THE MINIMUM DELTA-V LAMBERT'S PROBLEM

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RESUMO. Este trabalho tem por objetivo formular e resolver uma nova variante do conhecido "Problema de Lambert", um dos mais importantes e discutidos tópicos em mecânica celeste. A idéia é substituir a exigência de que a transferência seja completada num tempo dado (problema original) pela exigência de que o consumo de combustível envolvido nessa manobra seja mínimo. Esse problema é resolvido através do desenvolvimento de equações analíticas para as componentes do impulso aplicado e teoria de minimização de funções. A seguir, são feitas simulações para a comparação entre os resultados obtidos por essa teoria e resultados disponíveis na literatura. Esses resultados podem ser facilmente estendidos para o estudo de uma transferência bi-impulsiva entre duas órbitas Keplerianas coplanares com mínimo consumo de combustível.

ABSTRACT. This paper formulates and solves a new version of the well-known "Lambert's Problem," one of the most important topics in celestial mechanics. The idea is to replace the requirement that the transfer must be completed in a given time (original problem) by the requirement that the fuel expenditure involved in this transfer must be minimum. This problem is solved by developing analytical equations for the components of the impulse applied and theory of minimization of functions. Next, simulations are made to compare the results obtained from this theory with results available in the literature. Those results are easily extended to the study of bi-impulsive transfers between two Keplerian and coplanar orbits with minimum expenditure of fuel.

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1. INTRODUCTION

The original Lambert's problem is one of the most important and popular topics in celestial mechanics. Several important authors worked on it, trying to find better ways to solve the numerical difficulties involved (Breakwell *et alii* 1961; Battin, 1965 and 1968; Lancaster *et alii* 1966; Lancaster & Blanchard, 1969; Herrick, 1971; Prussing, 1979; Sun & Vinh, 1983; Taff & Randall, 1985; Gooding, 1990). It can be defined as: "A Keplerian orbit, about a given gravitational center of force is to be found connecting two given points (P_1 and P_2) in a given time Δt".

This paper formulates and proposes several forms to solve a problem that is related to the Lambert's problem. This new formulation is a little bit different from the original one, but it also has many important applications. This new problem is called "Minimum Delta-V Lambert's Problem" and it is formulated as follows: "A Keplerian orbit, about a given gravitational center of force is to be found connecting two given points (P_1 that belongs to an initial orbit and P_2 that belongs to a final orbit), such that the ΔV for the transfer is minimum".

To solve this problem, the analytical expressions for the total increment of the velocity required ΔV (as a function of only one independent variable) and for its first derivative with respect to this variable are obtained. Then, a numerical scheme to get the root of the first derivative and the numeric value of the ΔV at this point is used. From this information it is possible to get all the other parameters involved, like the components of the impulses, their locations, etc. This research is closely connected to the search for a minimum two-impulse transfer between two given coplanar orbits in the approach hat is used in Prado (1993) and Broucke & Prado (1993). The only difference is that the initial and final points of the transfer are now fixed.

2. DEFINITION OF THE PROBLEM

Suppose that there is a spacecraft in a Keplerian orbit that is called O_0 (the initial orbit). It is desired to transfer this spacecraft to a final Keplerian orbit O_2, that is coplanar with the orbit O_0. To perform this transfer, we start at the point P_1 (r_1, θ_1), where an impulse with magnitude ΔV_1 that has an angle ϕ_1 with the local transverse direction is applied. The transfer orbit crosses the final orbit at the point P_2 (r_2, θ_2), where an impulse with magnitude ΔV_2 making an angle ϕ_2 with the local transverse direction is applied. To define the basic problem (the "Minimum Delta-V Lambert's Problem"), it is necessary to specify the true anomaly (θ_1) of the departure point in the orbit O_0 (point P_1) and the true anomaly (θ_2) of the point of arrival in the orbit O_2 (point P_2). With these two values given and all the Keplerian elements of both orbits known, it is possible to determine both radius vectors $\vec{r}_1$ and $\vec{r}_2$ at the beginning and at the end of the transfer. Then the problem is to find which transfer orbit connecting these two vectors and using only two impulses is the one that requires the minimum ΔV for the maneuver. This problem is what is defined here as the "Minimum ΔV Lambert's Problem". The sketch of the transfer and the variables used are shown in Fig. 1.

