A LINEAR OPTIMIZATION APPROACH TO $\mathcal{H}_\infty$ AND MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROL FOR DISCRETE-TIME UNCERTAIN SYSTEMS*

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ABSTRACT - This paper addresses the problems of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control and $\mathcal{H}_\infty$ guaranteed cost control for discrete-time uncertain linear systems. The uncertainty is supposed to belong to convex bounded polyhedral domains, with no extra assumptions as matching conditions. First, the set of all quadratic stabilizing state feedback gains, providing a prespecified γ disturbance attenuation level, is described in terms of linear matrix inequalities. Then, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control is achieved via the minimization of a linear objective function, which turns out to be an upper bound to the $\mathcal{H}_2$ norm of the closed-loop transfer function. The $\mathcal{H}_\infty$ guaranteed cost is globally achieved by involving the γ uuper bound into the minimization procedure. The adopted parameter space allows the formulation of associated optimization procedures with linear objective functions under linear matrix inequalities constraints, assuring the algorithms good numerical behavior. Furthermore, additional constraints, for instance decentralization, can be easily incorporated. Examples illustrate the theoretical results.

Keywords: Discrete-Time Linear Systems; $\mathcal{H}_\infty$ Control; Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control; Convex Analysis.

Uma Abordagem de Otimização Linear para Controle $\mathcal{H}_\infty$ e Controle Misto $\mathcal{H}_2/\mathcal{H}_\infty$ de Sistemas Incertos Discretos no Tempo

RESUMO - Este trabalho trata dos problemas de controle misto $\mathcal{H}_2/\mathcal{H}_\infty$ e de controle $\mathcal{H}_\infty$ de custo garantido para sistemas lineares incertos discretos no tempo. A incerteza é suposta pertencente a domínios convexos poliedrais, sem hipóteses do tipo *matching conditions*. Primeiramente, o conjunto de todos os ganhos de realimentação de estado quadraticamente estabilizantes e que proporcionam um nível prescrito de atenuação γ é descrito em termos de desigualdades matriciais lineares. Então, o problema de controle robusto $\mathcal{H}_2/\mathcal{H}_\infty$ é resolvido através da minimização de uma função objetivo linear, que é também um limitante para a norma $\mathcal{H}_2$ da função de transferência em malha fechada. O custo garantido $\mathcal{H}_\infty$ é obtido de maneira global. envolvendo-se o limitante γ no processo de minimização. O espaço de parâmetros utilizado permite a formulação de procedimentos de otimização com funções objetivos lineares e restrições expressas em termos de desigualdades matriciais lineares, assegurando o bom comportamento numérico dos algoritmos. Além disso, restrições adicionais, como por exemplo a descentralização do controle, podem ser facilmente incorporadas. Os resultados teóricos são ilustrados através de exemplos.

Palavras-Chave: Sistemas Lineares Discretos no Tempo; Controle $\mathcal{H}_\infty$; Controle Misto $\mathcal{H}_2/\mathcal{H}_\infty$; Análise Convexa.

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1 - INTRODUCTION

In recent years, the $\mathcal{H}_2$ and/or the $\mathcal{H}_\infty$ performance measures have appeared as two of the most important objective functions in optimal control design, for both continuous-time and discrete-time linear systems. Defined in the time domain as the Linear Quadratic Problem, $\mathcal{H}_2$ control has been exhaustively exploited during the sixties, with its optimal solution readily obtained from the associated Riccati equation. In (Doyle *et al.*, 1989), the state space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems for continuous-time linear systems have been posed in terms of Riccati equations and, since then, a lot of work has been done involving $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ state feedback control.

Regarding discrete-time linear systems, in spite of the $\mathcal{H}_2$ state feedback control being immediately derived from the Linear Quadratic Problem and from the solution of its associated discrete-time Riccati equation, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and the optimal $\mathcal{H}_\infty$ control problems are more difficult to solve. A Riccati equation based approach for discrete-time $\mathcal{H}_\infty$ control can be found in (Iglesias and Glover, 1991) and (Yaesh and Shaked, 1991) and references therein. Similar results, based on the discrete-time bounded real Lemma and the strong solution to a discrete algebraic Riccati equation, are provided in (De Souza and Xie, 1992). Concerning the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem, in (Haddad *et al.*, 1991) necessary conditions are derived in terms of coupled Riccati-like equations. A very important result is provided in (Kaminer *et al.*, 1993), where the use of an appropriate matrix transformation, introduced in (Bernussou *et al.*, 1989), allows the solution of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem via convex optimization.

In fact, convexity has appeared as a major desirable property when numerical solutions are investigated and it plays a crucial role for uncertain linear systems control design (Boyd *et al.*, 1994), (Geromel *et al.*, 1991). A convex parameter space approach for quadratic stabilizability of uncertain systems is provided in (Geromel *et al.*, 1991), in both continuous-time and discrete-time cases, where the uncertainty is considered to belong to convex bounded domains. The adopted formulation can also handle additional constraints as, for instance, decentralized control. As it has been shown in (Geromel *et al.*, 1993) and (Peres *et al.*, 1991), $\mathcal{H}_2$ or $\mathcal{H}_\infty$ criteria can also be incorporated to the optimization procedure, as well as any other convex constraints, like robustness against actuator failure (Peres and Geromel, 1993).

