measured in sampling intervals, T_s. Moreover, $u(k)$ and $y(k)$ are respectively input and output time series obtained by sampling the continuous data $u(k)$ and $y(k)$ at T_s. Furthermore, $e(k)$ accounts for uncertainties, possible noise, unmodelled dynamics, etc. and $F^{\ell}[\cdot]$ is some nonlinear function of $y(k)$, $u(k)$ and $e(k)$ with nonlinearity degree $\ell \in \mathbb{Z}^+$. In this paper, the map $F^{\ell}[\cdot]$ is taken to be a polynomial of degree ℓ. In order to estimate the parameters of this map, equation (4) has to be expressed in prediction error form as

$$y(k) = \Psi^{\mathrm{T}}(k-1)\hat{\Theta} \ + \ \xi(k) \ , \tag{5}$$

where

$$\begin{aligned} \Psi^{\mathrm{T}}(k-1) &= \left[\Psi^{\mathrm{T}}_{yu}(k-1) \ \ \Psi^{\mathrm{T}}_{yu\xi}(k-1) \ \ \Psi^{\mathrm{T}}_{\xi}(k-1)\right] , \\ \hat{\Theta} &= \left[\hat{\Theta}^{\mathrm{T}}_{yu} \ \ \hat{\Theta}^{\mathrm{T}}_{yu\xi} \ \ \hat{\Theta}^{\mathrm{T}}_{\xi}\right]^{\mathrm{T}} , \end{aligned} \tag{6}$$

and where $\Psi^{\mathrm{T}}_{yu}(k-1)$ is a matrix which contains linear and nonlinear combinations of output and input terms up to and including time $k-1$. The matrices $\Psi^{\mathrm{T}}_{yu\xi}(k-1)$ and $\Psi^{\mathrm{T}}_{\xi}(k-1)$ are defined similarly. The parameters corresponding to each term in such matrices are the elements of the vectors $\hat{\Theta}_{yu}$, $\hat{\Theta}_{yu\xi}$ and $\hat{\Theta}_{\xi}$, respectively. Finally, $\xi(k)$ are the residuals which are defined as the difference between the measured data $y(k)$ and the one-step-ahead prediction $\Psi^{\mathrm{T}}(k-1)\hat{\Theta}$. The parameter vector Θ can be estimated by minimizing the following cost function (Chen *et alii*, 1989).

$$J_{\mathrm{LS}}(\hat{\Theta}) \doteq \| \, y(k) - \Psi^{\mathrm{T}}(k-1)\hat{\Theta} \, \| \qquad (7)$$

where $\| \cdot \|$ is the Euclidean norm. Moreover, least squares minimization is performed using orthogonal techniques in order to effectively overcome two major difficulties in nonlinear model identification, namely i) numerical ill-conditioning and ii) structure selection. In order to circumvent such problems orthogonal techniques may be used (Billings *et alii*, 1988; Korenberg and Paarmann, 1991).

2.4 Structure Selection

The number of terms in a polynomial grows very rapidly even for relatively low values of ℓ, n_y, n_u and n_e. This is too difficult a problem to be solved by trial and error. However, effective and elegant solutions to handle this problem are available. see (Billings *et alii*, 1988; Aguirre and Billings, 1995d) and the survey paper by Haber and Unbehauen (Haber and Unbehauen, 1990). One solution is the *error reduction ratio* (ERR) test (Billings *et alii*, 1988; Billings *et alii*, 1989; Korenberg *et alii*, 1988). Two advantages of this approach are i) it does not require the estimation of a complete model to determine the significance of a candidate term and its contribution to the output, and ii) the ERR test is derived as a by-product of the orthogonal estimation algorithm.

Because the final models will be composed of a reduced number of terms, which is a small fraction of the total number of candidate terms, the models in this paper can be viewed as 'simplified' or 'concise' global polynomials. It is believed that these models overcome some of the practical difficulties usually reported for non-simplified polynomials.

In a recent paper it has been shown that the ERR criterion gives qualitatively similar results as higher order spectrum techniques in detecting nonlinear interactions within the underlying dynamics (Aguirre and Billings, 1994a). A criterion similar to ERR has been used to generate radial basis functions with a small number of parameters (Mees, 1993). Other techniques can be used in connection with ERR such as the zeroing-and-refitting approach (Kadtke *et alii*, 1993) and the concept of term clustering (Aguirre and Billings, 1995d).

In a multivariable model with r inputs and m outputs, the entries in equation (4) are vectors, that is $u(k) = [u_1(k) \ \dots \ u_r(k)]^{\mathrm{T}}$, $y(k) = [y_1(k) \ \dots \ y_m(k)]^{\mathrm{T}}$ and , $e(k) = [e_1(k) \ \dots \ e_m(k)]^{\mathrm{T}}$, and both structure detection and parameter estimation can be performed in a way which is analogous to the monovariable case (Billings *et alii*, 1989).

The quantity ERR provides an indication of which terms to include in the model. In other words, the ERR test provides a means of ordering all the candidate terms according to a hierarchy which depends on the relative importance of each term. It should be noted that no trial-and-error is necessary for this. However, the following question arises: how many terms should be included in the model? A practical way of addressing this question is by means of *information criteria* such as the *final prediction error* (FPE) (Akaike, 1974), *Akaike's information criterion* (AIC) (Akaike, 1974), the *Bayesian information criterion* (BIC) (Kashyap, 1977), the *model entropy* (Crutchfield and McNamara, 1987) and the *Schwarz information criterion* (Mees, 1993). See (Gooijer *et alii*, 1985) for a survey of such techniques.