Using basic equations from the two-body celestial mechanics, it is possible to write an analytical expression for the total ΔV (= $\Delta V_1 + \Delta V_2$) required for this maneuver. To specify each of the three orbits involved in the problem, the elements D, h and k are used. They are defined by the following equations:

$$D = \frac{\mu}{C}; \quad k = eCos(\omega); \quad h = eSin(\omega) \tag{1}$$

where μ is the gravitational parameter of the central body; C is the angular momentum of the orbit, e is the eccentricity and ω is the argument of the periapse. The subscripts "0" for the initial orbit, "1" for the transfer orbit and "2" for the final orbit are also used. In those variables, the expressions for the radial (subscript r) and transverse (subscript t) components of the two impulses are:

$$\Delta V_{r1} = (D_1 k_1 - D_0 k_0)Sin(\theta_1) - (D_1 h_1 - D_0 h_0)Cos(\theta_1) \tag{2}$$

$$\Delta V_{t1} = D_1 - D_0 + (D_1 k_1 - D_0 k_0)Cos(\theta_1) + (D_1 h_1 - D_0 h_0)Sin(\theta_1) \tag{3}$$

$$\Delta V_{r2} = (D_2 k_2 - D_1 k_1)Sin(\theta_2) - (D_2 h_2 - D_1 h_1)Cos(\theta_2) \tag{4}$$

$$\Delta V_{t2} = D_2 - D_1 + (D_2 k_2 - D_1 k_1)Cos(\theta_2) + (D_2 h_2 - D_1 h_1)Sin(\theta_2) \tag{5}$$

The problem now is to find the transfer orbit that minimizes the total ΔV and that satisfies the two following constraints equations, expressing the fact that the orbits intersect:

$$g_1 = D_0^2(1 + k_0 Cos(\theta_1) + h_0 Sin(\theta_1)) - D_1^2(1 + k_1 Cos(\theta_1) + h_1 Sin(\theta_1)) = 0 \tag{6}$$

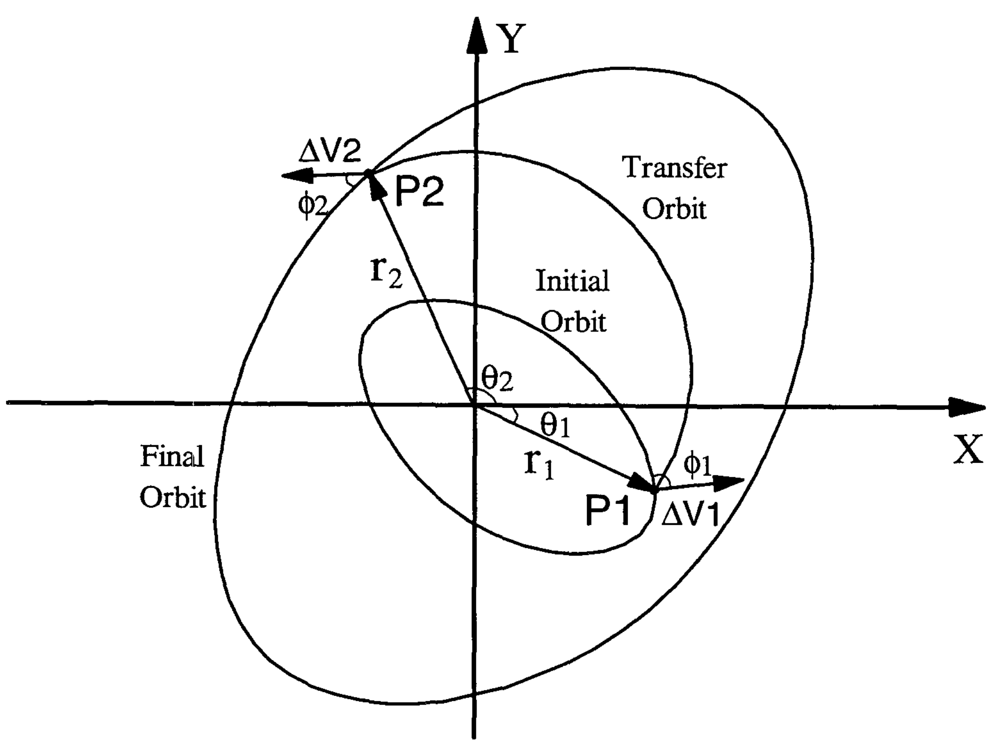

Fig. 1 - Geometry of the "Minimum ΔV Lambert's Problem".

$$g_2 = D_2^2\left(1+k_2 Cos(\theta_2)+h_2 Sin(\theta_2)\right)- \\ D_1^2\left(1+k_1 Cos(\theta_2)+h_1 Sin(\theta_2)\right)=0 \qquad (7)$$

The problem is reduced to the one of finding the value of D_1 that gives the minimum value for the expression $\Delta V = \sqrt{V_{r1}^2 + V_{t1}^2} + \sqrt{V_{r2}^2 + V_{t2}^2}$.