Concerning specifically uncertain discrete-time systems, the existing results for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_\infty$ control follow mainly two different ways. The first one is based on Riccati-like inequalities. The work of (De Souza *et al.*, 1993) proposes an $\mathcal{H}_\infty$ control synthesis based on the definition of an auxiliary dynamic system, depending on a vector of unknown parameters to be tuned. The authors, however, do not provide a procedure for the choice of these parameters. The second way is based on convex properties, that is, following the steps of (Bernussou *et al.*, 1989) and (Geromel *et al.*, 1991), the problems are restated in a convex programming framework. A convex approach to $\mathcal{H}_\infty$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of discrete-time uncertain systems in convex bounded domains is proposed, respectively, in (Geromel *et al.*, 1994) and (Geromel *et al.*, 1995), but an approximation had to be done to keep convexity, implying only sufficient conditions. The convex approach proposed in (Kaminer *et al.*, 1993) deals with necessary and sufficient conditions for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control and, thanks to convexity, it could be easily extended to deal with uncertain systems, but the auxiliary cost as well as the matrix inequalities involved are highly nonlinear, implying numerical difficulties.

This paper addresses the problems of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ (minimizing an upper bound to the $\mathcal{H}_2$ norm) and $\mathcal{H}_\infty$ state feedback control for discrete-time uncertain linear systems by means of a convex optimization approach. Indeed, the objective function as well as the matrix constraints related with the problems are **linear**, and the conditions proposed are necessary and sufficient. In this sense, this paper circumvents the nonlinearities and the drawbacks cited above, extending thus the results of (Geromel *et al.*, 1994), (Geromel *et al.*, 1995), (Kaminer *et al.*, 1993). The uncertainty is supposed to belong to convex polyhedral domains, thus covering a general class of uncertain discrete-time systems, including interval matrices. Thanks to the adopted formulation, additional constraints can be easily incorporated, as for instance robust decentralized state feedback control or robustness against actuator failure.

As another result, in case of precisely known systems, the relationship between the results of (Kaminer *et al.*, 1993) and (Yaesh and Shaked, 1991) is established. From the results of (Yaesh and Shaked, 1991), the solution of a Riccati equation yields a state feedback control guaranteeing a γ disturbance attenuation level and the minimization of a $\mathcal{H}_2$ upper bound. The same properties are assured by using the convex approach of (Kaminer *et al.*, 1993).

The paper is organized as follows. Section 2 presents the definitions and the basic results concerning $\mathcal{H}_2/\mathcal{H}_\infty$ norms, Riccati equations and linear matrix inequalities. For a review on $\mathcal{H}_2/\mathcal{H}_\infty$ control problems see, for instance, (Zhou *et al.*, 1995), and for a survey on control design problems through LMIs (linear matrix inequalities), see (Boyd *et al.*, 1994). Section 3 presents the main results, that is, the LMI formulation for the guaranteed $\mathcal{H}_\infty$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems; section 4 shows some examples, and the paper is closed by a concluding section.

2 - PRELIMINARIES

Let us consider the following linear time-invariant discrete-time system:

$$\begin{cases} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k) \\ u(k) &= Kx(k) \\ y(k) &= Cx(k) + Du(k) \end{cases} \tag{1}$$

where $x(k) \in \Re^n$ is the state, $u(k) \in \Re^m$ is the control, $w(k) \in \Re^s$ is the disturbance vector, $y(k) \in \Re^r$ is the controlled output. All matrices are of appropriate and known dimensions. The usual orthogonality hypothesis is made (that is, $C'D = \mathbf{0}$), and $D'D > \mathbf{0}$ is also assumed. Associated with the linear system given in (1), let us define augmented matrices $F \in \Re^{(n+r)\times(n+m)}$, $R \in \Re^{(n+m)\times(n+m)}$, $H \in \Re^{(n+m)\times(n+r)}$, $Q \in \Re^{(n+r)\times(n+r)}$ and $S \in \Re^{(n+r)\times(n+r)}$

$$F = \begin{bmatrix} A & B_2 \\ C & D \end{bmatrix} \quad , \quad R = \begin{bmatrix} C'C & \mathbf{0} \\ \mathbf{0} & D'D \end{bmatrix} \tag{2}$$

$$H = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad , \quad Q = \begin{bmatrix} B_1 B_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad , \quad S = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix} \tag{3}$$

The uncertainty is considered to only affect the pair (A, B_2); that is, the matrix F belongs to a convex polyhedral domain $\mathcal{D}$

$$\mathcal{D} \triangleq \left\{ F \ : \ F = \sum_{i=1}^{N} \xi_i F_i \ , \ \xi_i \geq 0 \ , \ \sum_{i=1}^{N} \xi_i = 1 \right\} \quad (4)$$

meaning that any feasible F can be written as an unknown convex combination of the N extreme matrices F_i, $i = 1 \cdots N$, where

$$F_i = \begin{bmatrix} A_i & B_{2i} \\ C & D \end{bmatrix} \quad (5)$$

Clearly, $N = 1$ describes a precisely known plant. For an arbitrary but fixed $F \in \mathcal{D}$, defining

$$A_{cl} \triangleq A + B_2 K \quad , \quad C_{cl} \triangleq C + DK \quad (6)$$

where K belongs to the set of all stabilizing state feedback gains, i.e.