2.5 Model Validation

The last step in any identification problem is the validation of the estimated models which is not a trivial problem. Most 'conventional' approaches to model validation are not particularly attractive when the models are chaotic and therefore alternative invariants should be used to quantify the quality and adequacy of the estimated models. When validating nonlinear models it is desirable that the criteria used should be sensitive to the 'fundamental' features of the models. In this respect it has been shown that bifurcation diagrams are far more sensitive to variations in model structure (Aguirre and Billings, 1994c) than many other nonlinear invariants used in model validation such as Poincaré maps (Casdagli, 1989; Gottwald *et alii*, 1992) correlation dimension (Grassberger *et alii*, 1991), Lyapunov exponents (Abarbanel *et alii*, 1989; Principe *et alii*, 1992), reconstructed phase-space plots (Adomaitis *et alii*, 1990). Recently, the concept of synchronization, which is rather well known in the field of control of chaos, has been suggested as a nontrivial test for validating estimated models (Brown *et alii*, 1994).

Finally, it should be pointed out that most of the results described so far in this section have been obtained using the tools described in the first paper of the series. Many of the commonly used statistical tools are of very limited use in assessing dynamical properties of estimated models.

2.6 NARMAX and Other Approaches

Unlike many localized techniques, the NARMAX approach does not involve finding neighborhoods, thus $N_{\mathrm{n}} = 1$ and all the data belong to a unique 'neighbourhood', that is, $\mathbf{y}(k) \in \mathcal{U}_1 \ k = d_{\mathrm{e}}, \dots, N$. This reduces the number of data required to estimate the dynamics. Moreover, the delay time is taken to be equal to the sampling period, thus $\tau = T_{\mathrm{s}}$.

There are a number of important differences between NARMAX polynomial identification and other methods. First, a NARMAX model includes input terms. This enables fitting data from non-autonomous systems and therefore estimating input/output maps, see (Casdagli, 1992; Hunter, 1992) for related ideas on this subject. An immediate consequence of this is that for input/output systems, it is not required that the output be on any particular attractor. Once an input/output model has been estimated, a particular input can be used to generate data on a specific attractor.

Another important difference is the presence of noise terms, that is, the moving average part of the model. It should be noted that equation (2) will only hold in the unlikely case when noise is absent. Any noise in the data or any imperfection in the estimate of the map f_T will result in an extra term in the right hand side of equation (2). Such a term would be responsible for modelling the mismatch introduced by the noise and unmodelled dynamics. It is a well known result in the theory of system identification that if such a term is omitted from the model structure, the estimate of the map f_T will become biased during parameter estimation (Söderström and Stoica, 1989) and nonlinear models are no exception to this rule (Billings and Voon, 1984).

It seems that when the noise is white and enters the system as a purely additive component, the division of the data into neighbourhoods and subsequent estimation reduces the bias. This will not be the case however if the model is global or if the noise is correlated. Thus in order to avoid bias a model for the noise and uncertainties, $\Psi_{yu\xi}^{\mathrm{T}}(k-1)\hat{\Theta}_{yu\xi} + \Psi_{\xi}^{\mathrm{T}}(k-1)\hat{\Theta}_{\xi}$, is included in the model structure before proceeding to parameter estimation. Once parameters have been estimated, only the deterministic part of the model is used, namely $\Psi_{yu}^{\mathrm{T}}(k-1)\hat{\Theta}_{yu}$. This procedure can handle moderate amounts of white and correlated noise.

Summarising, equation (5) is a hybrid model since it is composed of a deterministic part and a stochastic component. The latter is only used during parameter estimation in order to avoid bias on the former. Therefore, in this paper the models used to generate the surrogate data are purely deterministic although the stochastic part of the models is also represented for clarity. Thus, the deterministic component of the identified models is an approximation to the dynamics, that is, $f_T \approx \Psi_{yu}^{\mathrm{T}}(k-1)\hat{\Theta}_{yu}$ where $T=T_s$.

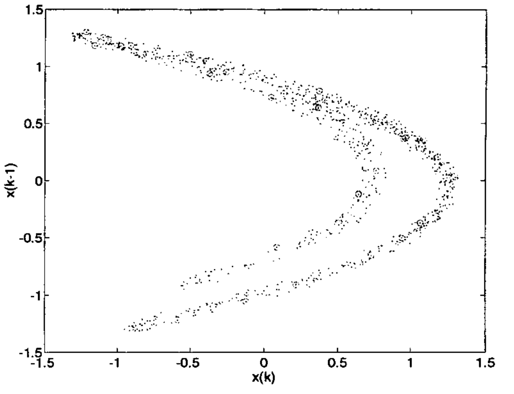
2.7 Modeling of Dynamical Systems

In this section some of examples are given to illustrate the performance of NARMAX polynomials in the modelling of nonliner systems. It is worth stressing that the stochastic component of some models is represented for greater clarity since such component is needed during parameter estimation to avoid bias. However, only the deterministic part is actually used to iterate the models in order to generate the figures. The emphasis is on the reconstruction of dynamical properties.

2.7.1 Poincaré Sections

The first return map for 1000 points taken from data obtained from the Hénon map (Hénon, 1976) is shown in figure 3a. The following model was estimated from the 50 points marked with circles

(a)



(b)

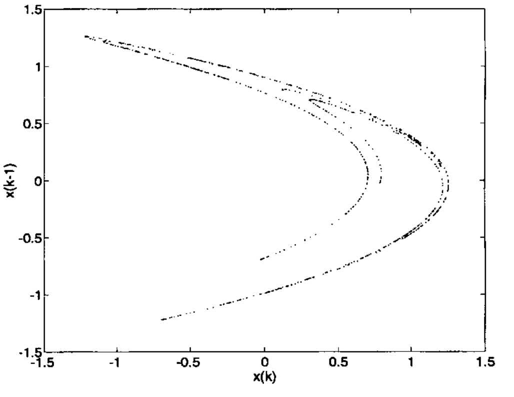


Figure 3 - First return map for (a) the Hénon map contaminated with noise. Only the encircled data were used in the estimation, (b) the identified map of equation (8).

$$x(k) = 0.99433 - 1.3818\,x(k-1)^2 + 0.29302\,x(k-2) \quad . \quad (8)$$

The first return map for this equation is shown in figure 3b and has a correlation dimension of $D_c = 1.11 \pm 0.22$ which shows good agreement with the original map for which $D_c = 1.21 \pm 0.01$. The correlation dimension estimated directly from 20000 data points with the same SNR as above was $D_c = 1.76 \pm 0.06$ revealing that the estimated value is quite sensitive to such levels of noise. Further improvement can be achieved by using more than 50 points, but the objective in this example was to show that the map can be estimated fairly accurately from a short and noisy time series.