3. USING THE CHAIN RULE FOR THE DERIVATIVES

In this approach (and in the next one), the constraints (6) and (7) are used to solve this system for two of our variables, making the equation for the ΔV a function of only one independent variable. The system formed by these two equations is symmetric and linear in the variables h_1 and k_1, so the system is solved for these two variables. The results are the equations (8) and (9).

$$k_1 = -Csc(\theta_1-\theta_2)\left[\left(\left(\frac{D_0^2}{D_1^2}\right)\left(1+k_0 Cos(\theta_1)+h_0 Sin(\theta_1)\right)-1\right)Sin(\theta_2)-\ldots \\ \ldots-\left(\left(\frac{D_2^2}{D_1^2}\right)\left(1+k_2 Cos(\theta_2)+h_2 Sin(\theta_2)\right)-1\right)Sin(\theta_1)\right] \qquad (8)$$

$$h_1 = -Csc(\theta_1-\theta_2)\left[\left(\left(\frac{D_2^2}{D_1^2}\right)\left(1+k_2 Cos(\theta_2)+h_2 Sin(\theta_2)\right)-1\right)Cos(\theta_1)-\ldots \\ \ldots-\left(\left(\frac{D_0^2}{D_1^2}\right)\left(1+k_0 Cos(\theta_1)+h_0 Sin(\theta_1)\right)-1\right)Cos(\theta_2)\right] \qquad (9)$$

Now that the ΔV is a function of only one variable (D_1), elementary calculus can be used to find its minimum. All that has to be done is to search for the root of the expression $\frac{\partial(\Delta V)}{\partial D_1}=0$. From the definition of ΔV it is possible to write:

$$\frac{\partial(\Delta V)}{\partial D_1} = 0 = \frac{1}{\Delta V_1}\left[\Delta V_{r1}\frac{\partial(\Delta V_{r1})}{\partial D_1}+\Delta V_{t1}\frac{\partial(\Delta V_{t1})}{\partial D_1}\right]+ \\ \frac{1}{\Delta V_2}\left[\Delta V_{r2}\frac{\partial(\Delta V_{r2})}{\partial D_1}+\Delta V_{t2}\frac{\partial(\Delta V_{t2})}{\partial D_1}\right] \qquad (10)$$

Now, the chain rule for derivatives is applied to obtain expressions for the quantities $\frac{\partial(\Delta V_{r1})}{\partial D_1}$; $\frac{\partial(\Delta V_{t1})}{\partial D_1}$; $\frac{\partial(\Delta V_{r2})}{\partial D_1}$; $\frac{\partial(\Delta V_{t2})}{\partial D_1}$. A general expression for them is:

$$\frac{\partial(\Delta V_{ij})}{\partial D_1} = \left.\frac{\partial(\Delta V_{ij})}{\partial D_1}\right|_{Direct} + \frac{\partial(\Delta V_{ij})}{\partial k_1}\frac{\partial k_1}{\partial D_1} + \frac{\partial(\Delta V_{ij})}{\partial h_1}\frac{\partial h_1}{\partial D_1} \qquad (11)$$

where i = r,t; j = 1,2 and the word "Direct" stands for the part of the derivative that comes from the explicit dependence of ΔV_{ij} in the variable D_1. The expressions for $\frac{\partial(\Delta V_{ij})}{\partial k_1}$ and $\frac{\partial(\Delta V_{ij})}{\partial h_1}$ can be obtained from the equations (2) to (5) and the expressions for $\frac{\partial k_1}{\partial D_1}$ and $\frac{\partial h_1}{\partial D_1}$ can be obtained from the equations (8) to (9).