$$\mathcal{K} \triangleq \left\{ K \in \Re^{m \times n} \ : \ A_{cl} \text{ asympt. stable} \right\} \quad (7)$$

the closed-loop transfer function from $w(k)$ to $y(k)$ is given by

$$G(z) \triangleq C_{cl} \left[z\mathbf{I} - A_{cl} \right]^{-1} B_1 \quad (8)$$

The $\mathcal{H}_2$-norm of the closed-loop transfer function (8) is given by

$$\|G\|_2^2 \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} Tr \left\{ G(e^{-j\omega})' G(e^{j\omega}) \right\} d\omega \quad (9)$$

and the $\mathcal{H}_\infty$-norm by

$$\| G\|_\infty \triangleq \max_{\omega \in [-\pi, \pi]} \sigma_{max} \left[G(z = e^{j\omega}) \right] \quad (10)$$

As is well known, the $\mathcal{H}_2$-norm can be easily calculated by

$$\|G\|_2^2 = Tr \ (C_{cl} L_c C_{cl}') = Tr \ (B_1' L_o B_1) \quad (11)$$

where L_c is the controllability Gramian and L_o is the observability Gramian, solutions to the discrete Lyapunov equations

$$A_{cl} L_c A_{cl}' - L_c + B_1 B_1' = \mathbf{0} \quad (12)$$

$$A_{cl}' L_o A_{cl} - L_o + C_{cl}' C_{cl} = \mathbf{0} \quad (13)$$

respectively.

Different transfer functions, for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, could be considered. Defining two different outputs

$$\begin{aligned} y_2(k) &= C_2 x(k) + D_2 u(k) \\ y_\infty(k) &= C_\infty x(k) + D_\infty u(k) \end{aligned} \quad (14)$$

and accordingly

$$G_2(z) \triangleq C_{cl2} \left[z\mathbf{I} - A_{cl} \right]^{-1} B_1 \ , \ G_\infty(z) \triangleq C_{cl2} \left[z\mathbf{I} - A_{cl} \right]^{-1} B_1 \quad (15)$$

In this case, matrices F and R should be redefined

$$F = \begin{bmatrix} A & B_2 \\ C_\infty & D_\infty \end{bmatrix} \ , \ R = \begin{bmatrix} C_2' C_2 & \mathbf{0} \\ \mathbf{0} & D_2' D_2 \end{bmatrix} \quad (16)$$

as well as F_i in equation (5). All the results on section 3 remain valid. For the sake of simplicity, it is assumed in the sequence that $y_2 = y_\infty$, that is, $C_2 = C_\infty$, $D_2 = D_\infty$ and $G_2 = G_\infty$.

The determination of the $\mathcal{H}_\infty$-norm is numerically more involved (Iglesias and Glover, 1991), usually performed through a unidimensional search procedure. A relationship between an upper bound to the $\mathcal{H}_\infty$-norm of a stable transfer function and a linear matrix inequality (LMI) is given by the following Lemma (Kaminer *et al.*, 1993) (Lemmas 1 to 5 apply only for systems without uncertainty):

Lemma 1: (Kaminer *et al.*, 1993) For a given $\gamma > 0$, the following statements are equivalent:

a) $\|G\|_\infty < \gamma$

b) There exists $Y = Y' > \mathbf{0}$ such that

$$\begin{bmatrix} A_{cl} \\ C_{cl} \end{bmatrix} Y \begin{bmatrix} A_{cl}' & C_{cl}' \end{bmatrix} + Q < \begin{bmatrix} Y & \mathbf{0} \\ \mathbf{0} & \gamma^2 \mathbf{I} \end{bmatrix} \quad (17)$$

A similar result can be found in (Yaesh and Shaked, 1991), where Riccati-like equations are related to $\mathcal{H}_\infty$-norm upper bounds, according to the following Lemma.

Lemma 2: (Yaesh and Shaked, 1991) $\|G\|_\infty < \gamma$ if and only if there exists a positive definite symmetric matrix P such that

$$A_{cl} P A_{cl}' - P + P C_{cl}' \left(\mathbf{I} + C_{cl} P C_{cl}' \right)^{-1} C_{cl} P + \gamma^{-2} B_1 B_1' = \mathbf{0} \quad (18)$$

The results stated in Lemma 1 and Lemma 2 can also be found in (De Souza and Xie, 1992). Notice that an iterative procedure could be stated in order to determine the $\mathcal{H}_\infty$-norm of (8), decreasing γ and testing for the existence of the positive definite symmetric matrix Y satisfying the conditions of Lemma 1 or, equivalently, $P = P' > \mathbf{0}$ such that (18) holds. Of course, conditions of Lemmas 1 and 2 are indeed the same, as shown in the following Lemma (the proof is straightforward, thus being omitted).

Lemma 3: Suppose $P = P' > \mathbf{0}$ satisfying Lemma 2 exists. Then,

a) The symmetric positive definite matrix Y given by

$$\hat{Y} = \gamma^2 \left(P^{-1} + C_{cl}' C_{cl} \right)^{-1} \quad (19)$$

is such that

$$A_{cl} \hat{Y} A_{cl}' - \hat{Y} + A_{cl} \hat{Y} C_{cl}' \left(\gamma^2 \mathbf{I} - C_{cl} \hat{Y} C_{cl}' \right)^{-1} C_{cl} \hat{Y} A_{cl}' + B_1 B_1' = \mathbf{0} \quad (20)$$

b) Any $Y = Y' > \mathbf{0}$ satisfying Lemma 1 is such that $\hat{Y} \leq Y$.