In the case of driven oscillators, the practical reconstruction of Poincaré sections is restricted to controlled experiments because of the large amount of data required since only one point in such sections is obtained for each forcing period. Moreover, small amounts of noise often blur the delicate fractal structure of the attractor and the Poincaré sections tend to become fuzzy.

The fuzziness in the Poincaré sections introduced by the noise is a direct consequence of superimposing a stochastic component on the top of a purely deterministic trajectory. The Poincaré sections reconstructed using NARX models do not suffer from such fuzziness because although the stochastic component was present during the model estimation, the effects of such a component were 'absorbed' by the moving average (MA) part of the model enabling unbiased estimation. The NARX part of the model, which is purely deterministic, will *not* induce any fuzziness in the Poincaré sections. The fact that the models are not perfect, however, will be revealed by possible distortions in the shape or the reconstructed attractors.

The following model was obtained from 1500 data points on the Duffing-Holmes attractor shown in figure 4a

$$\begin{aligned} y(k) &= 0.84725\, y(k-1) + 0.35713\, y(k-3) \\ &- 0.69431\times10^{-1} y(k-1)^3 + 0.12780\times10^{-1} y(k-4) \\ &+ 0.61319\times10^{-1} u(k-1) + 0.40325\, y(k-2) \\ &- 0.24349\times10^{-3} y(k-1)y(k-2)y(k-5) - 0.46215\, y(k-5) \\ &+ 0.096339\, u(k-3) - 0.15316\, y(k-2)y(k-3)y(k-4) \\ &- 0.73618\times10^{-2} y(k-1)^2 y(k-5) \\ &+ 0.071822\, y(k-1)y(k-3)y(k-4) \\ &+ \Psi_\xi^{\mathrm{T}}(k-1)\hat{\Theta}_\xi + \xi(k)\ . \end{aligned} \qquad (9)$$

An estimate of the original Poincaré section is shown in figure 4b which was obtained by iterating equation (9). This reconstructed Poincaré section is very similar to the original one.

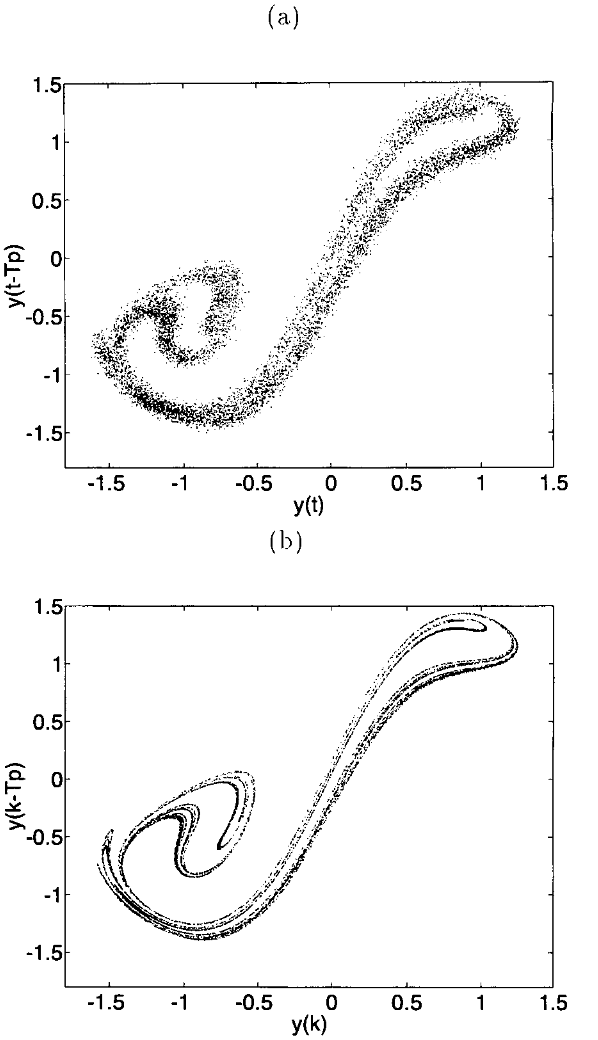


Figure 4 - Poincaré sections (a) obtained from a noisy orbit of the Duffing-Holmes oscillator, (b) of the identified model of equation (9). $A=0.3$ and $\omega=1\,\mathrm{rad/s}$, $T_{\mathrm{p}}=5$.