With all those equations available, a numerical algorithm can be built to iterate in the variable D_1 to find the unique real root of the equation $\frac{\partial(\Delta V)}{\partial D_1}=0$. To obtain the value of $\frac{\partial(\Delta V)}{\partial D_1}$ for a given D_1, necessary for the iteration process required, the following steps can be used:

i) Evaluate k_1 and h_1 from equations (8) and (9) for the given D_1;

ii) With D_1, h_1 and k_1 the equations (2) to (5) are used to evaluate ΔV_{r1}, ΔV_{t1}, ΔV_{r2}, ΔV_{t2}, ΔV_1 ($\sqrt{\Delta V_{r1}^2+\Delta V_{t1}^2}$) and ΔV_2 ($\sqrt{\Delta V_{r2}^2+\Delta V_{t2}^2}$);

iii) With all those quantities known, it is possible to evaluate $\frac{\partial(\Delta\ _{ij})}{\partial k_1}$ and $\frac{\partial(\Delta\ _{ij})}{\partial h_1}$ (obtained from equations (2) to (5)) and equation (10) to finally obtain $\frac{\partial(\Delta V)}{\partial D_1}$ for the given D_1.

4. SOLVING THE EQUATION $\frac{\partial(\Delta V)}{\partial D_1}=0$

At this point, it is important to remark that the function $\frac{\partial(\Delta V)}{\partial D_1}$ is very sensitive to small variations in D_1, specially

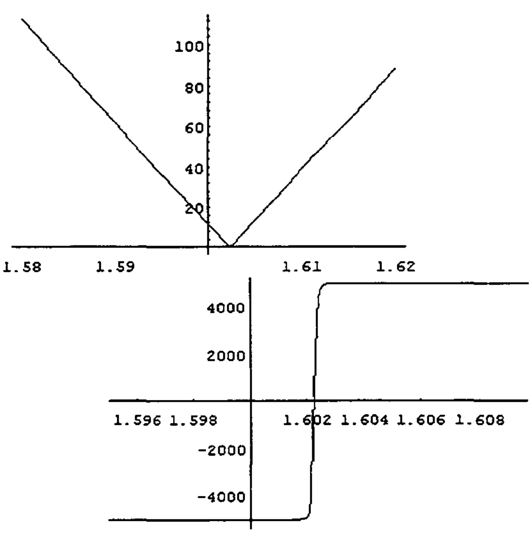

Fig. 2 - ΔV and its Derivative as a Function of D_1

hen close to the real root. Its curve is almost a straight line vith a slope that goes to infinity when θ_2 - θ_1 goes to 180°. ig. 2 shows the detail for a transfer where θ_2 - θ_1 = 3.14. rom that figure it is easy to see that this fact comes from the harpness of the curve ΔV x D_1, when close to the minimum. This characteristic is particular for the set of variables used ind it is not a physical problem. If another independent rariable is used, like the argument of the periapse of the ransfer orbit, the curve for the ΔV x D_1 has a much less sharp ninimum and, in consequence, its derivative has no big umps.

This behavior makes numerical methods to find the root based on derivatives (like the popular Newton-Raphson) inadequate. n this research, the method of dividing the interval in two parts in each iteration shows to be adequate, although not fast n convergence.

5. CALCULATING $\Delta V(D_1)$ EXPLICITLY

Another similar way to solve this problem is to use the equations for h_1 and k_1 (equations (8) and (9)) to find the equivalent of the equations (2) to (5) as a function of D_1 only. After some algebraic manipulations the following expressions (functions of D_1 only) can be obtained:

$$\begin{aligned}\Delta V_{r1} = -\frac{Csc(\theta_1-\theta_2)}{2D_1}\Big[&2\left(D_1^2-D_2^2\right)+\ldots\\ &\ldots+2\left(D_0^2-D_1^2\right)Cos(\theta_1-\theta_2)+\ldots\\ &\ldots+\left(D_0^2k_0-D_0D_1k_0\right)Cos(2\theta_1-\theta_2)+\ldots\\ &\ldots+\left(D_0^2k_0+D_0D_1k_0-2D_2^2k_2\right)Cos(\theta_2)+\ldots\\ &\ldots+\left(D_0^2h_0-D_0D_1h_0\right)Sin(2\theta_1-\theta_2)+\ldots\\ &\ldots+\left(D_0^2h_0+D_0D_1h_0-2D_2^2h_2\right)Sin(\theta_2)\Big]\end{aligned} \tag{12}$$

$$\Delta V_{t1} = \frac{D_0}{D_1}\left[(D_0-D_1)\left(1+k_0Cos(\theta_1)+h_0Sin(\theta_1)\right)\right] \tag{13}$$