The above results are related to $\mathcal{H}_\infty$ analysis of discrete-time systems. In other words, they state the relationship between a given $\gamma > 0$ (an upper bound to the $\mathcal{H}_\infty$-norm of a stable transfer function) and the existence of a certain positive definite matrix. Concerning $\mathcal{H}_\infty$ control synthesis, references (Kaminer *et al.*, 1993), (De Souza and Xie, 1992) and (Yaesh and Shaked, 1991) also present important results for precisely known systems. In (Kaminer *et al.*, 1993), using the matrix transformation introduced in (Bernussou *et al.*, 1989), they show that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for discrete-time systems can be solved by convex optimization. The next Lemma partially restates their result:

Lemma 4: (Kaminer *et al.*, 1993) Given a prespecified $\gamma > 0$, there exists a stabilizing state feedback control gain K such that $||G||_\infty < \gamma$ if and only if there exist $Y \in \Re^{n\times n}$, $Y = Y' > \mathbf{0}$, and $X \in \Re^{m\times n}$ such that the following convex inequality holds

$$F\begin{bmatrix} Y \\ X \end{bmatrix} Y^{-1} \begin{bmatrix} Y & X' \end{bmatrix} F' + Q < \begin{bmatrix} Y & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix} \qquad (21)$$

In the affirmative case, $K = XY^{-1}$.

For comparison purposes, a result of (Yaesh and Shaked, 1991) based on discrete-time Riccati-like equation, is reproduced here:

Lemma 5: (Yaesh and Shaked, 1991) Suppose the pair (A, B_2) is completely controllable and (C, A) is completely observable. Given a prespecified $\gamma > 0$, there exists a stabilizing state feedback control gain $K \in \Re^{m\times n}$ such that $||G||_\infty < \gamma$ if and only if there exists a positive definite matrix $P \in \Re^{n\times n}$ satisfying the following Riccati-like equation

$$P = A'PA + \gamma^{-2}PB_1\left(\mathbf{I} + \gamma^{-2}B_1'PB_1\right)^{-1}B_1'P - $$
$$- A'PB_2\left(D'D + B_2'PB_2\right)^{-1}B_2'PA + C'C \quad (22)$$

In the affirmative case, the control gain is given by

$$K = -\left(D'D + B_2'PB_2\right)^{-1}B_2'PA \qquad (23)$$

With the results of Lemma 5 (Yaesh and Shaked, 1991), the $\mathcal{H}_\infty$ control of precisely known systems can be achieved by solving (22). For a fixed γ, the control gain K also guarantees an upper bound of the closed-loop $\mathcal{H}_2$-norm. However, in spite of equation (22) being easily solved by eigenvalue methods (Yaesh and Shaked, 1991), the above result can not be extended to deal with uncertain systems. As a matter of fact, the control gain K depends explicitly on matrices A and B_2, and equation (22) involves the matrix P in highly nonlinear terms.

Note that the Riccati equation (22) can be obtained from the optimality conditions of min $Tr\{B_1'PB_1\}$ ($\geq ||G||_2^2$) subject to a closed-loop condition — "dual" to (18) — assuring $||G||_\infty < \gamma$.

Alternatively, in (Kaminer *et al.*, 1993) it is shown that by adequately choosing a convex criterion, an upper bound on the closed-loop $\mathcal{H}_2$-norm can be minimized, thus assuring mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control. Since the expressions are also convex on A and B_2, an extension to handle uncertain systems in convex and/or norm bounded domains can also be provided. Notice also that, contrary to Lemma 5, in Lemma 4 there is no explicit dependence of the control gain K on matrices A and B_2. However, the nonlinearities in (21) involve both Y and X.

The aim of this paper is to solve the $\mathcal{H}_2/\mathcal{H}_\infty$ control problem in the context of uncertain discrete-time systems in convex bounded domains using the optimization of linear criteria under LMI constraints. Two problems are investigated: the minimization of an $\mathcal{H}_2$ upper bound under an $\mathcal{H}_\infty$ disturbance attenuation constraint and the optimal $\mathcal{H}_\infty$ guaranteed cost control. First, the concept of quadratic stabilizability with γ disturbance attenuation is restated:

Definition 1: An uncertain system is said to be quadratically stabilizable with $\gamma > 0$ disturbance attenuation if there exist a control gain K and a matrix $Y = Y' > 0$ such that

$$\begin{bmatrix} A + B_2K \\ C + DK \end{bmatrix} Y \begin{bmatrix} (A + B_2K)' & (C + DK)' \end{bmatrix}$$
$$+ Q < \begin{bmatrix} Y & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix} \qquad (24)$$

holds for all $F \in \mathcal{D}$ (implying closed-loop quadratic stability and $||G||_\infty < \gamma$ for all $F \in \mathcal{D}$).