2.7.2 Bifurcation Diagrams

Using 1500 data points generated by simulation of the modified van der Pol equation (see first paper), with this SNR, sampled at $T_s=\pi/80$, the following NARMAX model was estimated

$$\begin{aligned} y(k) &= 0.83599\, y(k-1) + 0.87488\times10^{-1} y(k-4) \\ &+ 0.68539\times10^{-1} u(k-2) + 0.46776\times10^{-2} y(k-1)^3 \\ &- 0.47330\, y(k-6) + 0.12786\, y(k-2) \\ &+ 0.37341\, y(k-3) - 0.22840\times10^{-2} u(k-1) \\ &+ 0.49504\times10^{-1} y(k-5) - 0.014841\, y(k-1)^2 y(k-2) \\ &- 0.081389\, u(k-3) + 0.038305\, u(k-5) \\ &- 0.13554\times10^{-1} u(k-4) + 0.20404\times10^{-2} y(k-2)^2 y(k-3) \\ &- 0.34234\times10^{-2} y(k-1)y(k-6)^2 \\ &+ 0.35999\times10^{-2} y(k-2)y(k-4)y(k-6) \\ &+ \Psi_\xi^{\mathrm{T}}(k-1)\hat{\Theta}_\xi + \xi(k)\ . \end{aligned} \qquad (10)$$

This model also has a self-sustained oscillation with $\omega = 1.56\,\mathrm{rad/s}$.

Figure 5 shows that the identified model (10) does reproduce the major bifurcation patterns of the original system. See figure 14a of the first paper in the series for the original bifurcation diagram. This is usually more demanding than, for instance, requiring that a model should reproduce invariants associated with particular attractors (Aguirre and Billings, 1994c).

2.7.3 Original and Embedded Trajectories

This section reports some results concerning the use of NARMAX polynomials in reproducing embedded and original trajectories of strange attractors. To investigate this the well known Chua's double scroll attractor is used. If all variables are measured, multivariable NARMAX models can be fitted to the data and the iterated discrete-time outputs can be used to reconstruct the original attractor geometry in state-space.

The data in figure 6a were used to identify the following multivariable model

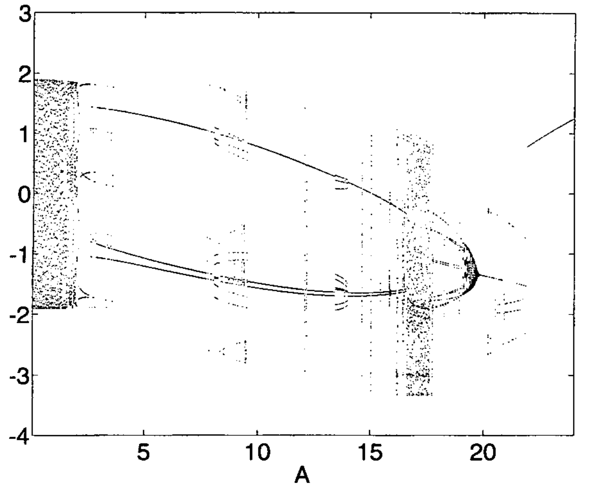


Figure 5 - Bifurcation diagram of the identified model of equation (10). $\omega = 4\,\text{rad/s}$.

$$
\begin{aligned}
x(k) &= 0.11282\times 10x(k-1) + 0.55867\,y(k-1) \\
&- 0.47190\times 10^{-1}x(k-1)^3 \\
&+ 0.39895\times 10^{-1}y(k-1)z(k-1)^2 \\
&- 0.31229\times 10^{-2}z(k-1)^3 + 0.18363\times 10^{-1}z(k-1) \\
&+ \Psi^{\mathrm{T}}_{\xi_x\xi_y\xi_z}(k-1)\hat{\Theta}_{\xi_x\xi_y\xi_z} + \xi_x(k) \\
y(k) &= 0.91948\,y(k-1) - 0.10392\times 10^{-3}z(k-1)^3 \\
&+ 0.70843\times 10^{-1}x(k-1) + 0.67800\times 10^{-1}z(k-1) \\
&- 0.13424\times 10^{-2}x(k-1)^3 \\
&+ 0.44206\times 10^{-3}x(k-1)^2y(k-1) \\
&+ \Psi^{\mathrm{T}}_{\xi_x\xi_y\xi_z}(k-1)\hat{\Theta}_{\xi_x\xi_y\xi_z} + \xi_y(k) \\
z(k) &= 0.96628\,z(k-1) - 0.95854\,y(k-1) \\
&- 0.36719\times 10^{-1}x(k-1) - 0.55765\times 10^{-1}y(k-1)^3 \\
&+ 0.10333\times 10^{-2}x(k-1)^3 \\
&+ 0.0020536\,x(k-1)y(k-1)z(k-1) \\
&+ \Psi^{\mathrm{T}}_{\xi_x\xi_y\xi_z}(k-1)\hat{\Theta}_{\xi_x\xi_y\xi_z} + \xi_z(k)\ . \qquad (11)
\end{aligned}
$$

This estimated model settles to a strange attractor which closely resembles the original double scroll Chua's attractor, see figure 6b.

Similar models for the Lorenz and Rössler attractors have been reported in (Aguirre and Billings, 1995e).

3 NOISE REDUCTION

A difficulty which appears to be common to most approaches for modelling nonlinear dynamical systems and chaotic attractors is that realistically noise will be present in the data. In particular, it has been conjectured that the local divergence of nearby orbits in a chaotic system seems to impose a natural limit on the accuracy of prediction-based identification algorithms when the data are noisy (Aguirre and Billings, 1995c). Consequently, there has been great motivation to develop filtering techniques for chaotic systems. This comprises one of the current phases in the investigation of chaos (Mitschke, 1990; Chennaoui *et alii*, 1990; Schreiber and Grassberger, 1991; Broomhead *et alii*, 1992; Davies, 1992; Grassberger *et alii*, 1993; Holzfuss and Kadtke, 1993).