$$\begin{aligned}\Delta V_{r2} = -\frac{Csc(\theta_1-\theta_2)}{2D_1}\Big[&2\left(D_1^2-D_0^2\right)+\ldots\\ &\ldots+2\left(D_2^2-D_1^2\right)Cos(\theta_1-\theta_2)+\ldots\\ &\ldots+\left(D_2^2k_2-D_1D_2k_2\right)Cos(\theta_1-2\theta_2)+\ldots\\ &\ldots+\left(D_2^2k_2+D_1D_2k_2-2D_0^2k_0\right)Cos(\theta_1)+\ldots\\ &\ldots+\left(D_1D_2h_2-D_2^2h_2\right)Sin(\theta_1-2\theta_2)+\ldots\\ &\ldots+\left(D_2^2h_2+D_1D_2h_2-2D_0^2h_0\right)Sin(\theta_1)\Big]\end{aligned} \tag{14}$$

$$\Delta V_{t2} = \frac{D_2}{D_1}\left[(D_1-D_2)\left(1+k_2Cos(\theta_2)+h_2Sin(\theta_2)\right)\right] \tag{15}$$

Those equations allow the calculation of the expression for $\frac{\partial(\Delta V)}{\partial D_1}$, that is given by expression (10). The partial derivatives involved are given by:

$$\begin{aligned}\frac{\partial(\Delta V_{r1})}{\partial D_1} = -\frac{Csc(\theta_1-\theta_2)}{2D_1^2}\Big[&2\left(D_1^2+D_2^2\right)-\ldots\\ &\ldots-2\left(D_0^2+D_1^2\right)Cos(\theta_1-\theta_2)-D_0^2k_0Cos(2\theta_1-\theta_2)+\ldots\\ &\ldots+\left(2D_2^2k_2-D_0^2k_0\right)Cos(\theta_2)-D_0^2h_0Sin(2\theta_1-\theta_2)+\ldots\\ &\ldots+\left(2D_2^2h_2-D_0^2h_0\right)Sin(\theta_2)\Big]\end{aligned} \tag{16}$$

$$\frac{\partial(\Delta V_{t1})}{\partial D_1} = -\left(\frac{D_0^2}{D_1^2}\right)\left[1+k_0Cos(\theta_1)+h_0Sin(\theta_1)\right] \tag{17}$$

$$\begin{aligned}\frac{\partial(\Delta V_{r2})}{\partial D_1} = -\frac{Csc(\theta_1-\theta_2)}{2D_1^2}\Big[&2\left(D_1^2+D_0^2\right)-\ldots\\ &\ldots-2\left(D_1^2+D_2^2\right)Cos(\theta_1-\theta_2)-D_2^2k_2Cos(\theta_1-2\theta_2)+\ldots\\ &\ldots+\left(2D_0^2k_0-D_2^2k_2\right)Cos(\theta_1)+D_2^2h_2Sin(\theta_1-2\theta_2)+\ldots\\ &\ldots+\left(2D_0^2h_0-D_2^2h_2\right)Sin(\theta_1)\Big]\end{aligned} \tag{18}$$

$$\frac{\partial(\Delta V_{t2})}{\partial D_1} = \left(\frac{D_2^2}{D_1^2}\right)\left[1+k_2Cos(\theta_2)+h_2Sin(\theta_2)\right] \tag{19}$$

Now, the same technique of dividing the interval in two parts in each iteration is used, to find the root of the equation (10).

6. USING LAGRANGE MULTIPLIERS

An elegant method to skip the algebraic work to solve equations (6) and (7) for h_1 and k_1 is to introduce two Lagrange multipliers λ_1 and λ_2. This is done by defining a new function to be minimized, given by the expression:

$$f(D_1,h_1,k_1,\lambda_1,\lambda_2) = \Delta V + \lambda_1 g_1 + \lambda_2 g_2 \tag{20}$$

where g_1 and g_2 are given by the equations (6) and (7).