The set of all quadratic stabilizing state feedback control gains assuring γ disturbance attenuation is denoted by $\mathcal{K}_{\mathcal{Q}}^{\gamma}$. Similarly to the continuous-time case, the quadratic stabilizability condition with a prespecified $\gamma > 0$ disturbance attenuation is stated in terms of the same K and the same $Y = Y' > 0$ holding for all feasible models $F \in \mathcal{D}$. The problems to be investigated are:

Problem 1: For a given $\gamma > 0$, obtain K and ρ^* such that

$$\rho^* = \min\left\{\rho \;:\; ||G||_2^2 \leq \rho\,,\; K \in \mathcal{K}_{\mathcal{Q}}^{\gamma}\right\} \qquad (25)$$

For $N = 1$, the above problem reduces to the solution of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of precisely known systems. In this particular case, a solution can be achieved by using the results of Lemma 5 (Yaesh and Shaked, 1991) or by minimizing an appropriately chosen nonlinear (but convex) criterion over the set of (X, Y) satisfying Lemma 4 (Kaminer *et al.*, 1993). For uncertain systems, however, the upper bounds must be satisfied for all $F \in \mathcal{D}$.

Problem 2: Obtain K and γ^* such that

$$\gamma^* = \min\left\{\gamma \;:\; K \in \mathcal{K}_{\mathcal{Q}}^{\gamma}\right\} \qquad (26)$$

Again, in the case $N = 1$, the optimal $\mathcal{H}_\infty$ control is achieved. It can be approximately solved by reducing the γ

parameter and testing the existence of $P = P' > \mathbf{0}$ in (22). From the numerical point of view, the difficulty in solving (22) may be increased due to the fact that γ^{-2} is not explicitly bounded. A discussion about an upper bound to γ^{-2} can be found in (Geromel *et al.*, 1995).

Alternatively, the optimal solution of (26) could be obtained using the convex properties of Lemma 4, by defining $\delta \triangleq \gamma^2$ and the convex problem $\min\{\delta : (21)$ holds with $Y = Y' > \mathbf{0}\}$ which can be adapted to cope with uncertain matrices $F \in \mathcal{D}$. The main difficulty of the above optimization problem stems from the nonlinearities involving its variables (X, Y, δ).

3 - MAIN RESULTS

In this section, Problems 1 and 2 are solved. Note that the results are stated in terms of guaranteed $\mathcal{H}_2/\mathcal{H}_\infty$ cost, i.e., for the uncertain case. In the case $N = 1$, the set $\mathcal{K}_Q^\gamma$ reduces to the set of stabilizing gains assuring γ disturbance attenuation.

First, a Theorem establishes the equivalence between $K \in \mathcal{K}_Q^\gamma$, for a given $\gamma > 0$, and the existence of a certain symmetric matrix $\mathcal{W} \in \Re^{(n+m)\times(n+m)}$ satisfying a set of linear matrix inequalities. This matrix is partitioned as

$$\mathcal{W} = \begin{bmatrix} W_1 & W_2 \\ W_2' & W_3 \end{bmatrix} \quad (27)$$

with $W_1 \in \Re^{n\times n}$ positive definite. Then, the solution of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and the $\mathcal{H}_\infty$ guaranteed cost control problems is addressed.

Theorem 6: Given $\gamma > 0$, system (1) is quadratically stabilizable with γ disturbance attenuation if and only if there exists a symmetric positive semidefinite matrix $\mathcal{W}$ partitioned as in (27) such that

$$F_i \mathcal{W} F_i' - H'\mathcal{W}H + Q - \gamma^2 S < \mathbf{0} \quad , \quad \forall\, i = 1\cdots N \quad (28)$$

where H, Q and S are defined in (3).

Proof: To prove the necessity, from Definition 1, there exist K and $Y = Y' > 0$ such that (for all $F \in \mathcal{D}$)

$$\begin{bmatrix} A + B_2K \\ C + DK \end{bmatrix} Y \begin{bmatrix} (A + B_2K)' & (C + DK)' \end{bmatrix} + Q - \begin{bmatrix} Y & \mathbf{0} \\ \mathbf{0} & \gamma^2 \mathbf{I} \end{bmatrix} < 0 \quad (29)$$

implying the existence of the symmetric matrix given by

$$\mathcal{W} = \begin{bmatrix} Y & YK' \\ KY & KYK' \end{bmatrix} \geq \mathbf{0} \quad (30)$$

such that

$$F\mathcal{W}F' - H'\mathcal{W}H + Q - \gamma^2 S < \mathbf{0} \quad (31)$$

holds for all feasible F. In particular, it also holds for the extreme matrices F_i, $i = 1\cdots N$, proving the necessity part of the Theorem.