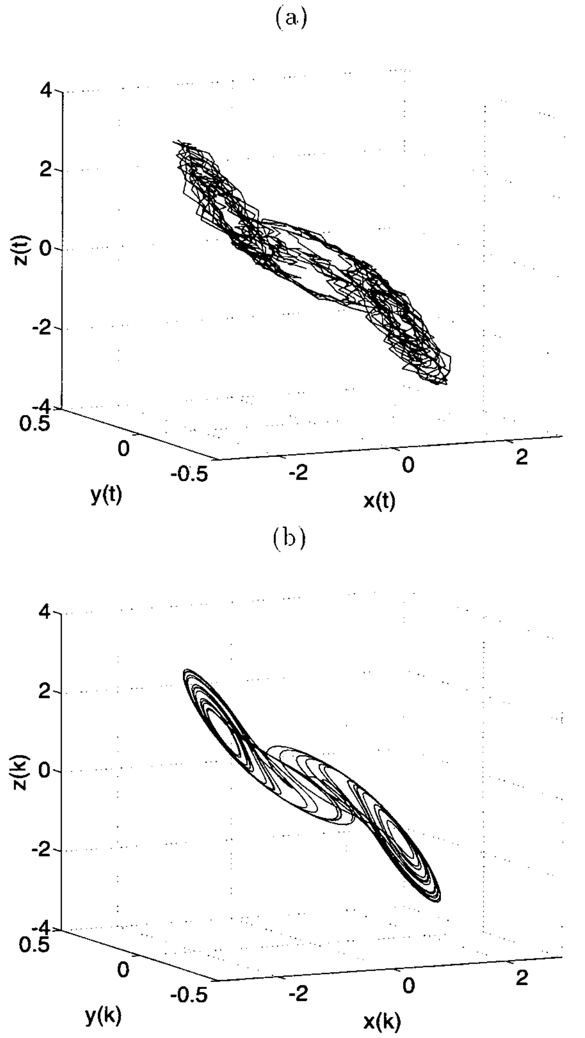


Figure 6 - (a) Noisy trajectory used for identification, SNR=72.9, 39.9 and 75.5 dB for x, y and z components, respectively. (b) Double scroll Chua's attractor reconstructed from the identified model in equation (11).

Some authors assume that some kind of *a priori* knowledge concerning the original system is available such as a piece of noise-free data (Marteau and Abarbanel, 1991), the structure of the maps describing the underlying dynamics (Davies, 1992), or even the complete maps, that is, structure and parameters are known (Hammel, 1990; Ozaki, 1993). However, a clear limitation in any real noise reduction problem is that the underlying dynamics are not usually known *a priori* and the map has to be estimated (learned) from the noisy data as an integral part of the noise reduction process. Consequently, the noise will pose limitations on the amount of noise which can effectively be eliminated. In the field of nonlinear dynamics, the main objective of filtering a chaotic time series is to enable the reconstruction and estimation of dynamical invariants such as Poincaré sections, Lyapunov exponents and fractal dimensions.

Another objective of filtering nonlinear data is to enable the identification of dynamically valid models which would reproduce the aforementioned invariants from sequences of filtered data. In a first attempt to solve this identification problem, global nonlinear predictors were used to filter the data (Aguirre and Billings, 1995c). In such a procedure the noise was separated from the signal by means of (very) short-term predictions. Because a chaotic predictor actually amplifies uncertainties in some 'direction' in state space, the aforementioned approach cannot be used to filter the noise by successive passes through the data and this therefore limits the achievable noise reduction.

This problem can be alleviated by using global smoothers because, unlike prediction-based techniques, smoothers are well suited for filtering chaotic data via successive noise-reduction iterations (Aguirre *et alii*, 1995).

3.1 Filtering Techniques

It is usually assumed that the noise is purely additive, or in other words the noise is entirely observational (Crutchfield and McNamara, 1987; Casdagli *et alii*, 1991; Grassberger *et alii*, 1991). Thus the noise reduction problem can be stated as follows: given a chaotic time series $x(t)$, it is desired to filter the measured data $y(t)=x(t)+e(t)$, where $e(t)$ is the additive noise, in order to recover $x(t)$. This is useful in 'cleaning' Poincaré sections and embedded attractors which have been blurred by noise.

Another aspect of this problem is to find a 'noise-reduced' orbit $\bar{y}(t)$ from which invariants such as λ_1, D_c and the attractor geometry can be more accurately estimated than if the noisy data $y(t)$ were used. This is sometimes referred to as *statistical noise reduction* as opposed to recovering $x(t)$ from $y(t)$ which has been called *detailed noise reduction* (Farmer and Sidorowich, 1991). In this paper, the objective is to be able to identify dynamically valid models from $\bar{y}(t)$.

Filtering based on model predicted outputs, whilst reducing the noise content in the data, will not guarantee that $\bar{y}(t)$ remains close to $y(t)$ (and ultimately close to $x(t)$) if the latter is chaotic. Thus, to ensure that $\bar{y}(t)$ remains close to $y(t)$, the following cost function can be used

$$J_{\rm NR} = \sum_{k=1}^{N} \{J_1\left[\bar{y}(k) - g_k(\bar{y}(k-1))\right] \ + \ J_2\left[\bar{y}(k) - y(k)\right]\} \tag{12}$$

where N is the number of points in the data, $J_1[\cdot]$ and $J_2[\cdot]$ indicate functions which are usually metric norms and $g_k(\cdot)$ are linear maps which describe the dynamics in a neighbourhood of a point on the true orbit. Clearly $J_1[\cdot]$ penalizes deviations from the true deterministic dynamics described by $g_k(\cdot)$ while $J_2[\cdot]$ guarantees that the cleaned orbit remains close to the measured orbit.