Then, using the standard theory for Lagrange multipliers, the five equations in the five unknowns D_1, h_1, k_1, λ_1, λ_2 that have to be satisfied are obtained by treating all the variables as independent of the others. The equations are:

$$\begin{aligned}\frac{\partial f}{\partial D_1} = \frac{1}{\Delta V_1}\Big[&\left(D_1-D_0-D_0h_0h_1+D_1h_1^2-D_0k_0k_1+D_1k_1^2\right)+\\ &\ldots+\left(2D_1k_1-D_0k_1-D_0k_0\right)Cos(\theta_1)+\ldots\end{aligned}$$

$$\ldots+\left(2D_1h_1 - D_0h_1 - D_0h_0\right)Sin(\theta_1)\Big]\ldots$$

$$\ldots+\frac{1}{\Delta V_2}\Big[\left(D_1 - D_2 - D_2h_1h_2 + D_1h_1^2 - D_2k_1k_2 + D_1k_1^2\right)+\ldots$$

$$\ldots+\left(2D_1k_1 - D_2k_1 - D_2k_2\right)Cos(\theta_2)+\ldots$$

$$\ldots+\left(2D_1h_1 - D_2h_1 - D_2h_2\right)Sin(\theta_2)\Big]-\ldots$$

$$\ldots-2\lambda_1 D_1\left(1+k_1Cos(\theta_1)+h_1Sin(\theta_1)\right)-\ldots$$
$$\ldots-2\lambda_2 D_1\left(1+k_1Cos(\theta_2)+h_1Sin(\theta_2)\right)=0 \quad (21)$$

$$\frac{\partial f}{\partial h_1} = \frac{D_1}{\Delta V_1}\left[D_1k_1 - D_0k_0 + (D_1 - D_0)Cos(\theta_1)\right]+$$
$$+\frac{D_1}{\Delta V_2}\left[D_1k_1 - D_2k_2 + (D_1 - D_2)Cos(\theta_2)\right]-\cdots \quad (22)$$
$$\ldots-\lambda_1 D_1^2 Sin(\theta_1) - \lambda_2 D_1^2 Sin(\theta_2) = 0$$

$$\frac{\partial f}{\partial k_1} = \frac{D_1}{\Delta V_1}\left[D_1h_1 - D_0h_0 + (D_1 - D_0)Sin(\theta_1)\right]+$$

1. $$\frac{D_1}{\Delta V_2}\left[D_1h_1 - D_2h_2 + (D_1 - D_2)Sin(\theta_2)\right].-\cdots \quad (23)$$
$$\lambda_1 D_1^2 Cos(\theta_1) - \lambda_2 D_1^2 Cos(\theta_2) = 0$$

$$\frac{\partial f}{\partial \lambda_1} = g_1 = 0 \quad (24)$$

$$\frac{\partial f}{\partial \lambda_2} = g_2 = 0 \quad (25)$$

After that, the system of equations (21) to (25) is solved by numerical means. This solution gives all the information required to consider the problem solved.

The disadvantage of this approach is the increase in the number of variables and equations from one to five. The advantage is that the algebraic work to derive the previous equations shown in this paper can be skipped.

7. RESULTS

To test those equations, codes in standard FORTRAN are developed to run some examples to get numerical results to compare with the ones available in the literature. For this purpose the initial and final orbit of the transfer are choose to be the same ones chosen by Lawden (1991), when solving the related problem of optimal two-impulse transfer. They are:

$D_0 = \sqrt{3}$; $h_0 = 0$; $k_0 = 1/3$

$D_2 = \sqrt{2}$; $h_2 = 1/4$; $k_2 = 0.4333$

Then equation (10) is solved (by any of the forms showed in this paper) to find D_1 and the respective ΔV for a given pair of θ_1 and θ_2. This process is repeated for values of θ_1 and θ_2 in the range $0 \le \theta_1 \le 360$ and $0 \le \theta_2 \le 360$. Contour-plots are made to show the behavior of the ΔV as a function of θ_1 and θ_2. Fig. 3 shows the results. Every point (θ_1, θ_2) in that plot is one particular case of the "Minimum Delta-V Lambert's