To show the sufficiency, from equation (28), $\mathcal{W}$ partitioned as in (27) and the definition of H, Q and S, it follows for all i, $i = 1\cdots N$

$$\begin{bmatrix} A_i & B_{2i} \\ C & D \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_2' & W_3 \end{bmatrix} \begin{bmatrix} A_i' & C' \\ B_{2i}' & D' \end{bmatrix} - \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_2' & W_3 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} B_1B_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \gamma^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} < \mathbf{0} \quad (32)$$

After some algebraic manipulation:

$$\begin{bmatrix} A_i + B_{2i}W_2'W_1^{-1} \\ C + DW_2'W_1^{-1} \end{bmatrix} W_1 \times \begin{bmatrix} (A_i + B_{2i}W_2'W_1^{-1})' & (C + DW_2'W_1^{-1})' \end{bmatrix} + \begin{bmatrix} B_1B_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} W_1 & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix} + \begin{bmatrix} B_{2i} \\ D \end{bmatrix} [W_3 - W_2'W_1^{-1}W_2] \begin{bmatrix} B_{2i}' & D' \end{bmatrix} < \mathbf{0} \quad (33)$$

for all i, $i = 1\cdots N$. Since $\mathcal{W} \geq \mathbf{0}$ with $W_1 > 0$, the submatrix W_3 is such that $W_3 \geq W_2'W_1^{-1}W_2$ implying that, with $K = W_2'W_1^{-1}$

$$\begin{bmatrix} A_{cli} \\ C_{cl} \end{bmatrix} W_1 \begin{bmatrix} A_{cli}' & C_{cl}' \end{bmatrix} + Q < \begin{bmatrix} W_1 & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix} \quad (34)$$

Finally, since (34) is convex on A_{cli} (see (Geromel *et al.*, 1991)), from (24) the conclusion is that $K = W_2'W_1^{-1} \in \mathcal{K}_Q^\gamma$, implying that $||G||_\infty < \gamma$ for all $F \in \mathcal{D}$. The Theorem is proved.

Based on the above Theorem, the set $\mathcal{K}_Q^\gamma$ can be entirely obtained from matrices $\mathcal{W}$ satisfying (28). Now, using this result, Problem 1 can be solved by minimizing a linear criterion subject to linear matrix inequalities:

Theorem 7: For $\gamma > 0$ given, the global optimal solution of the problem

$$\rho = \min \left\{ Tr\,(R\mathcal{W}) \;:\; F_i\mathcal{W}F_i' - H'\mathcal{W}H + Q - \gamma^2 S < \mathbf{0}\,, \;\forall\, i = 1\cdots N \right\} \quad (35)$$

with $\mathcal{W} = \mathcal{W}' \geq 0$ partitioned as in (27), yields $K = W_2'W_1^{-1}$ such that

a) $||G||_\infty < \gamma$

b) $||G||_2^2 < \rho$

for all feasible models $F \in \mathcal{D}$.

Proof: Part a) follows straightforwardly from Theorem 6. Let us prove part b). By developing the linear criterion given in (35), one gets

$$Tr\,(R\mathcal{W}) = Tr\,(CW_1C' + DW_3D') \quad (36)$$

which, since $\mathcal{W} \geq 0 \iff W_3 \geq W_2' W_1^{-1} W_2$, yields (using the orthogonality condition $C'D = \mathbf{0}$)

$$Tr\,(R\mathcal{W}) \quad \geq Tr\ ((C + DW_2'W_1^{-1})W_1(C + DW_2'W_1^{-1})') \tag{37}$$

Equation (34), with $K = W_2'W_1^{-1}$, implies

$$A_{cli} W_1 A_{cli}' - W_1 + B_1 B_1' < \mathbf{0} \tag{38}$$

for all i, $i = 1 \cdots N$. Since, for a fixed F and a given control gain K, the $\mathcal{H}_2$ norm is given by (11)-(13), from the above inequalities one can conclude that $W_1 > L_c$. Finally, equation (37) yields $Tr\ (R\mathcal{W}) > ||G||_2^2\ \forall\ F \in \mathcal{D}$, for all $\mathcal{W}$ satisfying Theorem 6. Since ρ is the minimum value of $Tr\ (R\mathcal{W})$ such that (28) holds, $||G||_\infty < \gamma$ and $||G||_2^2 < \rho$ for all $F \in \mathcal{D}$. The Theorem is proved.

Concerning the proposed upper bound to the $\mathcal{H}_2$ norm, two situations need to be considered. In the same transfer function case, the objective function assures the resulting controller to be as close as possible to the central $\mathcal{H}_\infty$ one. For precisely known systems, this means that the gap goes to zero as $\gamma \to \infty$, increasing as $\gamma \to \gamma^*$. For uncertain systems, the gap will be as small as possible but guaranteeing quadratic stability with γ disturbance attenuation. If different transfer functions are considered, as $\gamma \to \gamma^*$ the central $\mathcal{H}_\infty$ controller may yield better results or not. Nevertheless, the upper bound used in the paper yields a guaranteed $\mathcal{H}_2$ cost in all the cases.

With the results of Theorems 6 and 7, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be solved by means of powerful numerical procedures with sure convergence towards the global optimal solution (Boyd *et al.*, 1994), (Gahinet and Nemirovskii, 1994) For instance, a cutting plane like algorithm (based on the one proposed in (Geromel *et al.*, 1991) has been implemented and the numerical results will be shown in the next section. Thanks to the linearity, the numerical behavior in both uncertain and precisely known cases is very good. For comparison purposes (see section 4), a Fibonacci search sequence has also been programmed in order to iteratively solve equation (22), using the eigenvalue decomposition proposed in (Yaesh and Shaked, 1991) along with the upper bound to γ^{-2} proposed in (Geromel *et al.*, 1995). As a final remark, (35) obviously reduces to the $\mathcal{H}_2$ guaranteed cost control problem when $\gamma \to \infty$. This can be attested by verifying the equivalence, in this case, between equations (34) and (38).