In particular, the following cost functions have been used (Kostelich and Yorke, 1988; Kostelich and Yorke, 1990)

$$J_1[\cdot] = \| \ \bar{y}(k) - g_k(\bar{y}(k-1)) \ \|^2 + \| \ \bar{y}(k+1) - g_k(\bar{y}(k)) \ \|^2 \tag{13}$$

where $\| \cdot \|$ is the Euclidean norm, and

$$J_2[\cdot] = \| \ \bar{y}(k) - y(k) \ \|^2 \tag{14}$$

Another option is to choose $J_2[\cdot]$ as above and

$$J_1[\cdot] = 2 \ \| \ g_k(\bar{y}(k)) - \bar{y}(k+1) \ \|^{\rm T} \ \mu_k \tag{15}$$

where μ_k are Lagrange multipliers (Farmer and Sidorowich, 1991).

In the field of system identification, improving the *signal to noise ratio* (SNR) is also of interest because this facilitates both the unbiased estimation of the parameter vector and the correct determination of the model structure. The chief idea is to estimate the noise-free data and then use this estimate to perform parameter estimation. A way of doing this is to use the following predictor which can be derived from equation (5)

$$\hat{y}(t) = \Psi_{\hat{y}u}^{\rm T}(t-1)\hat{\Theta}_{yu} \tag{16}$$

It should be realised that in the last equation the parameter vector $\hat{\Theta}_{yu}$ was estimated from the original noisy data as is indicated by the absence of the hat on the subscript y. On the other hand, the matrix $\Psi_{\hat{y}u}^{\rm T}(t-1)$ was formed using predicted values of the data, that is $\hat{y}(t)$ up to and including time $t-1$. Because $\hat{y}(t)$ is an estimate of $x(t)$, equation (16) can be used in suboptimal parameter estimation schemes (Billings and Voon, 1984). However, if the data were chaotic after a few iterations $\hat{y}(t)$ would not be an accurate estimate of $x(t)$ because of the sensitive dependence on initial conditions. Therefore the use of $\hat{y}(t)$ in suboptimal schemes seems somewhat restricted for chaotic systems. The next two sections describe approaches which overcome some of these problems.

3.2 The Resetting Filter

The following predictor has been suggested to overcome some of the difficulties associated with the filtering of chaotic data (Aguirre and Billings, 1995c)

$$\hat{y}(t) = \Psi_{yu}^{\rm T}(t-1)\,\hat{\Theta}_{yu} + \Psi_{\xi}^{\rm T}(t-1)\,\hat{\Theta}_{\xi} \tag{17}$$

It should be noted that in this case $\hat{y}(t)$ is predicted based on previous values of the measured data $y(s)$, $s \leq t-1$, and not based on previously predicted values such as in equation (16). Moreover, since this predictor is used to predict only one step into the future, the predicted value $\hat{y}(t)$ is, in most cases, guaranteed to remain close to the data $y(t)$. This can be interpreted as being a consequence of the *resetting* effect achieved by using measured data to initialise the predictor at each step. The predictor in equation (17) will be referred to as the *resetting filter* (RF) and it is adequate for filtering chaotic signals.

The qualitative effect attained by the resetting filter is, in some respects, analogous to other methods. The resetting effect of the RF guarantees that $J_2[\cdot]$ (see equation (12)) is kept small. Moreover, the parameter vector of the RF is obtained by minimising J_{LS} in equation (7), which is clearly analogous to $J_1[\cdot]$ in equation (12). The main difference is that whilst $g_k(\cdot)$ usually represents local linear maps, $\Psi^T(t-1)\,\Theta$ is a global nonlinear map which may include inputs and residual in addition to output terms.

Predictor-based filtering for chaotic systems will not work in general because of the inability of making long-term accurate predictions along the unstable manifold. Therefore in such directions, the filter would actually amplify the noise (Schreiber and Grassberger, 1991). The same is valid for the RF, but to a much lesser extent because of the resetting effect which will guarantee that any noise amplification along the unstable manifold is kept to a minimum. However, if several passes through the data are required to attain the desired level of noise reduction, it is inevitable that the effect of positive Lyapunov exponents be manifest. Consequently, the filtered data may not resemble the original sequence and, in fact, might have a greater noise content than the raw data.

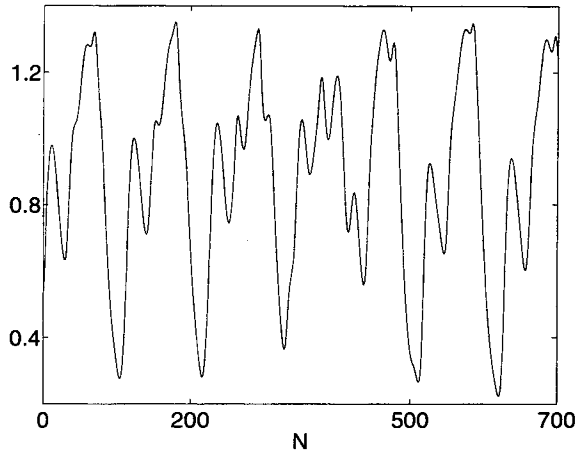


Figure 7 - Noise-free original time series for the Mackey-Glass model

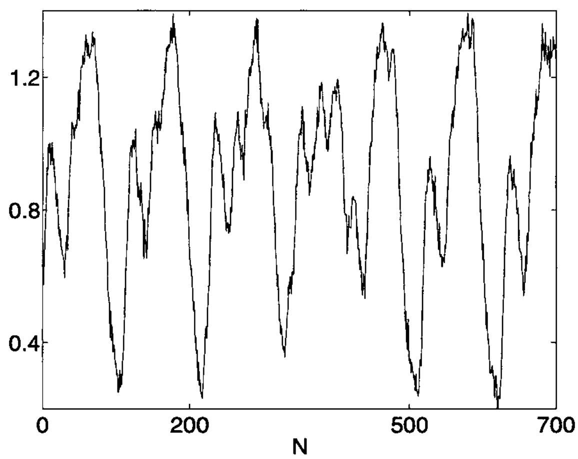


Figure 8 - Noisy (raw) time series for the Mackey-Glass model

3.3 Global Nonlinear Smoothers

The difficulty with the resetting filter in equation (17) is that it only uses past information to predict the future. However, the dynamics can only be predicted with any certainty as $t \rightarrow \infty$ along the stable manifold. Conversely, the dynamics can only be 'predicted' along the unstable manifold in reverse time, that is as $t \rightarrow -\infty$ (Schreiber and Grassberger, 1991). In other words, in order to estimate $y(t)$, future information as well as past information is required (Schreiber and Grassberger, 1991).