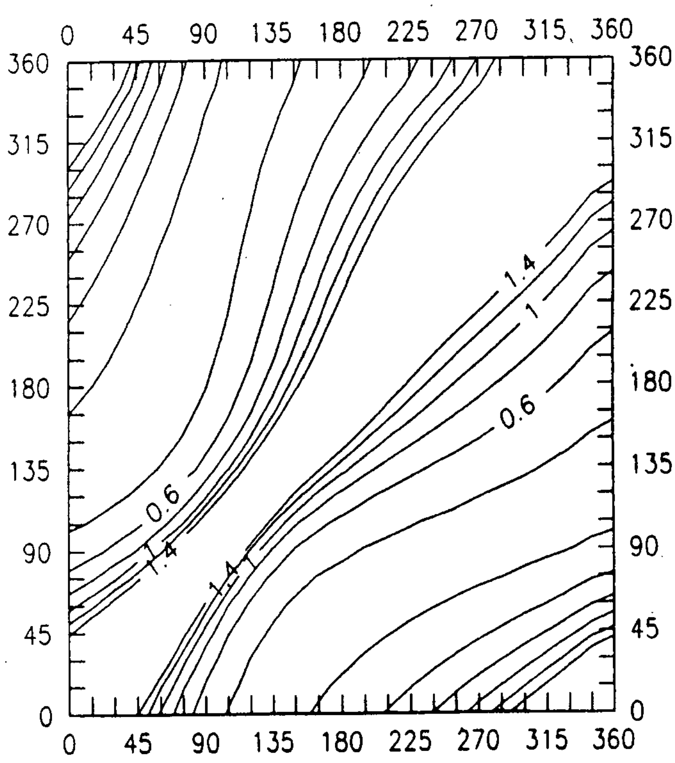

Fig. 3 - Contour-Plot for ΔV as a Function of θ_1 and θ_2.

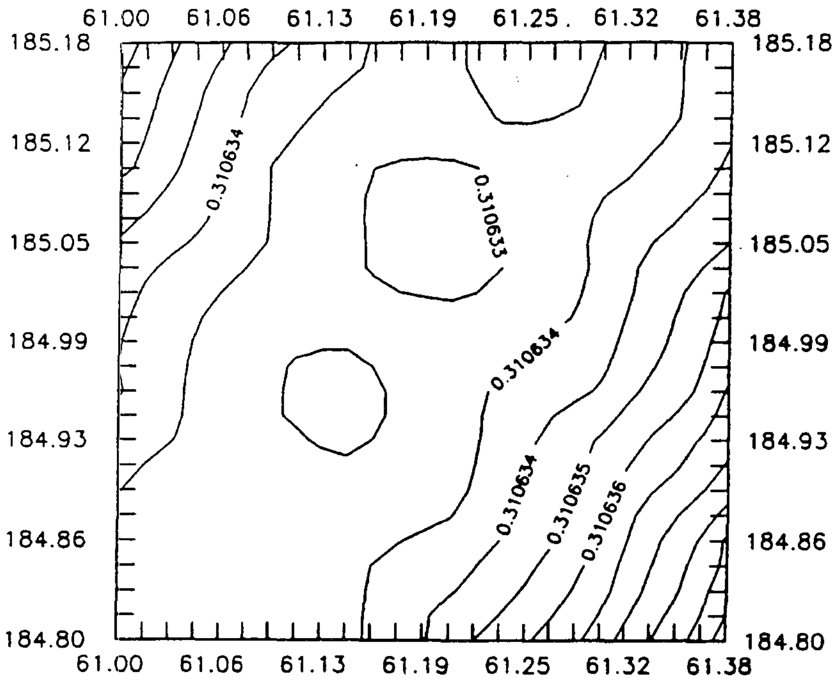

Fig. 4 - Contour-Plot for ΔV When θ_1 and θ_2 Are Close to the Absolute Minimum.

'roblem" and the ΔV associated is the solution of this case. :he whole picture is a collection of a large number of cases to over all the possibilities. Fig. 4 is an amplification of the egion close to the absolute minimum of ΔV, as shown in .awden (1991). From those plots it is possible to find the egions (in θ_1 and θ_2) that give us a minimum ΔV orbit ransfer between the two given orbits. If necessary, it is always ıossible to study in more detail a specific region, as done close ɔ the absolute minimum in Fig. 4. This procedure has the mportant practical applications of providing a good estimate to be used as a good first guess) for the minimum ΔV transfer ıetween the two orbits involved

}. CONCLUSIONS

[his paper formulates and proposes a solution to the Minimum Delta-V Lambert's Problem". This variant of the .ambert's problem has the original requirement of completing he transfer in a given time replaced by the new requirement hat the transfer has a minimum ΔV. The analytical :xpressions and numerical procedures to solve this problem ıre derived in different ways. Contour-plots for a test case are nade. It is also showed how to use this problem to solve the ›roblem of finding the minimum ΔV transfer orbit between wo given coplanar orbits.

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