Now, let us turn our attention to Problem 2. The next Theorem proposes a procedure to jointly optimize $\gamma > 0$ and $\mathcal{W}$ satisfying Theorem 6, in order to achieve an $\mathcal{H}_\infty$ guaranteed cost control.

Theorem 8: Define $\delta \triangleq \gamma^2$ and the linear problem

$$\min \left\{ \delta \ :\ F_i \mathcal{W} F_i' - H'\mathcal{W}H + Q - \delta S < \mathbf{0}\ ,\quad i = 1 \cdots N \right\} \tag{39}$$

with $\mathcal{W} = \mathcal{W}' \geq 0$ partitioned as in (27). Then, $K = W_2'W_1^{-1}$ is such that

a) $||G||_\infty < \sqrt{\delta}$ for all $F \in \mathcal{D}$.

b) In the case $N = 1$, the optimal $\mathcal{H}_\infty$ control is achieved.

Proof: It follows directly from Theorem 6. Since the above problem is convex (as a matter of fact, it is linear), the global optimal solution of problem (39) will be found, meaning that the upper bound $\gamma = \sqrt{\delta}$ reaches its lowest level.

Several remarks are now pertinent. First at all, the optimal $\mathcal{H}_\infty$ control and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control can be solved by using the results of Theorems 7 and 8 via the most powerful existing numerical methods (Boyd *et al.*, 1994), (Gahinet and Nemirovskii, 1994) since the criteria are linear and the constraints are linear matrix inequalities. Moreover, the upper bound γ can be explicitly involved into the optimization procedure, with no need of iterative line searches, and since γ is a positive scalar, this approach has no numerical problem regarding the upper bounds for γ, contrasting with the iterative application of the results presented in (Yaesh and Shaked, 1991). Notice also that, following the lines of (Geromel *et al.*, 1991), any additional convex constraint can be readily incorporated to the problems, yielding for instance a decentralized $\mathcal{H}_2/\mathcal{H}_\infty$ state feedback control or robustness against actuator failure (Peres and Geromel, 1993). A sufficient condition for decentralized state feedback control can be obtained by simply imposing to $\mathcal{W}$ the structure

$$W_D = \begin{bmatrix} W_{1D} & W_{2D} \\ W_{2D}' & W_3 \end{bmatrix} \tag{40}$$

where the subscript "D" denotes a block diagonal matrix and testing the conditions of Theorems 6, 7 or 8. If a solution is found, $K \equiv K_D = W_{2D}' W_{1D}^{-1}$ is the decentralized control gain. Notice that the above structural constraint preserves the linearity of the proposed solution.

As a final remark, the above results can be viewed as an extension of the results presented in (Peres *et al.*, 1991), (Geromel *et al.*, 1994) where the nonlinearities of equation (18) were circumvented by an approximation, providing only sufficient but convex conditions.

4 - EXAMPLES

Following the ideas of (Geromel *et al.*, 1991), a cutting plane based algorithm has been implemented in order to solve problems 1 and 2. The first example has been analyzed in (Geromel *et al.*, 1994) and also in a preliminary version of (Yaesh and Shaked, 1991). The matrices are

$$A = \begin{bmatrix} 0.9974 & a_{12} \\ -0.1078 & 1.1591 \end{bmatrix},\quad B_2 = \begin{bmatrix} 0.0013 \\ b_{22} \end{bmatrix}$$

$$a_{12},\ b_{22} \in [0.0270, 0.0809]$$

$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},\quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the nominal value for the uncertain parameters is $a_{12} = b_{22} = 0.0539$. For the nominal plant, the solution of the optimal $\mathcal{H}_\infty$ state feedback, i.e., Problem 2, yields $\gamma^* = 36.4438$ dB and

$$K = \begin{bmatrix} -37.2072 & -22.5218 \end{bmatrix}$$

Figure 1 shows the performance of the algorithm towards the optimum (the first 5 iterations have been neglected). In (Geromel *et al.*, 1994) (only sufficient conditions) the same example is solved, yielding $\gamma^*_{suf} = 38.65$ dB as the minimum value ($\approx 6\%$ greater). Using a Fibonacci line search procedure, the results of Lemma 2 (Yaesh and Shaked, 1991) and the upper bound to $\gamma^{-2} = 0.0069$ (see a discussion concerning the upper bound in (Geromel *et al.*, 1995)), we get

$$K_F = \begin{bmatrix} -37.2072 & -22.5218 \end{bmatrix} \quad , \quad \gamma_F^* = 36.4438 \text{ dB}$$

Notice that, within the numerical tolerance considered, there exists no difference between the control gains nor between the optimal values of γ. Now, in order to compare

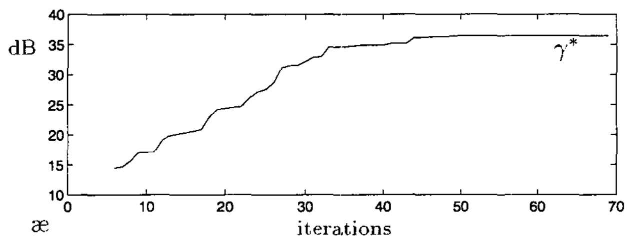

Fig. 1 - Evolution of γ.