This motivates the search for NARMAX smoothers of the form

$$\begin{aligned} y(t) \;\; = \;\; & F_s^{\ell}[\, y(t-n_y), \ldots, y(t-1), y(t+1), \ldots, y(t+n_y), \\ & u(t-d-n_u+1), \ldots, u(t-d), u(t+d), \ldots \\ & \quad \ldots, u(t+d+n_u-1), \\ & \xi(t-1), \ldots, \xi(t-n_e)\,] + \xi(t) \; . \end{aligned} \tag{18}$$

It should be noted that equations analogous to (5)–(7) can be derived for the smoother in equation (18). Moreover, the ERR criterion, used to select the most important terms to compose a NARMAX model, can also be used to select the structure of the smoother and the same least squares algorithm can be used to estimate the parameters.

From a dynamical point of view, the smoother in equation (18) will also succeed in predicting the dynamics along the unstable directions because it contains terms which relate to the future. Such terms will enable 'predicting' in reverse time since orbits converge along the unstable manifold as $t \rightarrow -\infty$. An iterative procedure for smoothing data using nonlinear smoothers has been given in (Aguirre *et alii*, 1995).

3.3.1 An Example

Zero-mean Gaussian noise was added to a set of data obtained from the Mackey-Glass model (Mackey and Glass, 1977) (see first paper in the series). The resulting records with SNR=40.2 dB were smoothed with global nonlinear smoothers. Figures 7–9 respectively show the noise-free, noisy and smoothed data for the Mackey-Glass model and figure 10 shows the resulting data filtered following the RF approach. These figures make it plainly clear that prediction-type approaches for noise reduction are not suitable for chaotic data.

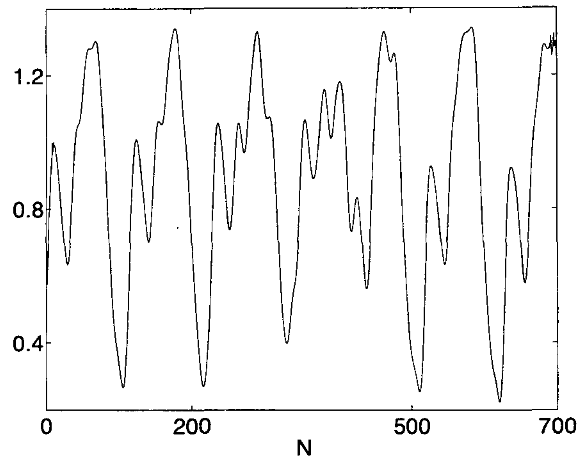


Figure 9 - Smoothed time series for the Mackey-Glass model. Terms with both negative and positive lags were used to compose the smoother, $N_p = 10$.

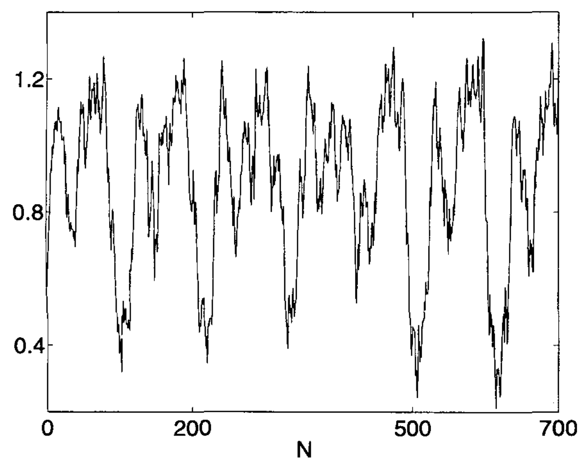


Figure 10 - Filtered time series for the Mackey-Glass model. Only terms with negative lags were used to compose the resetting filter. $N_p = 10$.

4 CONTROL AND SYNCHRONIZATION OF CHAOS

Is chaos a beneficial dynamical steady state? This is a central question in the control of chaotic systems. Of course, if the answer to the above question is yes, applied scientists and control engineers would be investigating ways of provoking chaos rather than suppressing it. A negative answer, on the other hand, would prompt researchers in the opposite direction.

Because of the sensitive dependence on initial conditions, displayed by chaotic systems, it is impossible to make accurate long-term predictions of such systems. In many situations, however, it is desirable that the system under investigation be predictable. Furthermore, the appearance of chaotic dynamics is not always welcome because in some situations it has been associated with abnormal behaviour (Glass and Mackey, 1988, pages 177, 179).

In other applications the onset of chaos seems to have several advantages. For instance, it has been argued that "a cognitive system *must* be chaotic in order to perform effective signal processing" (Nicolis, 1984). Further, chaos enhances heat transfer (Chang, 1992), improves mixing in chemical reactions (Ottino, 1992), reduces idle-channel tones in modulators (Schreier, 1994) and seems to have a promising future in secure communication systems (Cuomo *et alii*, 1993; Wu and Chua, 1993; Parlitz *et alii*, 1992). In addition, some authors have suggested that chaotic dynamics indicate a healthy state as opposed to the diseases which manifest as physiological periodic signals (Glass *et alii*, 1987; Goldberger *et alii*, 1990). The matter of how healthy chaos is, however, is far from settled (Pool, 1989). Consequently, techniques for controlling nonlinear dynamics are required in order to provoke or suppress chaos or any other dynamical regime according to the particular application at hand (Chen and Dong, 1993a; Ditto and Pecora, 1993; Hunt and Johnson, 1993).

Most of the works concerned with the control of chaos are devoted to stabilising a chaotic system to regular dynamics, that is, fixed points, periodic orbits or quasiperiodic regimes. The related problem of driving a system from a regular to a chaotic regime has received less attention. This type of control could be important in situations where chaos is not only welcome but also desirable (Goldberger *et alii*, 1990; Chang, 1992; Ottino, 1992; Cuomo *et alii*, 1993; Wu and Chua, 1993).

Clearly, chaos is *per se* neither beneficial nor harmful as described by James Gleick "In some applications, turbulence is desirable — inside a jet engine, for example, where efficient burning depends on rapid mixing. But in most, turbulence means disaster. Turbulent airflow over a wing destroys lift. Turbulent flow in an oil pipe creates stupefying drag" (Gleick, 1987, p. 122). Therefore it seems appropriate to search for control schemes which would perform well in both situations.

If on the one hand sensitivity to initial conditions hampers prediction-based control schemes, on the other hand such a property might turn out to be greatly advantageous from a control point of view. To see this it should be recalled that if a system is sensitive to initial conditions, a small perturbation at time t_0 can provoke relatively large effects at time $t > t_0$. This means that to achieve a certain control objective may require a much smaller control action if the system were chaotic. The problem of course is to determine how and when should the control action be applied. Some works in this direction have appeared in the literature (Ott *et alii*, 1990; Ditto *et alii*, 1990; Garfinkel *et alii*, 1992; Nitsche and Dressler, 1992; Romeiras *et alii*, 1992; Shinbrot *et alii*, 1990; Spano *et alii*, 1991).

Many different techniques have been investigated in the context of controlling chaos. Most methods can be grouped into two categories. When it is desired that chaos be suppressed the approaches are labelled under *control of chaos* and when the main objective is to make a system follow a chaotic trajectory the problem at hand is referred to as *synchronization of chaos*.

Chaos can be suppressed by the addition of small amplitude perturbations (Braiman and Goldhirsch, 1991), random perturbations (Kapitaniak, 1991), by parametric driving (Dorning *et alii*, 1992; Fronzoni *et alii*, 1991; Lima and

Pettini, 1990), by means of feedback (Liu *et alii*, 1994).

The problem of synchronization has been investigated in (Chua *et alii*, 1993; Kocarev *et alii*, 1993; Ogorzalek, 1993; Pecora, 1990; Wu and Chua, 1993).

The stabilization of chaotic systems has been achieved by applying feedback (Chen and Dong, 1993b; Dedieu and Ogorzalek, 1994; Hunt, 1991; Pyragas, 1992; Roy *et alii*, 1992), frequency harmonic balance techniques (Genesio and Tesi, 1993; Genesio and Tesi, 1992), conventional control techniques (Hartley and Mossayebi, 1993), open plus closed loop control (Jackson and Grosu, 1994), dynamical vibration absorbers (Kapitaniak *et alii*, 1993), adaptive control (Sinha *et alii*, 1990; Vassiliadis, 1993) and quantitative feedback design (QFD) (Yau *et alii*, 1993). The control of multiple attractor systems has been investigated in (Jackson, 1990).

Most of the references above are concerned with systems which are chaotic *before* control is applied. However, chaos has been detected in control systems in which the plant was not chaotic. Conditions for the occurrence of chaos in feedback systems (Genesio and Tesi, 1991), adaptive control (Mareels and Bitmead, 1986; Mareels and Bitmead, 1988; Golden and Ydstie, 1992) and in digital systems (Ushio and Hsu, 1987) have been reported in the literature.

The use of identified models in the design of control schemes has been addressed in (Aguirre and Billings, 1994b; Aguirre and Billings, 1995a; Aguirre, 1995b). It turns out that as long as an identified model reproduces some of the major dynamical features of the system, such a model can be used effectively in control problems. Many control schemes do not require a model or may even work with a model which is not *dynamically valid* but in such cases the control effort is usually greater and the control quality significantly poorer.

5 FINAL REMARKS

The subject of nonlinear dynamical systems has attracted great attention in recent years. It is therefore natural that various techniques for modeling and reconstructing such systems should be investigated. In this respect a landmark has been Taken's theorem and a number of subsequent results which today form the field of *embedology*. Parallel to these results, other techniques were developed by the engineering community. Such methods for the identification of nonlinear systems used other model structures such as Volterra and Wiener models, NARMAX models and neural networks. In the first part of this paper, the basic idea of embedding techniques has been reviewd. Similarly, the estimation on NARMAX polynomial models has been discussed and some differences between such approaches have been pointed out. The modeling of some nonlinear systems has been illustrated by numerical examples.

A major limitation in obtaining a good model for a nonlinear system is the noise present in the data. Noise can be, in a few cases, kept to a minimum but cannot be totally eliminated. Consequently there has been great interest in noise reduction algorithms. Some of such algorithms have been reviewed in a rather general framework and two algorithms, the resetting filter and nonlinear smoothers, have been described in some detail. It has been pointed out that if the data are chaotic special algorithms are usually required to achieve effective noise reduction.

Finally, a major issue in nonlinear dynamics nowadays is the control of chaotic systems. An enormous amount of papers have been published on this subject in the last years and a thorough review would be impossible. Nonetheless, several relevant references have been provided in order to enable the reader to further investigate this topic.

Throughout the paper it has been shown how NARMAX models can be used in the various problems concerning the modeling, noise reduction and control of nonlinear systems and chaos.

Acknowledgements

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