the results of Lemma 2 (Yaesh and Shaked, 1991) with the solution of Problem 1, $\gamma = 36.4450$ dB has been imposed and the solution, using the results of this paper, is

$$\mathcal{W} = \begin{bmatrix} 5.9687 & -10.0579 & 4.5668 \\ -10.0579 & 25.6248 & -202.7823 \\ 4.5668 & -202.7823 & 4,390.1423 \end{bmatrix}$$

$$K = W_2'W_1^{-1} = \begin{bmatrix} -37.1256 & -22.4856 \end{bmatrix}, \rho = 72.9513 \text{ dB}$$

and using the results of Lemma 2 (Yaesh and Shaked, 1991)

$$K_{YS} = \begin{bmatrix} -37.1242 & -22.4845 \end{bmatrix}$$

The upper bound to the $\mathcal{H}_2$-norm is also satisfied, since the closed-loop $\mathcal{H}_2$-norm calculated with K is $\| G\|_2^2 = 72.8525$ dB $< \rho$. A comparison with the sufficient conditions of (Geromel *et al.*, 1995) is not possible in this case, since even with $\gamma = 38$ dB the procedure in (Geromel *et al.*, 1995) fails to find a feasible solution.

To verify, according to Lemma 3, the equivalence between the above numerical results, the solution of equation (18) and the calculated $\hat{Y}$, using (19), are given respectively by

$$P = \begin{bmatrix} 0.0020 & -0.0272 \\ -0.0272 & 0.9604 \end{bmatrix}, \hat{Y} = \begin{bmatrix} 5.9687 & -10.0578 \\ -10.0578 & 25.6261 \end{bmatrix}$$

Note that $\hat{Y}$ practically coincides with the submatrix W_1, as expected. Finally, considering the whole uncertain domain, Problem 2 has been solved for $N = 4$ yielding the guaranteed $\mathcal{H}_\infty$ control gain

$$K = \begin{bmatrix} -56.4392 & -23.0584 \end{bmatrix} \quad , \quad \gamma^* = 50.4415 \text{ dB}$$

This result represents an improvement when compared with the ones provided in (Geromel *et al.*, 1994), where the approximate convex solution gives $\gamma = 51.46$ dB. Indeed, the calculated $\mathcal{H}_\infty$-norm at the vertices F_i, $i = 1 \cdots 4$, yields (dB values)

$$\|G_{F1}\|_\infty = 50.4289 \ , \quad \|G_{F2}\|_\infty = 49.7296$$

$$\|G_{F3}\|_\infty = 46.2520 \ , \quad \|G_{F4}\|_\infty = 46.3043$$

showing that our guaranteed cost is very close to the worst case $\mathcal{H}_\infty$-norm.

As a second example, a decentralized control problem is considered. The data (the entries of matrices A and B have been generated randomly, chosen from a normal distribution with mean zero and variance one) are

$$A = \begin{bmatrix} -0.7843 & 0.6101 & -0.3520 & -0.0609 \\ 0.3840 & 0.4785 & -0.2265 & -0.4703 \\ 0.8395 & 0.6206 & 1.3293 & 0.2635 \\ 0.4718 & -1.7705 & -1.6552 & 0.6473 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0.8143 & 1.0369 \\ -0.7091 & -0.0578 \\ -0.6700 & -0.8233 \\ -0.9889 & 1.9561 \end{bmatrix}$$

and

$$B_1 = \mathbf{I}_4 \ , \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ , \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problems 1 and 2 have been solved considering the additional structural constraint (40). Imposing $\gamma = 23.5218$ dB, the solution of Problem 1 gives

$$K_{P1} = \begin{bmatrix} 0.5293 & 0.2651 & 0 & 0 \\ 0 & 0 & 0.4175 & -0.7648 \end{bmatrix}$$

$$\rho = 42.3845 \text{ dB}$$

and the calculated norms are $\| G\|_2^2 = 33.0176$ dB $< \rho$ and $\| G\|_\infty = 23.0314$ dB $< \gamma$.

The solution of the decentralized $\mathcal{H}_\infty$ control is

$$K_{P2} = \begin{bmatrix} 0.5532 & 0.2120 & 0 & 0 \\ 0 & 0 & 0.4192 & -0.7836 \end{bmatrix}$$

$$\gamma^* = 22.9613 \text{ dB}$$

yielding $\| G\|_2^2 = 33.1066$ dB and $\| G\|_\infty = 22.6278$ dB $< \gamma^*$. Notice that, in the first case, the upper bound to the $\mathcal{H}_\infty$-norm is very close to the optimal value, justifying the gap between ρ and the $\mathcal{H}_2$-norm calculated. If the $\mathcal{H}_\infty$ constraint is relaxed, the problem tends to the decentralized $\mathcal{H}_2$ guaranteed control problem (Geromel *et al.*, 1993). Observe also that the sufficiency of the imposed decentralized structure may lead to larger gaps.

5 - CONCLUSION

The linear approach to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and the $\mathcal{H}_\infty$ control problems for discrete-time uncertain systems proposed in this paper has improved the existing results mainly in two ways. First, from a theoretical point of view, the parameter space used here describes the problems in a simple and direct style, keeping convexity (in fact, linearity) and allowing an immediate extension to handle uncertain convex bounded domains and decentralized control. The second point is that, thanks to the adopted formulation, powerful numerical procedures can be used to solve the proposed problems.

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